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UNIFORM DISTRIBUTION MODULO ONE ON SUBSEQUENCES

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ABSTRACT. Let $\mathcal{P}$ be a set of primes with a divergent series of reciprocals and let $\mathcal{K} = \mathcal{K}(\mathcal{P})$ denote the set of squarefree integers greater than one that are divisible only by primes in $\mathcal{P}$. G. Myerson and A. D. Pollington proved that $(u_n)_{n\geq 1} \subset [0, 1)$ is uniformly distributed (mod 1) whenever the subsequence $(u_{kn})_{n\geq 1}$ is uniformly distributed (mod 1) for every k in $\mathcal{K}$. We show that in fact $(u_n)_{n\geq 1}$ is uniformly distributed (mod 1) whenever the subsequence $(u_{pn})_{n\geq 1}$ is uniformly distributed (mod 1) for every $p \in \mathcal{P}$.

1. INTRODUCTION

In [4], G. Myerson and A. D. Pollington proved several intriguing results about uniform distribution modulo one, including the following theorem.

Theorem A (Myerson-Pollington). *Let $\mathcal{P}$ be a set of primes such that $\sum\{1/p : p \in \mathcal{P}\}$ diverges. Let $\mathcal{K} = \mathcal{K}(\mathcal{P})$ denote the set of squarefree integers greater than one divisible only by primes in $\mathcal{P}$. If $(u_{kn})_{n\geq 1}$ is uniformly distributed (mod 1) for every k in $\mathcal{K}$, then $(u_n)_{n\geq 1}$ is uniformly distributed (mod 1).*

The set $\mathcal{K}$ in Theorem A functions as a kind of "test set," used to probe the distribution of the sequence (u_n). The following definition formalizes this notion.

Definition. A set S of integers greater than 1 is called a *u.d. test set* if the sequence $(u_n)_{n\geq 1} \subset [0, 1)$ is uniformly distributed whenever the subsequence $(u_{sn})_{n\geq 1}$ is uniformly distributed for all $s \in S$.

Note that if S is a set of positive integers greater than 1 and S contains a u.d. test set, then *a fortiori* S is a u.d. test set.

It is natural to look for other u.d. test sets besides the sets $\mathcal{K}(\mathcal{P})$ and their supersets. Our main result, Theorem 1 below, shows that the set of primes $\mathcal{P}$ in Theorem A is itself a u.d. test set. Note that Theorem A is a consequence of Theorem 1, because in the former result $\mathcal{P} \subset \mathcal{K}$.

Theorem 1. *Let $\mathcal{P}$ be a set of primes. Then $\mathcal{P}$ is a u.d. test set if and only if $\sum\{1/p : p \in \mathcal{P}\}$ diverges.*

Myerson and Pollington proved Theorem A by using the Weyl criterion for uniform distribution and properties of the Möbius μ-function. The path that we take

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to Theorem 1 is quite different. We introduce a second type of integer set, called a *density test set*, whose definition makes no reference to uniform distribution. In Theorem 2, stated below, the property of being a u.d. test set is shown to be equivalent to that of being a density test set. After proving Theorem 2, we provide a sufficient condition for being a density test set. Finally, we prove that a set of primes is a density test set if and only if the series of reciprocals of its elements diverges.

We require some notation to state the definition of a density test set.

Notation. Let $A \subseteq \mathbb{N}$. We let $\mathbf{d}A = \lim_{N\to\infty} \frac{1}{N}\#\{n \le N : n \in A\}$ denote the natural density of A, provided the limit exists. For each $s \in \mathbb{N}$, we define $A_s = \{n \in \mathbb{N} : sn \in A\}$. One may think of $\mathbf{d}A_s$ (if it exists) as being the density of the set $\{n \in A : s|n\}$ *relative to* the set $\{s, 2s, 3s, \dots\}$.

Definition. Let S be a set of integers greater than 1. We say that S is a *density test set* if whenever a set A of positive integers has the property that there exists $\alpha \in [0,1]$ such that $\mathbf{d}A_s = \alpha$ for each $s \in S$, then $\mathbf{d}A = \alpha$.

Theorem 2. *S is a u.d. test set if and only if S is a density test set.*

2. Proof of Theorem 2

We need a well-known result from the theory of uniform distribution. A proof is given in [3, Theorem I.4.1].

