FOURIER MULTIPLIERS ON WEIGHTED $L^p$-SPACES

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Dedicated to Professor Leonard Y. H. Yap on the occasion of his sixtieth birthday

Abstract. In his 1986 paper in the Rev. Mat. Iberoamericana, A. Carbery proved that a singular integral operator is of weak type $(p, p)$ on $L^p(\mathbb{R}^n)$ if its lacunary pieces satisfy a certain regularity condition. In this paper we prove that Carbery’s result is sharp in a certain sense. We also obtain a weighted analogue of Carbery’s result. Some applications of our results are also given.

1. Introduction

We introduce some notation to be used in this paper. Let $S(\mathbb{R}^n)$ be the class of Schwartz functions on $\mathbb{R}^n$ and let $\phi$ be a non-negative $C^\infty(\mathbb{R}^n)$ function supported in $\{|x| \leq 1\}$ with $\int \phi = 1$. For $j \in \mathbb{Z}$ define the operator $P_j$ on $S(\mathbb{R}^n)$ by

$$P_j f = \phi_{2^j} * f,$$

where $\phi_{2^j}(x) = 2^{-jn} \phi(2^{-j}x), \ x \in \mathbb{R}^n$. Now let $\psi$ be a non-negative $C^\infty(\mathbb{R}^n)$ function supported in $\{|1 \leq |\xi| \leq 4\}$ such that $\sum_{j \in \mathbb{Z}} \psi(2^j \xi) = 1$ for $\xi \neq 0$. For $j \in \mathbb{Z}$, define the operator $Q_j$ on $S(\mathbb{R}^n)$ by

$$(Q_j f) \wedge (\xi) = \psi(2^j \xi) \hat{f}(\xi).$$

Let $\eta$ be a non-negative $C^\infty(\mathbb{R}^n)$ function supported in $\{|1 \leq |x| \leq 4\}$ such that $\sum_{j \in \mathbb{Z}} \eta(2^j x) = 1$ for $x \neq 0$. Let $T$ and $T_j$ be singular integral operators with kernels $K(x, y)$ and $K(x, y)\eta(2^{-j}(x - y))$, respectively.

Recently, Carbery proved the following theorem in [C, Theorem 1].

Theorem C. Let $T$ be a singular integral operator bounded on $L^2(\mathbb{R}^n)$ such that

$$\sum_{k \in \mathbb{Z}} \sup_{j \geq 0} \| \sum_{j \in \mathbb{Z}} Q_{j+k} T_{j+k}(I - P_j) \|_{\mathcal{M}_p} < \infty,$$

where $\| \cdot \|_{\mathcal{M}_p}$ is the operator norm on $L^p(\mathbb{R}^n)$ and $I$ is the identity operator on $L^2(\mathbb{R}^n)$. Then $T$ is of weak type $(p, p)$ on $L^p(\mathbb{R}^n)$.

We show in Theorem 4.1 that Theorem C is sharp. We also obtain in Theorem 2.1 a weighted analogue of Theorem C. Theorem 2.1 is then applied to obtain two multiplier results on power-weighted $L^p$-spaces; see Theorems 3.1 and 3.4.

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2. Weighted analogue of Theorem C

Let \( w \) be a non-negative locally integrable function on \( \mathbb{R}^n \). For \( E \subset \mathbb{R}^n \) define \( w(E) = \int_E w(x)dx \). For \( 1 \leq p < \infty \) let \( L^p_w(\mathbb{R}^n) \) be the space of all measurable functions \( f \) on \( \mathbb{R}^n \) such that \( \|f\|_{p,w} < \infty \), where

\[
\|f\|_{p,w} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x)dx \right)^{1/p}.
\]

We shall simply write \( L^p_w(\mathbb{R}^n) \) and \( \| \cdot \|_{p,w} \) if \( w(x) = |x|^\alpha \).

