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A SHARP EXPONENTIAL INEQUALITY FOR LORENTZ-SOBOLEV SPACES ON BOUNDED DOMAINS

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ABSTRACT. This paper generalizes an inequality of Moser from the case that ∇u is in the Lebesgue space L^n to certain subspaces, namely the Lorentz spaces $L^{n,q}$, where $1 < q \leq n$. The conclusion is that $\exp(\alpha u^p)$ is integrable, where $1/p + 1/q = 1$. This is a higher degree of integrability than in the Moser inequality when $q < n$. A formula for α is given and it is also shown that no larger value of α works.

For $n \geq 2$, let D be a bounded domain in R^n and let $W^n(D)$ be the Sobolev space of functions defined as the completion of the space of C^∞ functions compactly supported in D whose gradient is in $L^n(D)$. A well known result of J. Moser [7] is that, for functions u in the unit ball of $W^n(D)$, there is a sharp constant $\alpha = \alpha_n = n(\sigma_{n-1})^{1/(n-1)}$, where σ_{n-1} is the $n-1$ dimensional surface area of the unit sphere, such that

$$\int_D \exp\{\alpha u^{n/(n-1)}\} \leq A(n)m(D), \qquad A(n) \text{ is independent of } u.$$

We define a Lorentz-Sobolev space $W^{n,q}(D)$ using Lorentz norms and prove a similar sharp exponential inequality. When $q = n$, $W^n(D) = W^{n,n}(D)$.

For real valued functions f on R^n, let f^* be the nonincreasing rearrangement of f defined as $f^*(t) = \inf\{s\colon\ m\{|f| > s\} \leq t\}$. We define $f^\#(x)$ to be the spherically symmetric nondecreasing rearrangement of f defined as $f^\#(x) = f^*(\sigma_{n-1}|x|^n/n)$.

The Lorentz $L(n,q)$ norm is defined as

$$\|f\|_{n,q} = [(q/n)\int_0^\infty [f^*(t)t^{1/n}]^q dt/t]^{1/q}.$$

The constant (q/n) ensures that $\|\chi_E\|_{n,q} = (m(E))^{1/n}$. For $1 < q \leq n$, it is shown in [1] that $L(n,q)$ is an actual norm.

Definition. Define $W^{n,q}(D)$, $1 < q \leq n$, as the completion of the space of functions u in C^∞ compactly supported in D satisfying

$$\|\nabla u\|_{n,q} < \infty.$$

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Theorem 1. *Let D be a bounded domain in R^n. For functions u in $W^{n,q}(D)$ such that $\|\nabla u\|_{n,q} \leq 1$, $1 < q \leq n$, there is a sharp constant*

$$\alpha_{n,q} = q^{1/(q-1)} n^{1-p/n} (\sigma_{n-1})^{p/n}$$

such that for $p = q/(q-1)$,

$$\int_D \exp\{\alpha_{n,q}|u(x)|^p\}\, dx \leq A(q) m(D).$$

This theorem generalizes the above result of [7] and fits nicely with the results of [3] and [4]. Consider a result of Fusco, Lions, and Sbordone [4]. They have shown that the Zygmund-Sobolev space, defined as the set of functions u for which ∇u belongs to the Zygmund space $L^n(\log^{-r}(L))(D)$, $r > 0$, can be continuously imbedded into the Orlicz space $L_{\exp,n/(n-1+r)}(D)$, that is, the linear hull of the set of all functions f such that

$$\int_D \exp\{|f(x)|^{n/(n-1+r)}\}\, dx < \infty.$$

Since the exponent p for $1 < q \leq n$ satisfies $p \geq n/(n-1) > n/(n-1+r)$, Theorem 1 makes a stronger conclusion with a stronger hypothesis. However, simple examples show the theorem cannot be extended to the range $q > n$.

It is not known if Theorem 1 has extremals. The authors show in [5] that the closely related Theorem A below has extremals for all $q > 1$.

Proof of Theorem 1

Letting $t = |x|^n \sigma_{n-1}/n$, we may rewrite the $L(n,q)$ norm of f in terms of $f^{\#}$ as

$$\|f\|_{n,q} = \left[(n/\sigma_{n-1})^{1-q/n}(q/n)\int (f^{\#}(x))^q |x|^{q-n}\, dx\right]^{1/q}. \tag{1}$$

Our proof is based on the following one dimensional inequality.

