FINITE-DIMENSIONAL LEFT IDEALS IN SOME ALGEBRAS ASSOCIATED WITH A LOCALLY COMPACT GROUP

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Abstract. Let $G$ be a locally compact group, let $L^1(G)$ be its group algebra, let $M(G)$ be its usual measure algebra, let $L^1(G)$ be the second dual of $L^1(G)$ with an Arens product, and let $LUC(G)$ be the conjugate of the space $LUC(G)$ of bounded, left uniformly continuous, complex-valued functions on $G$ with an Arens-type product. We find all the finite-dimensional left ideals of these algebras. We deduce that such ideals exist in $L^1(G)$ and $M(G)$ if and only if $G$ is compact, and in $L^1(G)^{**}$ (except those generated by right annihilators of $L^1(G)^{**}$) and $LUC(G)^*$ if and only if $G$ is amenable.

1. Introduction

Let $G$ be a locally compact group, $L^1(G)$ be its group algebra, and $M(G)$ be its usual measure algebra. Other Banach algebras (usually larger than $L^1(G)$ and $M(G)$) can also be associated with $G$. For instance, the second dual $L^1(G)^{**}$ of $L^1(G)$ is a Banach algebra with an Arens product. One can also consider the space $LUC(G)$ of the bounded left uniformly continuous functions on $G$, or the space $WAP(G)$ of the weakly almost periodic functions on $G$. An Arens-type product can be introduced into the conjugate of each of these spaces of functions and make them into Banach algebras. Let $\mathcal{A}$ denote any of these algebras. Our main concern in this paper is with the finite-dimensional left ideals in $\mathcal{A}$. We start in Theorem 1 by giving examples of such ideals. These examples are obtained with the help of the $U$-invariant elements of $\mathcal{A}$ where $U$ is a continuous and bounded representation of $G$ on $\mathbb{C}^n$. This notion has already been introduced in earlier papers with $n = 1$. In [3], the so-called $\chi$-invariant elements (where $\chi$ is a character of $G$) were introduced in order to determine the minimal left ideals of these algebras when $G$ is abelian. In [4], these type of elements were referred to as $\lambda$-invariant elements with $\lambda \in \mathbb{T}$, and were used to solve some linear equations in $\ell^\infty(\mathbb{Z})^*$, then to determine the finite-dimensional left ideals in this case. In [5], the $U$-invariance was used to study the minimal ideals in these algebras. In Theorems 2 and 3, we determine all the finite-dimensional left ideals in $\mathcal{A}$ and show that they are in general of the form given in Theorem 1. This is a generalization of the result obtained in [4, 2.7(a)]. We deduce in the corollary which follows that such ideals exist in $LUC(G)^*$ and in $L^1(G)^{**}$ (apart from those generated by right annihilators of $L^1(G)^{**}$) if and only
if $G$ is amenable. In $L^1(G)$ and $M(G)$, the ideals of finite-dimension exist if and only if $G$ is compact.

2. Preliminaries

Let $G$ be a locally compact group with a left Haar measure $\lambda$, and a Haar modular function $\Delta$. For measurable functions $f$ and $\phi$ on $G$ and for $s \in G$, we write

$$f \ast \phi(s) = \int_G f(t)\phi(t^{-1}s) \, d\lambda(t)$$

whenever the integral exists. We shall be concerned with the following Banach algebras which are related to $G$. We begin by recalling, of course, the most intimate ones: the group algebra $L^1(G)$ and the measure algebra $M(G)$. The first one is the Banach algebra of all measurable complex-valued functions $\phi$ on $G$ satisfying

$$\int_G |\phi(s)| \, d\lambda(s) < \infty.$$

The product of two elements $\phi$ and $\psi$ in $L^1(G)$ is $\phi \ast \psi$. The second one is the Banach algebra of all bounded, regular, Borel measures on $G$. By the Riesz representation theorem (see [8, Chapter 3]), we shall regard a measure in $M(G)$ as an element of $C_0(G)^*$, where $C_0(G)$ is the space of continuous complex-valued functions on $G$ vanishing at infinity. The product of two elements $\mu$ and $\nu$ of $M(G)$ is given then by

$$(\mu \nu)(f) = \int_G \int_G f(st) \, d\mu(s) \, d\nu(t) = \int_G \int_G f(st) \, d\nu(t) \, d\mu(s) \quad \text{for} \quad f \in C_0(G).$$

