A SHARP EXPONENTIAL INEQUALITY
FOR LORENTZ-SOBOLEV SPACES
ON BOUNDED DOMAINS

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Abstract. This paper generalizes an inequality of Moser from the case that \( \nabla u \) is in the Lebesgue space \( L^n \) to certain subspaces, namely the Lorentz spaces \( L^{n,q} \), where \( 1 < q \leq n \). The conclusion is that \( \exp(\alpha u) \) is integrable, where \( \frac{1}{p} + \frac{1}{q} = 1 \). This is a higher degree of integrability than in the Moser inequality when \( q < n \). A formula for \( \alpha \) is given and it is also shown that no larger value of \( \alpha \) works.

For \( n \geq 2 \), let \( D \) be a bounded domain in \( \mathbb{R}^n \) and let \( W^n(D) \) be the Sobolev space of functions defined as the completion of the space of \( C^\infty \) functions compactly supported in \( D \) whose gradient is in \( L^n(D) \). A well known result of J. Moser [7] is that, for functions \( u \) in the unit ball of \( W^n(D) \), there is a sharp constant \( \alpha = \alpha_n = n(\sigma_{n-1})^{1/(n-1)} \), where \( \sigma_{n-1} \) is the \( n-1 \) dimensional surface area of the unit sphere, such that

\[
\int_D \exp\left\{ \frac{\alpha u^n}{(n-1)} \right\} \leq A(n)m(D), \quad A(n) \text{ is independent of } u.
\]

We define a Lorentz-Sobolev space \( W^{n,q}(D) \) using Lorentz norms and prove a similar sharp exponential inequality. When \( q = n \), \( W^n(D) = W^{n,n}(D) \).

For real valued functions \( f \) on \( \mathbb{R}^n \), let \( f^* \) be the nonincreasing rearrangement of \( f \) defined as \( f^*(t) = \inf \{ s : m(\{|f| > s\} \leq t \} \). We define \( f^\#(x) \) to be the spherically symmetric nondecreasing rearrangement of \( f \) defined as \( f^\#(x) = f^*(\sigma_{n-1}|x|^{n}/n) \).

The Lorentz \( L(n, q) \) norm is defined as

\[
\|f\|_{n,q} = (q/n) \int_0^\infty [f^*(t)^{1/n}]^q dt/t \right]^{1/q}.
\]

The constant \( (q/n) \) ensures that \( \|\chi_E\|_{n,q} = (m(E))^{1/n} \). For \( 1 < q \leq n \), it is shown in [1] that \( L(n,q) \) is an actual norm.

Definition. Define \( W^{n,q}(D) \), \( 1 < q < n \), as the completion of the space of functions \( u \) in \( C^\infty \) compactly supported in \( D \) satisfying

\[
\|\nabla u\|_{n,q} < \infty.
\]
Theorem 1. Let $D$ be a bounded domain in $\mathbb{R}^n$. For functions $u$ in $W^{n,q}(D)$ such that $\|\nabla u\|_{n,q} \leq 1$, $1 < q \leq n$, there is a sharp constant

$$\alpha_{n,q} = q^{1/(q-1)}n^{1-p/n}(\sigma_{n-1})^{p/n}$$

such that for $p = q/(q-1)$,

$$\int_D \exp\{\alpha_{n,q}|u(x)|^p\} \, dx \leq A(q)m(D).$$

This theorem generalizes the above result of [7] and fits nicely with the results of [3] and [4]. Consider a result of Fusco, Lions, and Sbordone [4]. They have shown that the Zygmund-Sobolev space, defined as the set of functions $u$ for which $\nabla u$ belongs to the Zygmund space $L^n(\log^{-r}(L))(D)$, $r > 0$, can be continuously imbedded into the Orlicz space $L_{\exp n/(n-1+r)}(D)$, that is, the linear hull of the set of all functions $f$ such that

$$\int_D \exp\{|f(x)|^{n/(n-1+r)}\} \, dx < \infty.$$ 

Since the exponent $p$ for $1 < q \leq n$ satisfies $p \geq n/(n-1) > n/(n-1+r)$, Theorem 1 makes a stronger conclusion with a stronger hypothesis. However, simple examples show the theorem cannot be extended to the range $q > n$.

It is not known if Theorem 1 has extremals. The authors show in [5] that the closely related Theorem A below has extremals for all $q > 1$.