Theorem A (Jodeit [6], Moser [7])**.** *Let $1 < q < \infty$, $1/p + 1/q = 1$. Let ω be a function in $C^1[0,\infty)$ such that $\omega(0) = 0$ and $\int_0^\infty |\omega'(t)|^q\, dt \leq 1$. Then $\sup_\omega \int_0^\infty \exp\{\omega^p(t) - t\}\, dt = A(q) < \infty$.*

We translate the statement of Theorem A to Theorem 1 by identifying $|x|/R = e^{-t/n}$ and $\omega(t) = \alpha^{1/p} u^{\#}(x)$, where $u^{\#}$ is the symmetrical rearrangement of a function u in $C^1(D)$ vanishing at the boundary of D and R is defined by $m(D) = m(B_R(0))$. We may assume $R = 1$. The constant α is determined after we carry out the change of variables. Observe $d|x|/dt = -|x|/n$ and

$$\omega'(t) = (\alpha^{1/p}|x|/n)|\nabla u|^{\#}(x), \qquad x \text{ in } B_1(0). \tag{2}$$

So the conclusion of Theorem A is that the (Moser) functional

$$F(u^{\#}) = F(u) = (n/\sigma^{n-1}) \int \exp\{\alpha u^p(x)\}\, dx \tag{3}$$

is bounded given that

$$G^q(u^{\#}) = \alpha^{q-1}(n^{1-q}/\sigma_{n-1}) \int (|\nabla u|^{\#}(x))^q |x|^{q-n}\, dx \leq 1. \tag{4}$$

We define

$$\alpha = \alpha_{n,q} = q^{1/(q-1)}(\sigma_{n-1})^{p/n} n^{1-p/n}. \tag{5}$$

This ensures that (4) resembles (1).

Lemma 1. *Let $u \geq 0$ be continuously differentiable on D, compactly supported in D, and all of the nonzero level sets have n-dim measure zero. For $n \geq 2$, $1 < q \leq n$, we have*

$$G(u^{\#}) \leq [\|\nabla u\|_{n,q}].$$

Proof. Let $c_{n,q}$ be the constant of (1).

$$\begin{aligned} G^q(u^{\#}) &= c_{n,q} \int |\nabla u^{\#}(x)|^{q-1} \cdot |x|^{q-n} |\nabla u^{\ \#}(x)|\, dx \\ &= c_{n,q} \int_0^\infty \int_{(u^{\#})^{-1}(t)} f(x) dH^{n-1} x\, dt, \end{aligned}$$

where $f(x) = |\nabla u^{\#}(x)|^{q-1} \cdot |x|^{q-n}$ is a radial function. Let Φ_1 and Φ_2 be defined by the equations

$$\begin{aligned} \Phi_1(u^{\#}(x)) &= f(x), \\ \Phi_2(u^{\#}(x)) &= |x|^{(n-q)/q}. \end{aligned}$$

Then, using the isoperimetric inequality,

$$\begin{aligned} G^q(u^{\#}) c_{n,q} &= \int_0^\infty \int_{(u^{\#})^{-1}(t)} \Phi_1(u^{\#}(x))\, dH^{n-1} x\, dt, \\ &\leq \int_0^\infty \int_{(u)^{-1}(t)} \Phi_1(u(x))\, dH^{n-1} x\, dt, \\ &= \int \Phi_1(u(x)) \Phi_2(u(x)) / \Phi_2(u(x)) |\nabla u(x)|\, dx, \\ &\qquad\qquad\qquad\qquad \text{by Holder's inequality} \\ &\leq \left[\int (\Phi_1(u(x)) \Phi_2(u(x)))^p\, dx \right]^{1/p} \left[\int |\nabla u(x) / \Phi_2(u(x))|^q\, dx \right]^{1/q}. \end{aligned} \tag{6}$$

Using the equimeasurability of u and $u^{\#}$, the first integral on the right is simply $[G^q(u^{\#})/c_{n,q}]^{1/p}$. The second integral is bounded by

$$\left[\int_0^\infty (|\nabla u|^*(t))^q [(1/\Phi_2(u))^*(t)]^q\, dt \right]^{1/q}. \tag{7}$$