Furthermore, for each $\phi \in L^1(G)$, we define $\lambda_{\phi} \in M(G)$ by

$$\lambda_{\phi}(f) = \int_G f(s)\phi(s) \, d\lambda(s) \quad \text{for} \quad f \in C_0(G).$$

Then under the map $\phi \mapsto \lambda_{\phi}$, we will regard $L^1(G)$ as a subalgebra of $M(G)$. In fact, $L^1(G)$ is a closed two-sided ideal of $M(G)$ (see for example [8, Theorem 19.18]). The other algebras which we shall consider are defined in the following way. Let $L^\infty(G)$ be the Banach space of all measurable complex-valued functions that are bounded almost everywhere with respect to $\lambda$, and for each $\phi \in L^\infty(G)$, let $\hat{\phi}$ be the function defined on $G$ by $\hat{\phi}(s) = \Delta(s^{-1})\phi(s^{-1})$. The Banach space $L^1(G)^{**}$ becomes a Banach algebra with the first Arens product. This product is obtained by first letting

$$f_\nu(\phi) = \nu(\hat{\phi} \ast f) \quad \text{for all} \quad \nu \in L^1(G)^{**}, \ f \in L^\infty(G) \text{ and } \phi \in L^1(G).$$

Then, for $\mu$ and $\nu$ in $L^1(G)^{**}$,

$$(\mu \nu)(f) = \mu(f_\nu) \quad \text{for all} \quad f \in L^\infty(G).$$

Let $C(G)$ denote the space of all bounded, complex-valued, continuous functions on $G$. The left translate of a function $f$ on $G$ by $s \in G$ is defined by $f_s(t) = f(st)$ for all $t \in G$. Let $LUC(G)$ be the space of left uniformly continuous functions in $C(G)$, i.e.,

$$LUC(G) = \{ f \in C(G) : s \mapsto f_s : G \to C(G) \text{ is norm continuous} \}.$$
Then $LUC(G)^*$ is also a Banach algebra under the product
\[(\mu \nu)(f) = \mu(f \nu) \quad \text{for all } f \in LUC(G), \quad \text{where} \]
\[f_\nu(s) = \nu(f_s) \quad \text{for all } s \in G \]
(the function $f_\nu$ is easily seen to be in $LUC(G)$). Note that the product in $M(G)$ (and so in $L^1(G)$) is defined in the same way with $C_0(G)$ instead of $LUC(G)$.

More generally, one can start with a norm closed, conjugate closed subspace $F$ of $C(G)$ containing the constant functions and having the property that the functions $f_s$ and $f_\mu$ are in $F$ for all $f \in F$, $s \in G$ and $\mu \in F^*$ (the functions $f_s$ and $f_\mu$ are defined as earlier in $LUC(G)$). Following [2, Definition 2.2.10], such an $F$ is said to be admissible; the space $F^*$ also becomes a Banach algebra under the product
\[(\mu \nu)(f) = \mu(f_\nu) \quad \text{for all } f \in F.\]

For more details, the reader is directed to [2, pages 72-78]. As we have already seen, $LUC(G)$ is admissible. Other examples are the space $WAP(G)$ of weakly almost periodic functions on $G$, and the space $AP(G)$ of almost periodic functions on $G$. These spaces are
\[WAP(G) = \{ f \in C(G) : f_G \text{ is weakly relatively compact} \}, \]
\[AP(G) = \{ f \in C(G) : f_G \text{ is norm relatively compact} \}, \]
where
\[f_G = \{ f_s : s \in G \}. \]

Let $\mathcal{A}$ denote each of the Banach algebras $L^1(G)$, $M(G)$, $L^1(G)^{**}$, and $F^*$, where $F$ is an admissible subspace of $C(G)$ with $AP(G)F \subseteq F$. Apart from $L^1(G)$, the algebra $\mathcal{A}$ is the dual of some Banach space of functions of $G$, which we shall denote by $\mathcal{F}$. In the case of $L^1(G)$, we let $\mathcal{F} = C_0(G)$. When $\mathcal{A} = F^*$ or $M(G)$, the group $G$ may be embedded continuously into $\mathcal{A}$ by the mapping $e : G \rightarrow \mathcal{A}$ which is defined by
\[e(s)(f) = f(s) \quad \text{for all } s \in G \text{ and } f \in \mathcal{F}. \]