Lemma 3. *Let $(a_n)_{n\ge1}$ be a given sequence of distinct integers. Then the sequence $(a_n x)_{n\ge1}$ is uniformly distributed (mod 1) for almost all real numbers x.*

Proof of Theorem 2. Assume that S is a density test set. Let $(u_n)_{n\ge1} \subset [0,1)$, and suppose that $(u_{sn})_{n\ge1}$ is uniformly distributed for each $s \in S$. Let $x \in (0,1]$ be fixed, and let $A = A(x) = \{n : 0 \le u_n < x\}$. For each $s \in S$ we have

$$\frac{1}{N}\#\{n \le N : n \in A_s\} = \frac{1}{N}\#\{n \le N : 0 \le u_{sn} < x\} \to x \quad \text{as } N \to \infty,$$

that is, $\mathbf{d}A_s = x$. Since S is a density test set, we may conclude that $\mathbf{d}A = x$. Then

$$\frac{1}{N}\#\{n \le N : 0 \le u_n < x\} = \frac{1}{N}\#\{n \le N : n \in A\} \to x \quad \text{as } N \to \infty.$$

It follows that (u_n) is uniformly distributed. Thus, S is a u.d. test set.

Next, we assume that S is a u.d. test set. Let A be a set of positive integers, and suppose that there is an $\alpha \in [0,1]$ such that $\mathbf{d}A_s = \alpha$ for each $s \in S$. It is convenient to consider the cases $0 < \alpha < 1$, $\alpha = 1$, and $\alpha = 0$ separately.

We assume first that $0 < \alpha < 1$. Let $A'_s = \mathbb{N}\backslash A_s$ for each $s \in S$. The sets A_s and A'_s have positive density, and hence are infinite. For each $s \in S$, let Γ_s denote the set of all real $x \in [0,1]$ for which the sequence $(snx : n \in A_s)$ is uniformly distributed (mod 1), and let Γ'_s denote the set of all real $x \in [0,1]$ for which $(snx : n \in A'_s)$ is uniformly distributed (mod 1). Let $\Gamma = (\bigcap_{s\in S}\Gamma_s) \cap (\bigcap_{s\in S}\Gamma'_s)$. By Lemma 3, each of the sets $[0,1]\backslash\Gamma_s$ and $[0,1]\backslash\Gamma'_s$ has measure 0, whence $[0,1]\backslash\Gamma$ has measure 0. In particular, Γ is not empty. Select any $\gamma \in \Gamma$. Put $w_n = \{n\gamma\}$ for $n \ge 1$, that is, w_n is the fractional part of $n\gamma$. By construction, the sequences $(w_{sn} : n \in A_s)$ and $(w_{sn} : n \in A'_s)$ are uniformly distributed for each $s \in S$. We now define a sequence

$(u_n)_{n\geq 1} \subset [0,1)$ by

$$u_n = \begin{cases} \alpha w_n, & \text{if } n \in A, \\ \alpha + (1-\alpha) w_n, & \text{otherwise.} \end{cases}$$

Let $s \in S$. We claim that $(u_{sn})_{n\geq 1}$ is uniformly distributed. Take $0 < x \leq 1$. If $x \leq \alpha$, then

$$\begin{aligned} \frac{1}{N}\#\{n \leq N : 0 \leq u_{sn} < x\} &= \frac{1}{N}\#\{n \leq N : sn \in A,\, 0 \leq \alpha w_{sn} < x\} \\ &= \frac{1}{N}\#\{n \leq N : n \in A_s,\, 0 \leq w_{sn} < x/\alpha\} \\ &\to \mathbf{d}A_s \cdot \frac{x}{\alpha} = x \quad \text{as } N \to \infty, \end{aligned}$$

since $(w_{sn} : n \in A_s)$ is uniformly distributed. On the other hand, if $x > \alpha$, then

$$\begin{aligned} \frac{1}{N}\#\{n \leq N : 0 \leq u_{sn} < x\} &= \frac{1}{N}\#\{n \leq N : sn \in A\} \\ &\quad + \frac{1}{N}\#\{n \leq N : sn \notin A,\, 0 \leq \alpha + (1-\alpha) w_{sn} < x\} \\ &= \frac{1}{N}\#\{n \leq N : n \in A_s\} \\ &+ \frac{1}{N}\#\{n \leq N : n \in A_s',\, 0 \leq w_{sn} < (x-\alpha)/(1-\alpha)\} \\ &\to \mathbf{d}A_s + \mathbf{d}A_s' \cdot \frac{x-\alpha}{1-\alpha} = x \quad \text{as } N \to \infty \end{aligned}$$

because $(w_{sn} : n \in A_s')$ is uniformly distributed. Hence, $(u_{sn})_{n\geq 1}$ is uniformly distributed, as claimed. We deduce that $(u_n)_{n\geq 1}$ is uniformly distributed, by virtue of S being a u.d. test set. Noting that $A = \{n : 0 \leq u_n < \alpha\}$, we see that $\mathbf{d}A$ exists and equals α.