The weight \( w \) is said to satisfy the Muckenhoupt \( A_p \) condition if there is a constant \( C \) such that

\[
\left( \frac{1}{|B|} \int_B w(x)dx \right)^{1/p} \left( \frac{1}{|B|} \int_B w(x)^{-1/(p-1)}dx \right)^{p-1} \leq C, \quad 1 < p < \infty;
\]

\[
\frac{1}{|B|} \int_B w(x)dx \leq C \inf_{x \in B} w(x), \quad p = 1,
\]

for all balls \( B \subset \mathbb{R}^n \), where \( |B| \) denotes the Lebesgue measure of \( B \).

The main result in this section is the following weighted version of Theorem C.

**Theorem 2.1.** Let \( w \in A_p, \ 1 < p < 2 \), and let \( T \) be a singular integral operator bounded on \( L^2_w(\mathbb{R}^n) \) such that

\[
\sum_{k \in \mathbb{Z}} \sup_{j \geq 2} \| \sum_{l \geq 0} Q_j + k T_{j+l} (I - P_j) \|_{\mathcal{M},p,w} < \infty,
\]

where \( \| \cdot \|_{\mathcal{M},p,w} \) is the operator norm on \( L^p_w(\mathbb{R}^n) \). Then \( T \) is of weak type \((p,p)\) on \( L^p_w(\mathbb{R}^n) \).

The proof of Theorem 2.1 depends on the following weighted version of Calderón-Zygmund decomposition whose proof is standard and is therefore omitted.

**Theorem 2.2.** Let \( w \in A_p, \ 1 \leq p < \infty \), and let \( f \in \mathcal{S}(\mathbb{R}^n) \). For \( \alpha > 0 \), there exist a sequence of mutually disjoint balls \( \{B_i\} \) and measurable functions \( g \) and \( b_i, \ i \in \mathbb{N} \), such that

(i) \( f = g + \sum_{i=1}^{\infty} b_i \);

(ii) \( b_i = f \chi_{B_i} \) and \( \|b_i\|_1 \leq 2^n \alpha |B_i| \);

(iii) \( \sum_{i=1}^{\infty} \|b_i\|_{p,w} \leq \|f\|_{p,w}, \ 1 \leq p < \infty \);

(iv) \( \sum_{i=1}^{\infty} w(B_i) \leq C \|f\|_{p,w}, \ 1 \leq p < \infty \);

(v) \( \|g\|_{\infty} \leq \alpha \);

(vi) \( \|g\|_{p,w} \leq C \|f\|_{p,w}, \ 1 \leq p < 2 \).

**Proof of Theorem 2.1.** Let \( f \in \mathcal{S}(\mathbb{R}^n) \). Fix \( \alpha > 0 \) and apply Theorem 2.2 to write \( f = g + \sum_i b_i \) with \( \|g\|_{\infty} \leq \alpha \) and each \( b_i \) supported in a ball \( B_i \) of radius \( 2^{i+1} \). Let

\[
G = g + \sum_i b_i \ast \phi_{2^{i+1}}
\]

and

\[
h = \sum_i (b_i - b_i \ast \phi_{2^{i+1}}),
\]
where $\phi$ is a non-negative radial $C^\infty(\mathbb{R}^n)$ function supported in $\{|x| \leq 1\}$ with $\int \phi = 1$ and $\phi_{2(i)}(x) = 2^{-nj(i)}\phi(2^{-j(i)}x)$. Then we have

$$\{x : |Tf(x)| > \alpha\} \subset \{x : |TG(x)| > \alpha/2\} \cup \{x : |Th(x)| > \alpha/2\}$$

$$:= E_\alpha \cup F_\alpha.$$ 

It follows from hypothesis (2.1) and Carbery’s arguments in [C, Theorem 1] that $w(E_\alpha) \leq C\alpha^{-p}\|f\|^p_{p,w}$. To estimate $w(E_\alpha)$, we note that $w(E_\alpha) \leq C\alpha^{-2}\|G\|^2_{2,w}$ because $T$ is bounded on $L^2_w(\mathbb{R}^n)$. Since $1 < p < 2$, Theorem 2.2(vi) implies that

