UNIFORM DISTRIBUTION MODULO ONE ON SUBSEQUENCES

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(Communicated by David E. Rohrlich)

Abstract. Let \( \mathcal{P} \) be a set of primes with a divergent series of reciprocals and let \( \mathcal{K} = \mathcal{K}(\mathcal{P}) \) denote the set of squarefree integers greater than one that are divisible only by primes in \( \mathcal{P} \). G. Myerson and A. D. Pollington proved that \( (u_n)_{n \geq 1} \subset [0,1) \) is uniformly distributed (mod 1) whenever the subsequence \( (u_{kn})_{n \geq 1} \) is uniformly distributed (mod 1) for every \( k \) in \( \mathcal{K} \). We show that in fact \( (u_n)_{n \geq 1} \) is uniformly distributed (mod 1) whenever the subsequence \( (u_{pn})_{n \geq 1} \) is uniformly distributed (mod 1) for every \( p \in \mathcal{P} \).

1. Introduction

In [4], G. Myerson and A. D. Pollington proved several intriguing results about uniform distribution modulo one, including the following theorem.

Theorem A (Myerson-Pollington). Let \( \mathcal{P} \) be a set of primes such that \( \sum \{1/p : p \in \mathcal{P} \} \) diverges. Let \( \mathcal{K} = \mathcal{K}(\mathcal{P}) \) denote the set of squarefree integers greater than one divisible only by primes in \( \mathcal{P} \). If \( (u_{kn})_{n \geq 1} \) is uniformly distributed (mod 1) for every \( k \) in \( \mathcal{K} \), then \( (u_n)_{n \geq 1} \) is uniformly distributed (mod 1).

The set \( \mathcal{K} \) in Theorem A functions as a kind of “test set,” used to probe the distribution of the sequence \( (u_n) \). The following definition formalizes this notion.

Definition. A set \( S \) of integers greater than 1 is called a u.d. test set if the sequence \( (u_n)_{n \geq 1} \subset [0,1) \) is uniformly distributed whenever the subsequence \( (u_{sn})_{n \geq 1} \) is uniformly distributed for all \( s \in S \).

Note that if \( S \) is a set of positive integers greater than 1 and \( S \) contains a u.d. test set, then a fortiori \( S \) is a u.d. test set.

It is natural to look for other u.d. test sets besides the sets \( \mathcal{K}(\mathcal{P}) \) and their supersets. Our main result, Theorem 1 below, shows that the set of primes \( \mathcal{P} \) in Theorem A is itself a u.d. test set. Note that Theorem A is a consequence of Theorem 1, because in the former result \( \mathcal{P} \subset \mathcal{K} \).

Theorem 1. Let \( \mathcal{P} \) be a set of primes. Then \( \mathcal{P} \) is a u.d. test set if and only if \( \sum \{1/p : p \in \mathcal{P} \} \) diverges.

Myerson and Pollington proved Theorem A by using the Weyl criterion for uniform distribution and properties of the Möbius \( \mu \)-function. The path that we take...
to Theorem 1 is quite different. We introduce a second type of integer set, called a density test set, whose definition makes no reference to uniform distribution. In Theorem 2, stated below, the property of being a u.d. test set is shown to be equivalent to that of being a density test set. After proving Theorem 2, we provide a sufficient condition for being a density test set. Finally, we prove that a set of primes is a density test set if and only if the series of reciprocals of its elements diverges.

We require some notation to state the definition of a density test set.

Notation. Let $A \subseteq \mathbb{N}$. We let $dA = \lim_{N \to \infty} \frac{1}{N} \#\{n \leq N : n \in A\}$ denote the natural density of $A$, provided the limit exists. For each $s \in \mathbb{N}$, we define $A_s = \{n \in \mathbb{N} : sn \in A\}$. One may think of $dA_s$ (if it exists) as being the density of the set $\{n \in A : s|n\}$ relative to the set $\{s, 2s, 3s, \ldots\}$.

Definition. Let $S$ be a set of integers greater than 1. We say that $S$ is a density test set if and only if $S$ is a u.d. test set.

Theorem 2. $S$ is a u.d. test set if and only if $S$ is a density test set.

2. Proof of Theorem 2

We need a well-known result from the theory of uniform distribution. A proof is given in [3, Theorem 1.4.1].