Suppose now that $\alpha = 1$. Define the sets Γ_s for $s \in S$ as above, and let $\Gamma = \bigcap_{s\in S} \Gamma_s$. Select $\gamma \in \Gamma$, and let $w_n = \{n\gamma\}$ for $n \geq 1$. We define a sequence $(u_n)_{n\geq 1} \subset [0,1)$ by

$$u_n = \begin{cases} w_n, & \text{if } n \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Arguing as in the last paragraph, we show that the subsequences $(u_{sn})_{n\geq 1}$ are uniformly distributed for each $s \in S$. Then (u_n) is itself uniformly distributed because S is a u.d. test set. In particular, $1 - \mathbf{d}A = \mathbf{d}\{n : u_n = 0\} = 0$, and hence, $\mathbf{d}A = \alpha$.

Finally, we assume that $\alpha = 0$. Define the sets Γ_s' for $s \in S$ as before, and let $\Gamma = \bigcap_{s\in S} \Gamma_s'$. Select $\gamma \in \Gamma$, and let $w_n = \{n\gamma\}$ for $n \geq 1$. We define $(u_n)_{n\geq 1} \subset [0,1)$ by

$$u_n = \begin{cases} 0, & \text{if } n \in A, \\ w_n, & \text{otherwise.} \end{cases}$$

As above, we show that the subsequences $(u_{sn})_{n\geq 1}$ are uniformly distributed for each $s \in S$, which implies that (u_n) is itself uniformly distributed because S is a u.d. test set. Consequently, $\mathbf{d}A = \mathbf{d}\{n : u_n = 0\} = 0 = \alpha$.

With all cases checked, we conclude that S is a density test set. $\square$

3. Proof of Theorem 1

We will deduce the principal implication in Theorem 1 from the following proposition.

Proposition 4. *Let S be a set of integers greater than 1 such that $\sum\{1/s : s \in S\} = \infty$, and let $f(y) := \sum\{1/s : s \in S, s \le y\}$. Let $[s,t]$ denote the least common multiple of positive integers s and t. Suppose that*

$$\sum_{\substack{s,t\in S\\ s,t\le y}} \frac{1}{[s,t]} \sim f(y)^2 \quad (y \to \infty). \tag{1}$$

Then S is a density test set.

Proof. Let $\tau_y(n,S)$ denote the number of $s \in S$ such that $s \le y$ and $s|n$. In the course of the proof, we use Turán's variance method (see [1, Chapter 4], for example) to show that $\tau_y(n,S)$ is close to $f(y)$ for "most" n.

Let A be a set of positive integers, and suppose that there exists $\alpha \in [0,1]$ such that $\mathbf{d}A_s = \alpha$ for each $s \in S$.

Let $1 \le y \le x$. Observe that

$$\begin{aligned}\frac{1}{x}\sum_{\substack{n\le x\\ n\in A}} \tau_y(n,S) &= \frac{1}{x}\sum_{\substack{n\le x\\ n\in A}}\sum_{\substack{s\le y\\ s\in S\\ s|n}} 1 = \sum_{\substack{s\le y\\ s\in S}} \frac{1}{x}\sum_{\substack{n\le x\\ n\in A\\ s|n}} 1 = \sum_{\substack{s\le y\\ s\in S}}\left(\frac{\alpha}{s} + o_s(1)\right)\\ &= \alpha f(y) + o_y(1) \quad (x\to\infty).\end{aligned}$$

It follows that

$$\frac{1}{x}\sum_{\substack{n\le x\\ n\in A}} 1 + \frac{1}{x}\sum_{\substack{n\le x\\ n\in A}}\left(\frac{\tau_y(n,S) - f(y)}{f(y)}\right) = \alpha + o_y(1) \quad (x\to\infty),$$

so that

$$\limsup_{x\to\infty}\left|\frac{1}{x}\sum_{\substack{n\le x\\ n\in A}} 1 - \alpha\right| \le \limsup_{x\to\infty}\frac{1}{x}\sum_{n\le x}\left|\frac{\tau_y(n,S) - f(y)}{f(y)}\right| \tag{2}$$

for all y large enough that $f(y) > 0$.