**Proof of Theorem 1**

Letting $t = |x|^n\sigma_{n-1}/n$, we may rewrite the $L(n,q)$ norm of $f$ in terms of $f^\#$ as

$$\|f\|_{n,q} = \left((n/\sigma_{n-1})^{1-q/n}(q/n) \int (f^\#(x))^{q/n}|x|^{q-n} \, dx \right)^{1/q}.\quad (1)$$

Our proof is based on the following one dimensional inequality.

**Theorem A** (Jodeit [6], Moser [7]). Let $1 < q < \infty$, $1/p + 1/q = 1$. Let $\omega$ be a function in $C^1[0,\infty)$ such that $\omega(0) = 0$ and $\int_0^\infty |\omega'(t)|^q \, dt \leq 1$. Then

$$\sup_{\omega} \int_0^\infty \exp\{\omega^p(t) - t\} \, dt = A(q) < \infty.$$

We translate the statement of Theorem A to Theorem 1 by identifying $|x|/R = e^{-t/n}$ and $\omega(t) = \alpha^{1/p}u^\#(x)$, where $u^\#$ is the symmetrical rearrangement of a function $u$ in $C^1(D)$ vanishing at the boundary of $D$ and $R$ is defined by $m(D) = m(B_R(0))$. We may assume $R = 1$. The constant $\alpha$ is determined after we carry out the change of variables. Observe $d|x|/dt = -|x|/n$ and

$$\omega'(t) = (\alpha^{1/p}|x|/n)|\nabla u|^\#(x), \quad x \in B_1(0).\quad (2)$$

So the conclusion of Theorem A is that the (Moser) functional

$$F(u^\#) = F(u) = (n/\sigma_{n-1}) \int \exp\{\alpha u^p(x)\} \, dx$$

is bounded given that

$$G^q(u^\#) = \alpha q^{-1}(n^{q-1}/\sigma_{n-1}) \int (|\nabla u|^\#(x))^q |x|^{q-n} \, dx \leq 1.\quad (4)$$
We define
\[(5) \quad \alpha = \alpha_{n,q} = q^{1/(q-1)}(\sigma_{n-1})^{p/n} n^{1-p/n}.
\]
This ensures that (4) resembles (1).

**Lemma 1.** Let \( u \geq 0 \) be continuously differentiable on \( D \), compactly supported in \( D \), and all of the nonzero level sets have \( n \)-dim measure zero. For \( n \geq 2 \), \( 1 < q \leq n \), we have
\[ G(u^\#) \leq \|\nabla u\|_{n,q}. \]

**Proof.** Let \( c_{n,q} \) be the constant of (1).
\[
G^q(u^\#) = c_{n,q} \int |\nabla u^\#(x)|^{q-1} \cdot |x|^{q-n} |\nabla u^\#(x)| \, dx
= c_{n,q} \int_0^\infty \int_{(u^\#)^{-1}(t)} f(x) dH^{n-1} x \, dt,
\]
where \( f(x) = |\nabla u^\#(x)|^{q-1} \cdot |x|^{q-n} \) is a radial function. Let \( \Phi_1 \) and \( \Phi_2 \) be defined by the equations
\[
\Phi_1(u^\#(x)) = f(x),
\Phi_2(u^\#(x)) = |x|^{(n-q)/q}.
\]
Then, using the isoperimetric inequality,
\[
G^q(u^\#)c_{n,q} = \int_0^\infty \int_{(u^\#)^{-1}(t)} \Phi_1(u^\#(x)) dH^{n-1} x \, dt,
\leq \int_0^\infty \int_{(u^\#)^{-1}(t)} \Phi_1(u(x)) dH^{n-1} x \, dt,
\]
by Holder’s inequality
\[
\leq \left[ \int (\Phi_1(u(x))\Phi_2(u(x)))^p \, dx \right]^{1/p} \left[ \int |\nabla u(x)/\Phi_2(u(x))|^q \, dx \right]^{1/q}.
\]
Using the equimeasurability of \( u \) and \( u^\# \), the first integral on the right is simply \( [G^q(u^\#)/c_{n,q}]^{1/p} \). The second integral is bounded by
\[
(7) \quad \left[ \int_0^\infty (|\nabla u^*(t)|)^q [(1/\Phi_2(u))^*(t)]^q \, dt \right]^{1/q}.
\]
We compute using \( (1/\Phi_2(u))^*(t) = \inf \{ s : m \{ x : 1/\Phi_2(u(x)) > s \} \leq t \} \), the equimeasurability of \( u \) and \( u^\# \), and the definition of \( \Phi_2 \) that
\[
(1/\Phi_2(u))^*(t) = [\sigma_{n-1}/(tn)]^{(n-q)/(qn)}.
\]
So (7) is equal to
\[
(8) \quad [(\sigma_{n-1}/n)^{(n-q)/n} \int (|\nabla u^*(t)|^{1/n})^q \, dt/t]^{1/q} = c_{n,q}^{-1/q} \|\nabla u\|_{n,q}.
\]
So, by (6) and (8),
\[ G^q(u^\#) \leq G^{q/p}(u^\#)\|\nabla u\|_{n,q}. \]
This proves Lemma 1 which implies Theorem 1 for a dense class of functions. An
application of Fatou’s Lemma completes the proof of Theorem 1. \(\square\)