We recall that an element $\mu$ of $\mathcal{A}$ is left invariant if
\[\mu(f_s) = \mu(f) \quad \text{for all } f \in \mathcal{F} \text{ and } s \in G. \]

When $L^1(G) \ast \mathcal{F} \subseteq \mathcal{F}$, we say that $\mu \in \mathcal{A}$ is topologically left invariant if
\[\mu(\phi \ast f) = \left( \int_G \phi(s) \, d\lambda(s) \right) \mu(f) \quad \text{for all } \phi \in L^1(G) \text{ and } f \in \mathcal{F}. \]

The notions of left invariance and topological left invariance are equivalent when $\mathcal{F} \subseteq LUC(G)$. But this is not so if $\mathcal{F} = L_\infty(G)$. We say that $\mathcal{F}$ is amenable if there is a non-zero left invariant (or equivalently a topologically left invariant) element in $\mathcal{A}$. When $G$ is a compact topological group, $C_0(G) = C(G)$ is amenable since $\lambda \in C(G)^*$. The spaces $WAP(G)$ and $AP(G)$ are always amenable. But this is not so for $LUC(G)$ and $L_\infty(G)$; for example, when $G$ is the free group on two generators (see [2, Example 3.4(e)]). So we say that the group $G$ is amenable if $L_\infty(G)$, or equivalently $LUC(G)$, is amenable. See [2] or [7].

We shall also need representations of $G$ on $\mathbb{C}^n$. Recall that a representation of $G$ on a Hilbert space $H$ is a homomorphism of $G$ into the semigroup of bounded operators on $H$. We say that $U$ is continuous when the function $s \mapsto U(s)\bar{x}$ is continuous on $G$ for each $\bar{x} \in H$. We say that $U$ is irreducible when $\{0\}$ and $H$ are the only invariant (closed) subspaces under all $U(s)$, i.e., there is no (closed)
subspace $E$ other than $\{0\}$ and $H$ satisfying $U(s)E \subseteq E$ for all $s \in G$. We also recall that there is a one-to-one correspondence between the representations of $L^1(G)$ on $H$ and those of $G$ on $H$. This is given by the formula

$$
\langle U(\phi)\bar{x}, \bar{y} \rangle = \int_G \langle U(s)\bar{x}, \bar{y} \rangle \phi(s) \, d\lambda(s) \quad \phi \in L^1(G), \; \bar{x}, \bar{y} \in H;
$$

see [8, Section 22]. Since we shall be concerned solely with representations $U$ on $H = \mathbb{C}^n$, we fix a basis $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ for $\mathbb{C}^n$, and correspond to $U$ the matrix representation $U = (u_{ij})_{i,j=1}^n$, where $u_{ij}$ are the coordinate functions defined on $G$ by

$$
u_{ij}(s) = \langle U(s)\bar{x}_i, \bar{y}_j \rangle.
$$

Note that the letter $U$ is used to denote the representations of $G$, $L^1(G)$, and their corresponding matrices, but we promise the reader that this will cause no confusion.

### 3. Left ideals of finite-dimension

The notion of $\chi$-invariance, where $\chi$ is a continuous character of $G$ was introduced in [3] to determine the minimal left ideals of $\mathcal{A}$ when $G$ is abelian. In this section, we generalize this notion by defining the $U$-invariant vectors of $\mathcal{A}^n$, where $U$ is a representation of $G$ on $\mathbb{C}^n$. The $U$-invariance is essential to determine the finite-dimensional left ideals of these algebras. Some of the arguments in Theorems 1 and 2 have already been in [1]; they are, however, simplified here with the help of the $U$-invariance. Before we state our main definition, we introduce the following notations.