$$\|g\|^2_{2,w} \leq C\alpha^{2-p}\|f\|^p_{p,w}. \tag{2.2}$$

Note that

$$\|b_i * \phi_{2(i)}\|_\infty \leq \|b_i\|_1\|\phi_{2(i)}\|_\infty$$

$$\leq C2^n\alpha|B_i|2^{-nj(i)}$$

$$\leq C\alpha$$

for all $i \in \mathbb{N}$. Since $1 < p < 2$ we have

$$\sum_i \|b_i * \phi_{2(i)}\|^2_{2,w} \leq C\alpha^{2-p}\sum_i \|b_i * \phi_{2(i)}\|_{p,w}^p$$

$$\leq C\alpha^{2-p}\sum_i \|b_i\|_{p,w}^p$$

$$\leq C\alpha^{2-p}\|f\|^p_{p,w},$$

where the second inequality follows from [ST, Theorem 6, p. 162] and the last inequality is by Theorem 2.2 (iii). It now follows from (2.2) and (2.3) that

$$\|G\|^2_{2,w} \leq C\alpha^{2-p}\|f\|^p_{p,w}.$$ 

Hence we also have

$$w(E_\alpha) \leq C\alpha^{-p}\|f\|^p_{p,w}.$$ 

Consequently we have $w\{x : |Tf(x)| > \alpha\} \leq C\alpha^{-p}\|f\|^p_{p,w}$; that is to say, $T$ is of weak type $(p,p)$ on $L^p_w(\mathbb{R}^n)$.

Theorem 2.1 has the following simple corollary.

**Corollary 2.3.** Let $\{\alpha(k)\}_{k=0}^\infty$ be a sequence of positive real numbers satisfying $\sum_{k=0}^{\infty} |k|\alpha(k) < \infty$. Let $w \in A_p$, $1 < p < 2$, and let $m \in L^\infty(\mathbb{R}^n)$ be such that $m$ is a multiplier on $L^2_w(\mathbb{R}^n)$. Suppose that for all $j > i$, we have

$$\|m_i * (\tilde{\eta})_{2^{-j}}\|_{M_{p,w}} \leq \alpha(i-j),$$

where $m_i(\xi) = m(\xi)\psi(2^i\xi)$. Then $m$ is a multiplier of weak type $(p,p)$ on $L^p_w(\mathbb{R}^n)$.

**Proof.** Let $T$ be the convolution operator defined on $S(\mathbb{R}^n)$ by $(Tf)^\wedge(\xi) = m(\xi)\hat{f}(\xi)$. For $j \in \mathbb{Z}$ define the operator $T_j$ on $S(\mathbb{R}^n)$ by

$$(T_jf)^\wedge(\xi) = (m * (\tilde{\eta})_{2^j})\hat{f}(\xi).$$

Using Theorem 2.1 and [ST, Theorem 6, p.162], we can prove the corollary in the same manner as Theorem 3 is proved in [C]. Details of the proof are therefore omitted.
3. Multipliers on power-weighted $L^p(\mathbb{R}^n)$

Corollary 2.3 indicates the amount of regularity needed for each $m_i$ so that $m$ is a multiplier of weak type $(p, p)$ on $L^p_w(\mathbb{R}^n)$. Our next theorem shows that for certain power weights, such a regularity condition is implied by $m_i$ satisfying a certain Lipschitz condition. We say a distribution $f$ is in the Lipschitz space $\Lambda_{\alpha}^{\beta}(\mathbb{R}^n)$ for $\beta > 0$, $1 \leq r, s \leq \infty$, if $\|f\|_{\Lambda_{\alpha}^{\beta}} < \infty$, where

$$\|f\|_{\Lambda_{\alpha}^{\beta}} = \left\{ \sum_{j \in \mathbb{Z}} 2^{-sj\beta} \|f \ast (\hat{\eta})_{2^j}\|_r \right\}^{1/s},$$

with the usual modification if $s = \infty$.

**Theorem 3.1.** Let $\beta > 0$ and let $m \in L^\infty(\mathbb{R}^n)$ be such that $m_i \in \Lambda_{\alpha}^{\beta}(\mathbb{R}^n)$ and

$$\sup_{i \in \mathbb{Z}} 2^{-i\beta} \|m_i\|_{\Lambda_{\alpha}^{\beta}} < \infty,$$

where $m_i$ is as in Corollary 2.3.