**The constant \(\alpha^{n,q}\) of Theorem 1 is sharp**

In [7], Moser shows with a simple example that Theorem A is sharp. Unfortunately,
the sharpness of Theorem A implies the sharpness of Theorem 1 only for
\(q = n\). However the computations below allow us to modify Moser’s example and
establish the sharpness of \(\alpha_{n,q}\).

Let \(D\) be \(B_1(0)\). Let \(a > 1\) and \(0 < \delta < 1\). We will choose \(a\) and \(\delta\) later. Define
\(\omega(0) = 0, \omega(t) = \delta a^{-1/q}\) for \(0 \leq t \leq a\), and 0 otherwise. Then we use equation
(2), with \(\alpha\) replaced by \(\alpha_{n,q}\) to define a radial Lorentz-Sobolev function \(u\) whose
gradient is supported in the annulus of radii \(e^{-a/n}\) and 1 centered at \(x_0\). We claim that
\[ \|\nabla u\|_{n,q}^q \leq \delta^q[1 + n/(qa)]. \]
Assuming the claim, we now construct an example to show that \(\alpha_{n,q}\) is maximal.
Let \(\alpha_2 > \alpha_{n,q}\). Then for \(\beta = \alpha_2/\alpha_{n,q} > 1, c = \sigma_{n-1}/n\)
\[ \int_B \exp\{\alpha_2u^p\} \ dx = c \int_0^\infty \exp\{\beta \omega^p(t) - t\} \ dt,
\geq c \int_a^\infty \exp\{\beta \delta a - t\} \ dt,
= c \exp\{a(\beta \delta - 1)\}. \]
Now choose \(\delta < 1\) so that \(\delta^\beta > 1/\beta\). Then for all large enough \(a\), \(\|\nabla u\|_{n,q} \leq 1\), yet
\(\exp\{a(\beta \delta - 1)\}\) is unbounded.

To establish our claim, be begin by computing \((\|\nabla u\|)^*\). Let \(C = a\delta/\alpha_{n,q}\).
Observe for \(e^{-a/n} \leq |x| \leq 1\), \(\max\|\nabla u\| = Ca^{-1/q}e^{a/n}\) and \(\min\|\nabla u\| = Ca^{-1/q}\). So, for \(Ca^{-1/q} \leq s \leq Ca^{-1/q}e^{a/n}\), we have
\(t = m\{\|\nabla u\| > s\} = (\sigma_{n-1}/n)(Ca^{-1/q}/s) - e^{-a}\). Therefore, solving for \(s\) gives us
\[ (\|\nabla u\|)^*(t) = \begin{cases} Ca^{-1/q} & 0 \leq t \leq \sigma_{n-1}[1 - e^{-a}]/n, \\ 0, & \text{otherwise.} \end{cases} \]
We compute
\[ \|\nabla u\|_{n,q}^q = C^q q/(an) \int_0^{\sigma_{n-1}[1 - e^{-a}]/n} t^{q/n} \ dt
= \delta^q/a \int_0^{(1 - e^{-a})} s^{q/n} \ ds
= \delta^q/a \int_0^{e^{-a} - 1} \left[ w/(w + 1) \right]^{q/n} \ dw. \]
By considering the integration over \([0, 1]\) and \([1, e^a - 1]\) separately, the above is less
than \((\delta^q/a)[n/q + a]\).
REFERENCES


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