**Notations.** Let $\mu \in \mathcal{A}$, $f \in \mathcal{F}$, $\bar{\mu} = (\mu_i)_{i=1}^n$ be a column vector in $\mathcal{A}^n$, and $A$ be an $n \times n$ matrix with entries $a_{ij}$ ($i, j = 1, \ldots, n$) in $\mathcal{F}$. Then we write $\mu(A) = (\mu(a_{ij}))_{i,j=1}^n$, $\bar{\mu}(f)$ is the column vector $((\mu_i(f)))_{i=1}^n$ of $\mathbb{C}^n$, $(A)\bar{\mu}$ is the column vector $\left(\sum_{j=1}^n \mu_j(a_{ij})\right)_{i=1}^n$ of $\mathbb{C}^n$ (note that this is obtained with the matrix multiplication relative to the product given by the duality between $\mathcal{A}$ and $\mathcal{F}$), and $\mu \bar{\mu}$ is the column vector $(\mu \mu_i)_{i=1}^n$ of $\mathcal{A}^n$. Furthermore, when $A$ and $B$ are $n \times n$ matrices whose entries are measurable functions on $G$ then, whenever the integrals exist, we write

$$
A \cdot B = \int_G A(t)B(t) \, d\lambda(t),
$$

$$
A \ast B(s) = \int_G A(t)B(t^{-1}s) \, d\lambda(t),
$$

$$
A \bar{\ast} B = \int_G B(t^{-1}s)A(t) \, d\lambda(t).
$$

**Definition.** We say that a vector $\bar{\mu}$ of $\mathcal{A}^n$ is $U$-invariant if there exists a continuous and bounded representation $U$ of $G$ on $\mathbb{C}^n$ such that

$$
\bar{\mu}(f_s) = U(s^{-1})\bar{\mu}(f) \quad \text{for each} \; f \in \mathcal{F} \text{ and } s \in G.
$$

When $\mathcal{A} = L^1(G)^{**}$, we say that $\bar{\mu}$ be topologically $U$-invariant if

$$
\bar{\mu}(f \ast f) = U(\phi)\bar{\mu}(f) \quad \text{for each} \; f \in L^\infty(G) \text{ and } \phi \in L^1(G).
$$

Let $\widetilde{U}$ be defined on $G$ by $\widetilde{U}(s) = U(s^{-1})$. A direct computation leads to the following lemma. We omit the proof.
Lemma 1. Let $A$ be an $n \times n$ matrix with entries in $F$, and for $s \in G$, let $A_s$ be the matrix whose entries are the left translates of those of $A$ by $s$. Let $\Phi$ be an $n \times n$ matrix with entries in $L^1(G)$.

1. If $\mu \in A^n$ is $U$-invariant, then $(A_s)\mu = \left(AU(s)\right)\mu$.

2. If $A = L^1(G)^{**}$ and $\mu \in A^n$ is topologically $U$-invariant, then

\[
(\Phi \ast A)\mu = (\Phi \cdot A(\cdot U))\mu \quad \text{and} \quad (\Phi \ast A)\mu = (A(\Phi \cdot U))\mu,
\]

where $\Phi \cdot A(\cdot U)$ is the matrix-valued function defined on $G$ by $\Phi \cdot A(s)U = \int_G \Phi(t)A(s)U(t)\,d\lambda(t)$.

Lemma 2. Let $U$ be a continuous and bounded representation of $G$ on $\mathbb{C}^n$, and let $I$ be the identity representation. Let $\mu$ and $\nu$ be in $A^n$ such that $\mu(f) = (fU)\nu$ for all $f \in F$; or equivalently, $\nu(f) = (fU)\mu$ for all $f \in F$. Then

1. $\nu$ is $I$-invariant if and only if $\mu$ is $U$-invariant,

2. when $A = L^1(G)^{**}$, $\nu$ is topologically $I$-invariant if and only if $\mu$ is topologically $U$-invariant.

Remark. Observe that $(fU)\nu$ and $(fU)\mu$ are well defined in the lemma. This is due to the fact that the coordinate functions of $U$ and $\tilde{U}$ are almost periodic (which is easy to verify, or see [1, Lemma 1]) and the extra assumption that $AP(G)F \subseteq F$.

Proof of Lemma 2. Let $\nu \in A^n$ be $I$-invariant, $s \in G$ and $f \in F$. Then, by Lemma 1,

\[
\tilde{\mu}(s) = (sU)\nu = (\tilde{U}(s)fs)\nu = \tilde{U}(s)fU)\nu = \tilde{U}(s)\mu(f).
\]

So $\tilde{\mu}$ is $U$-invariant. For the converse, let $\tilde{\mu}$ be $U$-invariant, $s \in G$ and $f \in F$. Then

\[
\tilde{\nu}(s) = (sUU(s))\tilde{\mu} = (sUU(s))\nu = (\tilde{U}U(s))\nu = (\tilde{U}U(s))\tilde{\mu} = \tilde{\nu}(s).
\]

So $\tilde{\nu}$ is $I$-invariant.