Lemma 3. Let $(a_n)_{n \geq 1}$ be a given sequence of distinct integers. Then the sequence $(a_nx)_{n \geq 1}$ is uniformly distributed (mod 1) for almost all real numbers $x$.

Proof of Theorem 2. Assume that $S$ is a density test set. Let $(u_n)_{n \geq 1} \subseteq [0, 1)$, and suppose that $(usn)_{n \geq 1}$ is uniformly distributed for each $s \in S$. Let $x \in (0, 1]$ be fixed, and let $A = A(x) = \{n : 0 \leq u_n < x\}$. For each $s \in S$ we have

$$\frac{1}{N} \#\{n \leq N : n \in A_s\} = \frac{1}{N} \#\{n \leq N : 0 \leq u_{sn} < x\} \to x \quad \text{as } N \to \infty,$$

that is, $dA_s = x$. Since $S$ is a density test set, we may conclude that $dA = x$. Then

$$\frac{1}{N} \#\{n \leq N : 0 \leq u_n < x\} = \frac{1}{N} \#\{n \leq N : n \in A\} \to x \quad \text{as } N \to \infty.$$

It follows that $(u_n)$ is uniformly distributed. Thus, $S$ is a u.d. test set.

Next, we assume that $S$ is a u.d. test set. Let $A$ be a set of positive integers, and suppose that there is an $\alpha \in [0, 1]$ such that $dA_s = \alpha$ for each $s \in S$. It is convenient to consider the cases $0 < \alpha < 1$, $\alpha = 1$, and $\alpha = 0$ separately.

We assume first that $0 < \alpha < 1$. Let $A'_s = \mathbb{N}\backslash A_s$ for each $s \in S$. The sets $A_s$ and $A'_s$ have positive density, and hence are infinite. For each $s \in S$, let $\Gamma_s$ denote the set of all real $x \in [0, 1]$ for which the sequence $(snx : n \in A_s)$ is uniformly distributed (mod 1), and let $\Gamma'_s$ denote the set of all real $x \in [0, 1]$ for which $(snx : n \in A'_s)$ is uniformly distributed (mod 1). Let $\Gamma = (\bigcap_{s \in S} \Gamma_s) \cap (\bigcap_{s \in S} \Gamma'_s)$. By Lemma 3, each of the sets $[0, 1]\backslash \Gamma_s$ and $[0, 1]\backslash \Gamma'_s$ has measure 0, whence $[0, 1]\backslash \Gamma$ has measure 0. In particular, $\Gamma$ is not empty. Select any $\gamma \in \Gamma$. Put $w_n = \{n\gamma\}$ for $n \geq 1$, that is, $w_n$ is the fractional part of $n\gamma$. By construction, the sequences $(w_{sn} : n \in A_s)$ and $(w_{sn} : n \in A'_s)$ are uniformly distributed for each $s \in S$. We now define a sequence
Let $s \in S$. We claim that $(u_{sn})_{n \geq 1}$ is uniformly distributed. Take $0 < x \leq 1$. If $x \leq \alpha$, then
\[
\frac{1}{N} \# \{ n \leq N : 0 \leq u_{sn} < x \} = \frac{1}{N} \# \{ n \leq N : sn \in A, 0 \leq \alpha w_{sn} < x \}
\]
\[
= \frac{1}{N} \# \{ n \leq N : n \in A_s, 0 \leq w_{sn} < x/\alpha \}
\]
\[
\to dA_s \cdot \frac{x}{\alpha} = x \quad \text{as} \quad N \to \infty,
\]
since $(w_{sn} : n \in A_s)$ is uniformly distributed. On the other hand, if $x > \alpha$, then
\[
\frac{1}{N} \# \{ n \leq N : 0 \leq u_{sn} < x \} = \frac{1}{N} \# \{ n \leq N : sn \notin A, 0 \leq \alpha + (1-\alpha)w_{sn} < x \}
\]
\[
+ \frac{1}{N} \# \{ n \leq N : n \notin A_s, 0 \leq w_{sn} < (x/\alpha)/(1-\alpha) \}
\]
\[
\to dA_s + dA'_s \cdot \frac{x}{1-\alpha} = x \quad \text{as} \quad N \to \infty
\]
because $(w_{sn} : n \in A'_s)$ is uniformly distributed. Hence, $(u_{sn})_{n \geq 1}$ is uniformly distributed, as claimed. We deduce that $(u_n)_{n \geq 1}$ is uniformly distributed, by virtue of $S$ being a u.d. test set. Noting that $A = \{ n : 0 \leq u_n < \alpha \}$, we see that $dA$ exists and equals $\alpha$.