We compute using $(1/\Phi_2(u))^*(t) = \inf\{s : m\{x : 1/\Phi_2(u(x)) > s\} \leq t\}$, the equimeasurability of u and $u^{\#}$, and the definition of Φ_2 that

$$(1/\Phi_2(u))^*(t) = [\sigma_{n-1}/(tn)]^{(n-q)/(qn)}.$$

So (7) is equal to

$$[(\sigma_{n-1}/n)^{(n-q)/n} \int (|\nabla u|^*(t) t^{1/n})^q\, dt/t]^{1/q} = c_{n,q}^{-1/q} \|\nabla u\|_{n,q}. \tag{8}$$

So, by (6) and (8),

$$G^q(u^\#) \le G^{q/p}(u^\#)\|\nabla u\|_{n,q}. \tag{9}$$

This proves Lemma 1 which implies Theorem 1 for a dense class of functions. An application of Fatou's Lemma completes the proof of Theorem 1. □

The constant $\alpha^{n,q}$ of Theorem 1 is sharp

In [7], Moser shows with a simple example that Theorem A is sharp. Unfortunately, the sharpness of Theorem A implies the sharpness of Theorem 1 only for $q = n$. However the computations below allow us to modify Moser's example and establish the sharpness of $\alpha_{n,q}$.

Let D be $B_1(0)$. Let $a > 1$ and $0 < \delta < 1$. We will choose a and δ later. Define $\omega(0) = 0$, $\omega'(t) = \delta a^{-1/q}$ for $0 \le t \le a$, and 0 otherwise. Then we use equation (2), with α replaced by $\alpha_{n,q}$, to define a radial Lorentz-Sobolev function u whose gradient is supported in the annulus of radii $e^{-a/n}$ and 1 centered at x_0. We claim that

$$[\|\nabla u\|_{n,q}]^q \le \delta^q[1 + n/(qa)].$$

Assuming the claim, we now construct an example to show that $\alpha_{n,q}$ is maximal. Let $\alpha_2 > \alpha_{n,q}$. Then for $\beta = \alpha_2/\alpha_{n,q} > 1$, $c = \sigma_{n-1}/n$

$$\begin{aligned}\int_B \exp\{\alpha_2 u^p\}\,dx &= c\int_0^\infty \exp\{\beta\omega^p(t) - t\}\,dt,\\ &\ge c\int_a^\infty \exp\{\beta\delta^p a - t\}\,dt,\\ &= c\,\exp\{a(\beta\delta^p - 1)\}.\end{aligned}$$

Now choose $\delta < 1$ so that $\delta^p > 1/\beta$. Then for all large enough a, $\|\nabla u\|_{n,q} \le 1$, yet $\exp\{a(\beta\delta^p - 1)\}$ is unbounded.

To establish our claim, be begin by computing $(|\nabla u|)^*$. Let $C = n\delta/\alpha_{n,q}^{1/p}$. Observe for $e^{-a/n} \le |x| \le 1$, that $\max|\nabla u| = Ca^{-1/q}e^{a/n}$ and $\min|\nabla u| = Ca^{-1/q}$. So, for $Ca^{-1/q} \le s \le Ca^{-1/q}e^{a/n}$, we have $t = m\{|\nabla u| > s\} = (\sigma_{n-1}/n)[(Ca^{-1/q}/s)^n - e^{-a}]$. Therefore, solving for s gives us

$$(|\nabla u|)^*(t) = \begin{cases}\dfrac{Ca^{-1/q}}{[(n/\sigma_{n-1})t + e^{-a}]^{1/n}}, & 0 \le t \le \sigma_{n-1}[1 - e^{-a}]/n,\\ 0, & \text{otherwise.}\end{cases}$$

We compute

$$\begin{aligned}[\|\nabla u\|_{n,q}]^q &= C^q q/(an)\int_0^{\sigma_{n-1}[1-e^{-a}]/n} \frac{t^{q/n}}{[(n/\sigma_{n-1})t + e^{-a}]^{q/n}}\,\frac{dt}{t}\\ &= \delta^q/a\int_0^{(1-e^{-a})} \frac{s^{q/n}}{[s + e^{-a}]^{q/n}}\,\frac{ds}{s}\\ &= \delta^q/a\int_0^{e^a-1} [w/(w+1)]^{q/n}\,\frac{dw}{w}.\end{aligned}$$

By considering the integration over $[0,1]$ and $[1, e^a - 1]$ separately, the above is less than $(\delta^q/a)[n/q + a]$.

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