Statement (2) follows with the help of statement (2) of Lemma 1. Let $\tilde{\nu}$ be topologically $I$-invariant, $\phi \in L^1(G)$ and $f \in L^\infty(G)$. We remark first that

\[
(\phi \ast f)U(s) = U(s) \int_G \phi(t)f(t^{-1}s)\,d\lambda(t)
\]

\[
= \int_G U(tU(t^{-1}s))\phi(t)f(t^{-1}s)\,d\lambda(t) = (\phi U) \ast (fU)(s).
\]

Therefore,

\[
\tilde{\mu}(\phi \ast f) = ((\phi \ast f)U)\nu = ((\phi U) \ast (fU))\nu = ((\phi U) \cdot (fU)(\cdot U))\nu
\]

\[
= \left(\int_G U(s)\phi(s)d\lambda(s)\right)(fU)\nu = U(\phi)\tilde{\mu}(f),
\]

and so $\tilde{\mu}$ is topologically $U$-invariant.

For the converse, let $\tilde{\mu}$ be topologically $U$-invariant, $f \in L^\infty(G)$ and $\phi \in L^1(G)$. Then, for each $s \in G$,

\[
(\phi \ast f)\tilde{U}(s) = \tilde{U}(s) \int_G \phi(t)f(t^{-1}s)\,d\lambda(t)
\]

\[
= \int_G \tilde{U}(t^{-1}s)\tilde{U}(t)\phi(t)f(t^{-1}s)\,d\lambda(t) = (\phi \tilde{U})\tilde{\mu}(fU)(s),
\]
Thus, $(\phi \ast f)\overline{U} \mu = \left((\phi \overline{U})(f \overline{U})\right) \overline{U} = \left(\int_G \phi(s)U(s)d\lambda(s)\right) \overline{U} = \left(\int_G \phi(s)d\lambda(s)\right) (f \overline{U} \mu) = \int_G f \overline{U} d\lambda = \overline{U}(f)$.

So $\widetilde{U}$ is topologically $I$-invariant.

**Theorem 1.** Let $G$ be a locally compact group. Let $\overline{\mu} \in \mathcal{A}^n$ and $M$ be the linear span of the coordinates $\mu_1, \mu_2, ..., \mu_n$ of $\overline{\mu}$. Let $U$ be a continuous and bounded representation of $G$ on $\mathbb{C}^n$. Then $M$ is a left ideal of $\mathcal{A}$ of dimension less or equal to $n$ in each of the following cases:

1. $F$ is amenable, $\mathcal{A} = F^*$ and $\overline{\mu}$ is $U$-invariant.
2. $G$ is amenable, $\mathcal{A} = L^1(G)^{\ast\ast}$ and $\overline{\mu}$ is topologically $U$-invariant.
3. $G$ is compact, $\mathcal{A} = L^1(G)$ or $M(G)$, and $\overline{\mu}$ is $U$-invariant.

Furthermore, $M$ is minimal and of dimension $n$ when $U$ is irreducible.

**Proof.** We consider only the first two statements. The proof of statement (3) is similar. Let $\mathcal{A} = F^*$, and let $\overline{\mu}$ be $U$-invariant. That $M$ is a left ideal follows directly from the lemma above. Let $\mu$ be arbitrary in $F^*$ and $f \in F$. Then $(\mu \overline{\mu})(f) = \mu(f\overline{\mu})$, where $f\overline{\mu}(s) = \overline{\mu}(f_s) = \overline{U}(s)\overline{\mu}(f)$, and so

$$\mu \overline{\mu}(f) = \mu(\overline{U}\overline{\mu}(f)) = \mu(\overline{U})\overline{\mu}(f).$$

Thus, $\mu \overline{\mu} = \mu(\overline{U})\overline{\mu}$, which means obviously that $M$ is a left ideal of $F^*$.