Suppose now that $\alpha = 1$. Define the sets $\Gamma_s$ for $s \in S$ as above, and let $\Gamma = \bigcap_{s \in S} \Gamma_s$. Select $\gamma \in \Gamma$, and let $w_n = \{n\gamma\}$ for $n \geq 1$. We define a sequence $(u_n)_{n \geq 1} \subset [0,1)$ by
\[
u_n = \begin{cases} w_n, & \text{if } n \in A, \\ 0, & \text{otherwise}. \end{cases}
\]
Arguing as in the last paragraph, we show that the subsequences $(u_{sn})_{n \geq 1}$ are uniformly distributed for each $s \in S$. Then $(u_n)$ is itself uniformly distributed because $S$ is a u.d. test set. In particular, $1 - dA = d\{ n : u_n = 0 \} = 0$, and hence, $dA = \alpha$.

Finally, we assume that $\alpha = 0$. Define the sets $\Gamma'_s$ for $s \in S$ as before, and let $\Gamma = \bigcap_{s \in S} \Gamma'_s$. Select $\gamma \in \Gamma$, and let $w_n = \{n\gamma\}$ for $n \geq 1$. We define $(u_n)_{n \geq 1} \subset [0,1)$ by
\[
u_n = \begin{cases} 0, & \text{if } n \in A, \\ w_n, & \text{otherwise}. \end{cases}
\]
As above, we show that the subsequences $(u_{sn})_{n \geq 1}$ are uniformly distributed for each $s \in S$, which implies that $(u_n)$ is itself uniformly distributed because $S$ is a u.d. test set. Consequently, $dA = d\{ n : u_n = 0 \} = 0 = \alpha$.

With all cases checked, we conclude that $S$ is a density test set. \qed
3. Proof of Theorem 1

We will deduce the principal implication in Theorem 1 from the following proposition.

**Proposition 4.** Let $S$ be a set of integers greater than 1 such that $\sum \{1/s : s \in S\} = \infty$, and let $f(y) := \sum \{1/s : s \in S, s \leq y\}$. Let $[s,t]$ denote the least common multiple of positive integers $s$ and $t$. Suppose that

$$\sum_{s,t \leq y} \frac{1}{s,t} \sim f(y)^2 \quad (y \to \infty).$$

Then $S$ is a density test set.

**Proof.** Let $\tau_y(n,S)$ denote the number of $s \in S$ such that $s \leq y$ and $s|n$. In the course of the proof, we use Turán’s variance method (see [1, Chapter 4], for example) to show that $\tau_y(n,S)$ is close to $f(y)$ for “most” $n$.

Let $A$ be a set of positive integers, and suppose that there exists $\alpha \in [0,1]$ such that $d_{Ax} = \alpha$ for each $s \in S$.

Let $1 \leq y \leq x$. Observe that

$$\frac{1}{x} \sum_{n \in A, n \leq x} \tau_y(n,S) = \frac{1}{x} \sum_{s \leq y} \sum_{n \in A} \frac{1}{s} = \sum_{s \leq y} \sum_{n \in A} \frac{1}{s} \left( \frac{\alpha}{s} + o_s(1) \right)$$

$$= \alpha f(y) + o_y(1) \quad (x \to \infty).$$

It follows that

$$\frac{1}{x} \sum_{n \in A, n \leq x} 1 + \frac{1}{x} \sum_{s \leq y} \left( \frac{\tau_y(n,S) - f(y)}{f(y)} \right) = \alpha + o_y(1) \quad (x \to \infty),$$

so that

$$\limsup_{x \to \infty} \frac{1}{x} \sum_{n \leq x} 1 - \alpha \leq \limsup_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \left| \frac{\tau_y(n,S) - f(y)}{f(y)} \right|$$

for all $y$ large enough that $f(y) > 0$.