We now further restrict y so that $1 \le y \le \sqrt{x}$. We have

$$\sum_{n\le x}\tau_y(n,S)^2 = \sum_{n\le x}\sum_{\substack{[s,t]|n\\ s,t\in S\\ s,t\le y}} 1 = \sum_{\substack{[s,t]\le x\\ s,t\in S\\ s,t\le y}}\left[\frac{x}{[s,t]}\right] = x\sum_{\substack{s,t\in S\\ s,t\le y}}\frac{1}{[s,t]} + O(x).$$

A similar but simpler argument shows that

$$\sum_{n\le x}\tau_y(n,S) = xf(y) + O(y).$$

We combine our last two results to obtain

$$\begin{aligned}\sum_{n\le x}(\tau_y(n,S) - f(y))^2 &= \sum_{n\le x}\tau_y(n,S)^2 - 2f(y)\sum_{n\le x}\tau_y(n,S) + [x]f(y)^2\\ &= x\left(\sum_{\substack{s,t\in S\\ s,t\le y}}\frac{1}{[s,t]} - f(y)^2\right) + O(x).\end{aligned}$$

Using the hypothesis (1) we deduce that

$$\limsup_{x\to\infty} \frac{1}{x} \sum_{n\le x} \left(\frac{\tau_y(n,S) - f(y)}{f(y)} \right)^2 = o(1) \qquad (y \to \infty). \tag{3}$$

From (2), the Cauchy-Schwarz inequality, and (3), we have

$$\begin{aligned} \limsup_{x\to\infty} \left| \frac{1}{x} \sum_{\substack{n\le x \\ n\in A}} 1 - \alpha \right| &\le \limsup_{x\to\infty} \frac{1}{x} \sum_{n\le x} \left| \frac{\tau_y(n,S) - f(y)}{f(y)} \right| \cdot 1 \\ &\le \limsup_{x\to\infty} \frac{1}{x} \left\{ \sum_{n\le x} \left| \frac{\tau_y(n,S) - f(y)}{f(y)} \right|^2 \right\}^{1/2} \cdot x^{1/2} \\ &= o(1) \qquad (y \to \infty). \end{aligned}$$

We let $y \to \infty$ to conclude that $\mathbf{d}A = \alpha$. Thus, S is a density test set. □

While Proposition 4 provides us with a sufficient condition for being a density test set, the next lemma offers a necessary condition. To state the result concisely, we use some terminology from the theory of sets of multiples. For $S \subseteq \mathbb{N}$, let $\mathcal{M}(S) = \{ms : m \ge 1, s \in S\}$ be the set of multiples of numbers in S. A set $S \subseteq \mathbb{N}\backslash\{1\}$ is said to be *Behrend* if $\mathcal{M}(S)$ has natural density 1. The interested reader can find a detailed discussion of Behrend sets in [2].

Lemma 5. *If S is a density test set, then S is Behrend.*

Proof. Let S be a density test set, and set $A = \mathcal{M}(S)$. Since $\mathbf{d}A_s = 1$ for each $s \in S$, we have $\mathbf{d}A = 1$. Thus, S is Behrend. □

The proof of Theorem 1 uses a basic property of Behrend sets, which we state in Lemma 6. A proof is given in [2, Corollary 0.10].

Lemma 6. *The series of reciprocals of elements from a Behrend set diverges.*

Proof of Theorem 1. In view of Theorem 2, it suffices to show that $\mathcal{P}$ is a density test set if and only if $\sum\{1/p : p \in \mathcal{P}\}$ diverges.

If $\mathcal{P}$ is a density test set, then $\sum\{1/p : p \in \mathcal{P}\}$ diverges by Lemmas 5 and 6.

Now assume that $\sum\{1/p : p \in \mathcal{P}\} = \infty$. Let $f(y) = \sum\{1/p : p \in \mathcal{P}, p \le y\}$. We have

$$\sum_{\substack{p,q\in\mathcal{P} \\ p,q\le y}} \frac{1}{[p,q]} = f(y)^2 + f(y) + O(1) \sim f(y)^2 \quad (y \to \infty).$$

Then $\mathcal{P}$ is a density test set by Proposition 4. □

4. Concluding remarks

A problem that is complementary to the one considered in this paper is the following. If $(u_n) \subset [0,1)$ is uniformly distributed, need the subsequence $(u_{kn})_{n\ge1}$ be uniformly distributed for some $k > 1$? Myerson and Pollington [4] have shown that the answer is a resounding "no." They constructed a uniformly distributed sequence (u_n) such that the subsequence $(u_{kn+j})_{n\ge1}$ is not uniformly distributed for *all* integers $k \ge 2$ and $j \ge 0$.

It would be of interest to find a simple characterization of density test sets (or equivalently, of u.d. test sets). One might conjecture that a set S is a density test set if and only if S is Behrend. Indeed, Theorem 1 and Lemmas 5 and 6 establish this equivalence when S consists only of primes. We can prove that the equivalence also holds when the elements of S have at most two prime factors, but at this point we are unable to settle the general case.

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