For statement (2), let $\mu \in L^1(G)^{\ast\ast}$ and $f \in L^\infty(G)$. Then $(\mu \overline{\mu})(f) = \mu(f\overline{\mu})$, where

$$f\overline{\mu}(\phi) = \overline{\mu}(\phi \ast f) = U(\phi)\overline{\mu}(f),$$

and

$$U(\phi) = \int_G U(s)\phi(s)d\lambda(s) = \int_G \overline{U}(s)\phi(s)d\lambda(s) = \overline{U}(\phi) \text{ for all } \phi \in L^1(G).$$

Thus, $(\mu \overline{\mu})(f) = \mu(f\overline{\mu}) = \mu(\overline{U})\overline{\mu}$, and so $M$ is a left ideal of $L^1(G)^{\ast\ast}$. That $M$ is of dimension less or equal to $n$ is clear in each case.

Suppose now that $U$ is irreducible, and let us prove that $M$ is minimal. We start with the algebra $F^*$. Let $\mu \in M$ be arbitrary, and write $\mu = \sum_{i=1}^n x_i \mu_i = \overline{\mu} = \overline{x}_\mu$, where $\overline{x} = (x_1, x_2, ..., x_n)$ is a non-zero vector of $\mathbb{C}^n$. For each vector $\overline{y} \in \mathbb{C}^n$, we can find $s_1, s_2, ..., s_k$ in $G$ and $\alpha_1, \alpha_2, ..., \alpha_k$ in $\mathbb{C}$ such that

$$\sum_{i=1}^n \alpha_i \overline{U}(s_i)\overline{y} = \overline{y}$$

since $\overline{U}$ is also irreducible. It follows that

$$\sum_{i=1}^n \alpha_i e(s_i)\mu = \sum_{i=1}^n \alpha_i e(s_i)(\overline{\mu} \overline{x}) = \sum_{i=1}^n \alpha_i (e(s_i)\overline{\mu})\overline{x}$$

$$= \sum_{i=1}^n \alpha_i (\overline{U}(s_i)\overline{\mu})\overline{x} = \sum_{i=1}^n \alpha_i (\overline{U}(s_i)\overline{x})\overline{\mu} = \overline{y} \overline{\mu}$$
(remember that \( e(s) \in F^* \) and \( e(s)(f) = f(s) \) for \( s \in G \) and \( f \in F \)). This means first that \( F^* \mu = M \), and so \( M \) is minimal. Secondly, if \( \overline{g} \in \mathbb{C}^n \) is such that \( \overline{g}_\mu \neq 0 \), then this argument shows also that

\[
\sum_{i=1}^n \alpha_i e(s_i)\overline{\mu} = \overline{g}_\mu \neq 0,
\]

and implies that \( \overline{\mu} \neq 0 \). So the elements \( \mu_1, \mu_2, ..., \mu_n \) are linearly independent and \( M \) is of dimension \( n \).

In \( L^1(G)^{**} \), the corresponding representation \( U : L^1(G) \rightarrow \mathbb{C}^n, f \rightarrow \overline{f} \) is also irreducible, and so we can find, for each \( \overline{g} \in \mathbb{C}^n \), \( \phi \in L^1(G) \) such that \( U(\phi)\overline{f} = \overline{g} \). As above, this shows that \( M \) is minimal and is of dimension \( n \).

**Remark.** Statement (1) means that \( G \) needs to be amenable if \( F = LUC(G) \).

**Theorem 2.** Let \( A \) be \( L^1(G), M(G) \) or \( F^* \), and let \( M \) be a left ideal of \( A \) of dimension \( n \). Then there exist \( m \) vectors \( \mu^i \in A^n \) and \( m \) irreducible, unitary, bounded and continuous representations \( U^i \) \((i = 1,2,...,m)\) of \( G \) such that

1. each \( \mu^i \) is \( U^i \)-invariant,
2. for each \( i = 1,2,...,m \), the coordinates of \( \mu^i \) span a minimal left ideal \( M_i \) of \( A \) of dimension \( n_i \), and
3. \( M = M_1 \oplus M_2 \oplus ... \oplus M_m \).