We now further restrict $y$ so that $1 \leq y \leq \sqrt{x}$. We have

$$\sum_{n \leq x} \tau_y(n,S)^2 = \sum_{n \leq x} \sum_{[s,t] \mid n, s,t \leq y} 1 = \sum_{[s,t] \leq x} \left[ \frac{x}{[s,t]} \right] = x \sum_{[s,t] \leq y} \frac{1}{[s,t]} + O(x).$$

A similar but simpler argument shows that

$$\sum_{n \leq x} \tau_y(n,S) = xf(y) + O(y).$$

We combine our last two results to obtain

$$\sum_{n \leq x} (\tau_y(n,S) - f(y))^2 = \sum_{n \leq x} \tau_y(n,S)^2 - 2f(y) \sum_{n \leq x} \tau_y(n,S) + [x] f(y)^2$$

$$= x \left( \sum_{n \leq x} \frac{1}{[s,t]} - f(y)^2 \right) + O(x).$$
Using the hypothesis (1) we deduce that
\[ \limsup_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \left( \frac{\tau_y(n, S) - f(y)}{f(y)} \right)^2 = o(1) \quad (y \to \infty). \]

From (2), the Cauchy-Schwarz inequality, and (3), we have
\[ \limsup_{x \to \infty} \left| \frac{1}{x} \sum_{n \leq x} 1 - \alpha \right| \leq \limsup_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \left| \frac{\tau_y(n, S) - f(y)}{f(y)} \right| \cdot 1 \]
\[ \leq \limsup_{x \to \infty} \frac{1}{x} \left( \sum_{n \leq x} \left| \frac{\tau_y(n, S) - f(y)}{f(y)} \right|^2 \right)^{1/2} \cdot x^{1/2} \]
\[ = o(1) \quad (y \to \infty). \]

We let \( y \to \infty \) to conclude that \( dA = \alpha \). Thus, \( S \) is a density test set.

While Proposition 4 provides us with a sufficient condition for being a density test set, the next lemma offers a necessary condition. To state the result concisely, we use some terminology from the theory of sets of multiples. For \( S \subseteq \mathbb{N} \), let \( \mathcal{M}(S) = \{ ms : m \geq 1, s \in S \} \) be the set of multiples of numbers in \( S \). A set \( S \subseteq \mathbb{N} \setminus \{1\} \) is said to be **Behrend** if \( \mathcal{M}(S) \) has natural density 1. The interested reader can find a detailed discussion of Behrend sets in [2].

**Lemma 5.** If \( S \) is a density test set, then \( S \) is Behrend.

**Proof.** Let \( S \) be a density test set, and set \( A = \mathcal{M}(S) \). Since \( dA_s = 1 \) for each \( s \in S \), we have \( dA = 1 \). Thus, \( S \) is Behrend. \( \square \)

The proof of Theorem 1 uses a basic property of Behrend sets, which we state in Lemma 6. A proof is given in [2, Corollary 0.10].

**Lemma 6.** The series of reciprocals of elements from a Behrend set diverges.

**Proof of Theorem 1.** In view of Theorem 2, it suffices to show that \( P \) is a density test set if and only if \( \sum \{1/p : p \in P\} \) diverges.

If \( P \) is a density test set, then \( \sum \{1/p : p \in P\} \) diverges by Lemmas 5 and 6.

Now assume that \( \sum \{1/p : p \in P\} = \infty \). Let \( f(y) = \sum \{1/p : p \in P, p \leq y\} \). We have
\[ \sum_{p, q \in P, p \leq y} \frac{1}{[p, q]} = f(y)^2 + f(y) + O(1) \sim f(y)^2 \quad (y \to \infty). \]

Then \( P \) is a density test set by Proposition 4. \( \square \)

**4. Concluding remarks**

A problem that is complementary to the one considered in this paper is the following. If \( (u_n) \subset [0, 1) \) is uniformly distributed, need the subsequence \((u_{kn})_{n \geq 1}\) be uniformly distributed for some \( k > 1 \)? Myerson and Pollington [4] have shown that the answer is a resounding “no.” They constructed a uniformly distributed sequence \((u_n)\) such that the subsequence \((u_{kn+j})_{n \geq 1}\) is not uniformly distributed for all integers \( k \geq 2 \) and \( j \geq 0 \).

It would be of interest to find a simple characterization of density test sets (or equivalently, of u.d. test sets). One might conjecture that a set \( S \) is a density test
set if and only if \( S \) is Behrend. Indeed, Theorem 1 and Lemmas 5 and 6 establish this equivalence when \( S \) consists only of primes. We can prove that the equivalence also holds when the elements of \( S \) have at most two prime factors, but at this point we are unable to settle the general case.

References


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