(i) Let $1 < q < 2$ and let $n(2-q)/2q < \beta < n/2$. Then $m$ is a multiplier on $L^q_{\alpha}(\mathbb{R}^n)$ for all $q < r \leq 2$ and $|\alpha| \leq 2\beta(r-q)/(2-q)$.

(ii) If $n/2 \leq \beta$, then $m$ is a multiplier on $L^q_{\alpha}(\mathbb{R}^n)$ for all $1 \leq r \leq 2$ and $|\alpha| < n(r-1)$.

We need the following two lemmas in the proof of Theorem 3.1.

**Lemma 3.2.** Let $1 < q < 2$ and let $n(2-q)/2q < \beta < n/2$. Let $m \in L^\infty(\mathbb{R}^n)$ be such that $m_i \in \Lambda_{\alpha}^{\beta}(\mathbb{R}^n)$ for all $i \in \mathbb{Z}$ and

$$\sup_{i \in \mathbb{Z}} 2^{-i\beta} \|m_i\|_{\Lambda_{\alpha}^{\beta}} < \infty,$$

where $m_i$ is as in Corollary 2.3. Then $m$ is a multiplier on $L^q(\mathbb{R}^n)$.

**Proof.** Recall that $\sum_{k \in \mathbb{Z}} \psi(2^k \xi) = 1$ for $\xi \neq 0$ and write

$$m_i \ast (\hat{\eta})_{2^{-i}}(\xi) = \sum_{k \in \mathbb{Z}} m_i \ast (\hat{\eta})_{2^{-i}}(\xi)\psi(2^k \xi)$$

$$= \sum_{k < i-1} + \sum_{k = i-1}^{i+1} + \sum_{k > i+1} m_i \ast (\hat{\eta})_{2^{-i}}(\xi)\psi(2^k \xi)$$

$$= I + II + III.$$

Let $\tau_{ijk}$ be defined by $\tau_{ijk}(\xi) = (m_i \ast (\hat{\eta})_{2^{-i}}(\xi))\psi(2^k \xi)$. Then $II = \sum_{k=i-1}^{i+1} \tau_{ijk}$.

Choose $p$ such that $1 < p < q$ and $\beta > n(2-p)/2p$. We shall estimate $\|\tau_{ijk}\|_{M_p}$ for $k = i$ by interpolation between $\|\tau_{ij}||_{M_2}$ and $\|\tau_{ij}||_{M_1}$. Clearly, we have

$$\|\tau_{ij}||_{M_2} = \|\tau_{ij}||_{M_1}$$

$$\leq \|m_i \ast (\hat{\eta})_{2^{-i}}||_{\infty} \|\psi(2^k \xi)||_{\infty}$$

$$\leq C(2^{-i})^\beta,$$
where the last inequality follows from (3.2) and \( \| \psi \|_\infty \leq 1 \). Note that \( \| \tau_{ij} \|_{M_1} = \| (\tau_{ij})^\vee \|_1 \). Let \( (\tau_{ij})^\vee = (\tau_{ij})^\vee \chi_A + (\tau_{ij})^\vee \chi_B \), where \( A = \{ |x| \leq 2^{j+3} \} \), \( B = \{ |x| > 2^{j+3} \} \). Then

\[
\| (\tau_{ij})^\vee \chi_A \|_1 \leq \| (\tau_{ij})^\vee \|_2 \| \chi_A \|_2
\leq \| m_\delta \ast (\hat{\eta})_{2^{-i}} \|_\infty \| \psi(2^j \cdot) \|_2 \| \chi_A \|_2
\leq C 2^{(i-j)\beta} 2^{-nj/2} 2^{n(j+3)/2} \leq C 2^{(i-j)(\beta-n/2)},
\]

where the penultimate inequality follows from (3.2) and \( \| \psi(2^j \cdot) \|_2 = 2^{-jn/2} \| \psi \|_2 \).