**Proof.** In the case of \( A = L^1(G) \), we regard \( M \) as a left ideal of \( M(G) \). This is possible because \( M \) is closed and \( L^1(G) \) is a closed ideal of \( M(G) \). We start with a set of elements \( \mu_1, \mu_2, ..., \mu_n \) which generate \( M \), and let \( \bar{\mu} = (\mu_i)_{i=1}^n \). Then, for each \( \mu \in A \) and for each \( i = 1, 2, ..., n \), there exist \( \alpha_{i1}(\mu), \alpha_{i2}(\mu), ..., \alpha_{in}(\mu) \in \mathbb{C} \) such that

\[
\mu \mu_i = \alpha_{i1}(\mu)\mu_1 + \alpha_{i2}(\mu)\mu_2 + ... + \alpha_{in}(\mu)\mu_n.
\]

Put \( A(\mu) = (\alpha_{ij}(\mu))_{i,j=1}^n \). Then, for \( \mu \) and \( \nu \) in \( A \),

\[
A(\mu \nu)\bar{\mu} = (\mu \nu) A(\mu) = A(\nu)(\mu \bar{\mu}) = A(\nu)A(\mu)\bar{\mu}.
\]

Hence, \( A(\mu \nu) = A(\nu)A(\mu) \), i.e., \( A \) is an antirepresentation of \( A \). Moreover, for each \( 1 \leq j \leq n \), let \( f \in \mathcal{F} \) be such that \( \mu_j(f) = 1 \) and \( \mu_k(f) = 0 \) for \( k \neq j \). Then, for each \( 1 \leq i \leq n \), \( \mu_\mu_i(f) = a_{ij}(\mu) \), so

\[
|a_{ij}(\mu)| \leq \|\mu_\mu_i\||f|| \leq \|\mu\| \|\mu_i\||f||,
\]

which implies that \( A \) is bounded. Since the product in \( A \) is \( \sigma(A, \mathcal{F}) \)-continuous on the left side, one can also see that the functions \( \mu \mapsto a_{ij}(\mu) \) are \( \sigma(A, \mathcal{F}) \)-continuous. When \( A = L^1(G) \), we let \( B \) be the antirepresentation of \( G \) associated to \( A \); and when \( \mathcal{F} = F \) is an amenable subspace of \( C(G) \), we let

\[
V(s) = (a_{ij}(e(s)))_{i,j=1}^n.
\]

Then \( \overline{V} \) is a bounded and continuous representation of \( G \) in each case. Furthermore, for \( f \in \mathcal{F} \), we have

\[
\bar{\mu}(f) = e(s)\bar{\mu}(f) = A(e(s))\bar{\mu}(f) = V(s)\bar{\mu}(f),
\]

i.e., \( \bar{\mu} \) is \( \bar{V} \)-invariant. Now it is easy to verify (or see [1, Lemma 3]) that \( \bar{V} \) is in fact equivalent to a unitary representation \( U \) in the sense that \( P\bar{V}(s) = U(s)P \) for all \( s \in G \), where \( P \) is an invertible operator on \( \mathbb{C}^n \). (This result is also true for the infinite-dimensional representations when \( G \) is amenable; see [9] or [7].)
\(\tilde{\gamma} = P\tilde{\mu}.\) It is clear that the coordinates of \(\tilde{\gamma}\) also generate the ideal \(M.\) We have also
\[
\tilde{\gamma}(f_s) = P\tilde{\mu}(f_s) = PV(s)\tilde{\mu}(f) = \tilde{U}(s)P\tilde{\mu}(f) = \tilde{U}(s)\tilde{\gamma}(f),
\]
and so \(\tilde{\gamma}\) is \(U\)-invariant. Since \(U\) is unitary, it follows by [8, 21.40(a)] that \(U\) is a direct sum of continuous, irreducible representations \(U^1, U^2, \ldots, \) and \(U^m.\) This means that \(\mathbb{C}^n\) is the direct sum of some invariant subspaces \(H_i,\) and each \(U^i\) is the restriction of \(U\) to \(H_i (i = 1, 2, \ldots, m).\) For each \(f \in \mathcal{F},\) we write \(\tilde{\gamma}(f) = \sum_i \tilde{\mu}^i(f).\) Then
\[
\sum_{i=1}^m \tilde{\mu}^i(f_s) = \tilde{\gamma}(f_s) = \tilde{U}(s)\tilde{\gamma}(f) = \sum_{i=1}^m \tilde{U}^i(s)\tilde{\mu}^i(f).
\]
It follows that, for each \(i = 1, 2, \ldots, m,\) \(\tilde{\mu}^i(f_s) = \tilde{U}^i(s)\tilde{\mu}^i(f).\) This yields statement (1). Statement (2) follows from Theorem 1. Statement (3) is clear.