To estimate \( \| (\tau_{ij})^\vee \chi_B \|_1 \) we write \( B_\ell = \{ 2^\ell < |x| \leq 2^{\ell+1} \} \) for \( \ell \geq j + 3 \). Then

\[
\| (\tau_{ij})^\vee \chi_B \|_1 = \sum_{\ell > j+2} \int_{B_\ell} |(\tau_{ij})^\vee (x)| dx.
\]

Now for \( x \in B_\ell \) and \( t \in \mathbb{N} \), there exists a constant \( C_\ell \) such that

\[
|(\tau_{ij})^\vee (x)| \leq 2^{-ni} \int_{|1 \leq |y| \leq 1|} |(m_\delta)^\vee (y)\eta(2^{-j} y)| |(\psi)^\vee (2^{-i} (x-y))| dy
\]

\[
\leq C_\ell 2^{-2ni+nj+(i-\ell)t},
\]

where the last inequality follows from \( \| (m_\delta)^\vee \|_\infty \leq \| m_\delta \|_1 \leq C 2^{-ni} \). Now choose \( t = 3n \) and we have

\[
\| (\tau_{ij})^\vee \chi_B \|_1 \leq C \sum_{\ell > j+2} 2^{ni+nj-2n\ell} \leq C 2^{(i-j)n}.
\]

Since \( i < j \) and \( \beta < n/2 \), we have \( \| (\tau_{ij})^\vee \|_1 \leq C 2^{(i-j)(\beta-n/2)} \). It follows that

\[
\| \tau_{ij} \|_{M_1} \leq C 2^{(i-j)(\beta-n/2)}.
\]

Interpolating between (3.3) and (3.4) yields \( \| \tau_{ij} \|_{M_p} \leq C 2^{(i-j)(\beta-n(2-p)/2p)} \) for \( 1 < p < 2 \).

Similar estimates of \( \| \tau_{ij} \|_{M_p} \) for \( k = i-1, i+1 \) give

\[
\| \Pi \|_{M_p} \leq C 2^{(i-j)(\beta-n(2-p)/2p)}.
\]

Routine calculations as in [C, p.395] show that for \( |\gamma| \leq n \) and \( t > 2n \), there exists a constant \( C_\ell \) such that \( \| \Pi \|_{M_\infty} \leq C_\ell \| m_\delta \|_\infty 2^{(j-i)(n+|\gamma|-t)} \) and \( \| \Pi \|_{M_p} \leq C_\ell \| m_\delta \|_\infty 2^{(j-i)(n+|\gamma|-t)} \).

Consequently for \( j > i \) and \( t = 3n \), we have

\[
\| m_\delta \ast (\hat{\eta})_{2^{-i}} \|_{M_p} \leq \| \Pi \|_{M_p} + \| \Pi \|_{M_p} + \| \Pi \|_{M_p}
\leq C 2^{(i-j)(\beta-n(2-p)/2p)} + 2^{(i-j)n}.
\]

It follows from Corollary 2.3 that \( m \) is a multiplier of weak type \((p,p)\) on \( L^p(\mathbb{R}^n) \). Since \( 1 < p < q < 2 \) and \( m \) is a multiplier on \( L^2(\mathbb{R}^n) \), we have \( m \) a multiplier on \( L^q(\mathbb{R}^n) \).

**Lemma 3.3.** Let \( 0 < \beta < n/2 \). Let \( m \in L^\infty(\mathbb{R}^n) \) be such that \( m_i \in \Lambda^\beta_{n/\beta, 2}(\mathbb{R}^n) \) and

\[
\sup_{i \in \mathbb{Z}} \| m_i \|_{\Lambda^\beta_{n/\beta, 2}} < \infty,
\]

where \( m_i \) is as in Corollary 2.3. Then \( m \) is a multiplier on \( L^2_{2\beta}(\mathbb{R}^n) \).
Proof. Let \( f \in \mathcal{S}(\mathbb{R}^n) \). Since \( \|(m_i\hat{f})^\vee\|_{2,2;\beta} \sim \|m_i\hat{f}\|_{\Lambda^\beta_{2,2}} \), it follows from Herz [He, Lemma 1.5*] that
\[
\|m_i\hat{f}\|_{\Lambda^\beta_{2,2}} \leq C\|m_i\|_{\infty} + \|m_i\|_{\Lambda^\alpha_{\beta/\beta,2}} \|\hat{f}\|_{\Lambda^\beta_{2,2}} 
\leq C\|f\|_{2,2;\beta}.
\]
Thus we have \( \sup_{i \in \mathbb{Z}} \|m_i\|_{\Lambda_{2,2;\beta}} < \infty \). Hence \( m \) is a multiplier on \( L^2_{2;\beta}(\mathbb{R}^n) \).

We now turn to the proof of Theorem 3.1.

Proof of Theorem 3.1. (i). Note that \( m_i \subset \{ 2^{-i} \leq |\xi| \leq 2^{-i+2} \} \). Now Theorem 6.3.1 of [BL] and hypothesis (3.1) of Theorem 3.1 imply that \( m \) satisfies the hypotheses of both Lemmas 3.2 and 3.3. Thus \( m \) is a multiplier on \( L^q(\mathbb{R}^n) \), \( 1 < q < 2 \), and a multiplier on \( L^2_{2;\beta}(\mathbb{R}^n) \). The result now follows from the Stein-Weiss interpolation of \( L^p \) spaces with change of measures; see [SW, Theorem 2.11].

(ii). The proof of Lemma 3.2 can be easily modified to show that \( m \) is a multiplier on \( L^p(\mathbb{R}^n) \) for all \( 1 < p < 2 \). Now \( \Lambda^\beta_{\alpha/\beta,2}(\mathbb{R}^n) \subset \Lambda^\alpha_{\omega,2}(\mathbb{R}^n) \) for all \( 0 < \alpha < n/2 \). It follows from Lemma 3.3 that \( m \) is a multiplier on \( L^2_{2;\alpha}(\mathbb{R}^n) \) for all \( |\alpha| < n/2 \). The result again follows from the Stein-Weiss interpolation with change of measures.

As an application of Theorem 3.1 we consider the following multiplier discussed in (3.6) of Baernstein and Sawyer [BS, p.22].

Let \( a \) and \( b \) be positive real numbers and let \( s \) be an integer larger than \( b/a \). Let \( m \) be a strongly singular multiplier such that
\[
|D^\beta m(\xi)| \leq C|\xi|^{-b|\beta|}|\beta|^{(a-1)|\beta|}, \quad 0 \leq |\beta| \leq s, \quad |\xi| \leq 1,
\]
where \( \beta = (\beta_1, \ldots, \beta_n) \), \( D^\beta = D_{\beta_1} \cdots D_{\beta_n} \), and \( D = \partial/\partial \xi_j \).

The prototypical example is the function \( m_{\alpha,b} \) defined by \( m_{\alpha,b}(\xi) = \Theta(\xi)|\xi|^{-b\epsilon|\xi|^\alpha} \), where \( \Theta \in C^\infty(\mathbb{R}^n) \), \( \Theta = 0 \) on \( |\xi| < 1 \), \( \Theta = 1 \) on \( |\xi| \geq 2 \).

The next theorem shows that \( m \) is a multiplier on certain power-weighted \( L^p(\mathbb{R}^n) \).

Theorem 3.4. Let \( m \in L^\infty(\mathbb{R}^n) \) satisfy (3.5) above. Then we have

(i) if \( 1 < q < 2 \) and \( n(2-q)/2q < b/a < n/2 \), then \( m \) is a multiplier on \( L^q_\alpha(\mathbb{R}^n) \)
for all \( q < r \leq 2 \) and \( |\alpha| \leq 2b(r-q)/a(2-q) \);

(ii) if \( n/2 \leq b/a \), then \( m \) is a multiplier on \( L^\infty_\alpha(\mathbb{R}^n) \) for all \( 1 < r \leq 2 \) and \( |\alpha| < n(r-1) \).