**Remark.** In \(L^1(G)^{**},\) the situation is slightly different. In fact, one can also produce finite-dimensional left ideals with the use of the right annihilators of \(L^1(G)^{**}.\) These are elements \(\mu\) in \(L^1(G)^{**}\) which satisfy \(L^1(G)^{**}\mu = \{0\};\) see [6]. In such a situation, the representation \(\tilde{V}\) is trivial in the proof above.

**Theorem 3.** Let \(M\) be a left ideal of \(L^1(G)^{**}\) of dimension \(n,\) and suppose that \(M\) contains \(l\) linearly independent right annihilators \(\gamma_1, \gamma_2, \ldots, \gamma_l\) of \(L^1(G)^{**}\). Then there exist \(m\) vectors \(\tilde{\mu}^i \in (L^1(G)^{**})^{n_i}\) and \(m\) irreducible, unitary, bounded and continuous representations \(U^i (i = 1, 2, \ldots, m)\) of \(G\) such that

1. each \(\tilde{\mu}^i\) is topologically \(U^i\)-invariant,
2. for each \(i = 1, 2, \ldots, m,\) the coordinates of \(\tilde{\mu}^i\) span a minimal left ideal \(M_i\) of \(L^1(G)^{**}\) of dimension \(n_i,\) and
3. \(M = \mathbb{C}\gamma_1 \oplus \mathbb{C}\gamma_2 \oplus \ldots \oplus \mathbb{C}\gamma_l \oplus M_1 \oplus M_2 \oplus \ldots \oplus M_m.\)

**Proof.** We take \(n - l\) linearly independent elements \(\mu_1, \mu_2, \ldots, \mu_{n-l}\) in \(M\) which are not right annihilators of \(L^1(G)^{**},\) let \(\tilde{\mu} = (\mu_i)_{i=1}^{n-l}.\) Then, form the matrices \(A(\mu)\) such that \(A\tilde{\mu} = A(\mu)\tilde{\mu}\) for \(\mu \in L^1(G)^{**}\), restrict \(A\) to \(L^1(G),\) and let \(\tilde{V}\) be the corresponding representation of \(G.\) Then, for \(\phi \in L^1(G)\) and \(f \in L^\infty(G),\) we have
\[
\tilde{\mu}(\phi * f) = \tilde{\mu}(\hat{\phi} * f) = \hat{\phi} \tilde{\mu}(f) = A(\hat{\phi})\tilde{\mu}(f) = V(\hat{\phi})\tilde{\mu}(f) = \tilde{V}(\phi)\tilde{\mu}(f),
\]
and so \(\tilde{\mu}\) is topologically \(V\)-invariant. The proof is completed as that of Theorem 2.

**Corollary.** Let \(G\) be a locally compact group. Then

1. finite-dimensional (left) ideals exist in \(M(G)\) and \(L^1(G)\) if and only if \(G\) is compact,
2. finite-dimensional left ideals exist in \(LUC(G)^*\) if and only if \(G\) is amenable,
3. finite-dimensional left ideals which are not generated by right annihilators of \(L^1(G)^{**}\) exist in \(L^1(G)^{**}\) if and only if \(G\) is amenable.

**Proof.** This follows from Lemma 2 and Theorems 2 and 3.

**Remark.** The finite-dimensional right ideals in \(WAP(G)^*\) are determined in the same way because the two Arens product coincide in this case; see [2, Section 4.2]. When \(G\) is compact, one proceeds also in the same way to find these ideals in \(L^1(G)\) and \(M(G).\) These facts were already observed for the minimal right ideals in [1,
Section 4]. However, in [4, Remark 2.7(b)], we have proved that the non-trivial right ideals are all of infinite dimension in $LUC(\mathbb{Z})^* = \ell^\infty(\mathbb{Z})^*$, where $\mathbb{Z}$ is the additive group of the integers. In [1, Section 4], we have given a class of locally compact abelian groups, which includes $\mathbb{Z}$, for which the non-trivial right ideals are all of infinite dimension in $LUC(G)^*$. Now we can prove that, for a locally compact abelian group $G$, the finite-dimensional right ideals exist in $LUC(G)^*$ if and only if $G$ is compact. We hope to publish this result in another paper.

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References