Proof. Let \( s \) be an integer larger than \( b/a \). Since \( m_i(\xi) = m(\xi)\psi(2^i\xi) \) and \( m = 0 \)
for \( |\xi| \leq 1 \), we have \( m_i = 0 \) for \( i \geq 2 \). For \( i < 2 \) our hypothesis (3.5) implies that
\[
\|D^\beta m_i\|_{\infty} \leq C2^{2\epsilon|b|+|\beta|}, \quad 0 \leq |\beta| \leq s,
\]
where \( C \) is independent of \( i \). Assume that \( b/a \) is not an integer and let \( b/a = \nu + \sigma \),
where \( \nu \) is a non-negative integer and \( 0 < \sigma < 1 \). For \( 1 \leq j \leq n \) we have
\[
\left\| \Delta_h \frac{\partial^\nu m_i}{\partial \xi_j^{\nu+1}} \right\|_{\infty} \leq C \min\{ |h| \max_{|\beta|=\nu+1} \|D^\beta m_i\|_{\infty}, \|\partial^\nu m_i/\partial \xi_j^{\nu+1}\|_{\infty} \}
\leq C \min\{ |h|2^{2\epsilon(b-(a-1)(\nu+1))}, 2^{b-(a-1)\nu} \},
\]
where $\Delta_h f(x) = f(x + h) - f(x)$. Let $\omega_\infty(t, \frac{\partial^\nu m_i}{\partial \xi_j^\nu}) = \sup_{|h|<t} \| \Delta_h \frac{\partial^\nu m_i}{\partial \xi_j^\nu} \|_\infty$. Then we have

$$\sum_{j=1}^{n} \left( \int_0^\infty \left( t^{-\sigma} \omega_\infty(t, \frac{\partial^\nu m_i}{\partial \xi_j^\nu}) \right)^2 \frac{dt}{t} \right)^\frac{1}{2} \leq C 2^{b/a}.$$

It follows from [BL, Theorem 6.3.1] that $m_i \in \Lambda_{\infty,2}^{b/a}$ and $\|m_i\|_{\Lambda_{\infty,2}^{b/a}} \leq C 2^{b/a}$.

If $b/a$ is an integer, we write $b/a = \nu + 1$, where $\nu \geq 0$. Then for $1 \leq j \leq n$ we have

$$\left\| \Delta_h^2 \frac{\partial^\nu m_i}{\partial \xi_j^\nu} \right\|_{\infty} \leq C \min\{2^{\nu(b-(\nu+1)(\nu+2))}, 2^{\nu(b-(\nu+1)\nu)}\},$$

where $\Delta_h^2 f = \Delta(\Delta_h f)$.

Routine calculation as in the case where $b/a$ is a non-integer then shows that $m_i \in \Lambda_{\infty,2}^{b/a}$ and $\|m_i\|_{\Lambda_{\infty,2}^{b/a}} \leq C 2^{b/a}$. The theorem now follows from Theorem 3.1.

4. Sharpness of Theorem C

In this section we prove that Theorem C is sharp in the following sense.

**Theorem 4.1.** Let $1 < p < 2$. There exists a convolution operator $T$ bounded on $L^2(\mathbb{R}^n)$ so that

(i) $\sum_{k \in \mathbb{Z}, j \in \mathbb{Z}} \sup_{\ell \in \mathbb{N}} \|Q_{\ell+k} T_{j+l}(I - P_j)\|_{\mathcal{M}_p} < \infty$;

(ii) $T$ is bounded on $L^p(\mathbb{R}^n)$;

(iii) $T$ is not of weak type $(r,r)$ on $L^r(\mathbb{R}^n)$ for $1 < r < p$.

**Proof.** Let $1 < r < p < 2$. Choose $q$ such that $r < q < p$ and choose $\alpha$ such that $0 < \alpha < 1$, $q < 2/(2 - \alpha) < p$. Following Onneweer [O, p.56] we construct a sequence $(P_k)_{k \geq 0}$ of Rudin-Shapiro-like polynomials on the $n$-dimensional torus $T^n$ such that $\|P_k\|_\infty \leq 2^{n(k+1)/2}$ and $|\hat{P}_k(j)| = 1$ for $j = (j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n$, where $0 \leq j_i < 2^{n(k+1)/2}$ for $i = 1, \ldots, n$. Note that $\hat{P}_k$ is supported on $\mathbb{Z}^n$. Now for $k \geq 0$ define $\gamma_k$ on $\mathbb{Z}^n$ by $\gamma_k = 2^{n(\alpha-1)/2} \hat{P}_k$. Let $\hat{k}$ denote the center of the cube $[2^{k+2}, 2^{k+3}]^n$ and define $\Phi_k$ on $\mathbb{Z}^n$ by $\Phi_k(j) = \gamma_k(j - \hat{k})$. Write $\Phi = \sum_{k \geq 0} \Phi_k$ and it can be shown as in Figà-Talamanca and Gaudry [FG, Theorem B] that $\Phi$ is a multiplier on $L^p(T^n)$ but not a multiplier on $L^q(T^n)$.

Let $m$ be the function defined on $\mathbb{R}^n$ by $m(\xi) = \sum_{j \in \mathbb{Z}^n} S(\xi - j) \Phi(j)$, where $S(\xi) := (\max\{1 - |\xi|^2, 0\})^n$ for $\xi \in \mathbb{R}^n$. Now define the convolution operator $T$ on $S(R^n)$ by $(Tf)^\wedge = m \hat{f}$. Then $T$ is bounded on $L^2(\mathbb{R}^n)$. Since $n(1 - \alpha)/2 >$
\[ n(2-p)/2p, \] it follows from the proofs of Theorem 3 of \[ C \] and Lemma 3.2 that \[ T \] will satisfy (i) if we have

\[ \sup_{k \in \mathbb{Z}} 2^{-nk(1-\alpha)/2} \| m_k \|_{A^\infty(\mathbb{Z}^d)/2} < \infty, \]

where \[ m_k(\xi) = m(\xi) \psi(2^k \xi). \] Note that \[ \text{supp} \ m_k \subseteq \{ 2^{-k} \leq |\xi| \leq 2^{-k+2} \} \] and \[ m_k = 0 \] for \[ k \geq 0. \] Thus we only need to prove (4.1) for \[ k < 0. \] Let \[ \xi \in \mathbb{R}^d \] and let \[ A_\xi = \{ j \in \mathbb{Z}^d : |\xi - j| \leq 1 \}. \] Then \[ A_\xi \] is a finite set with at most \[ 2^n \] elements. Let \[ \beta = n(1-\alpha)/2 \] and we have \[ |\Phi(j)| \leq 2^{(k+2)\beta} \] for \[ j \in A_\xi, \] \[ \xi \in \text{supp} \ m_k. \] Furthermore, \[ k < 0 \] implies that there exists a constant \[ C \] so that \[ \| D^\mu S(\cdot - j) \psi(2^k \xi) \|_\infty \leq C \] for all multi-indices \[ \mu \] with \[ 0 \leq |\mu| \leq n \] and all \[ j \in A_\xi, \] \[ \xi \in \text{supp} \ m_k. \] Consequently we have

\[ |D^\mu m_k(\xi)| \leq \sum_{j \in A_\xi} |D^\mu S(\xi - j) \Phi(j) \psi(2^k \xi)| \leq C 2^{k\beta}. \]

If \[ \beta \] is not an integer, we write \[ \beta = \nu + \sigma \] with \[ 0 < \sigma < 1. \] Then \[ \nu \] is an integer less than \[ n/2. \] For \[ i = 1, 2, \ldots, n \] we have

\[ \left\| \Delta_h \frac{\partial^\nu m_k}{\partial \xi_i^\nu} \right\|_\infty \leq C |h|^\sigma 2^{k\beta}. \]

Thus we have \[ \| m_k \|_{A^\beta_{\infty}} \leq C 2^{k\beta} \] if \[ \beta \] is not an integer. If \[ \beta \] is an integer, then routine calculation as in the proof of Theorem 3.4 shows that \[ \| m_k \|_{A^\beta_{\infty}} \leq C 2^{k\beta}. \]

Consequently, the sequence \[ \{ m_k \}_{k=\infty} \] satisfies (4.1) and we have \[ T \] satisfying (i).

Since \[ \beta = n(1-\alpha)/2 > n(2-p)/2p, \] (4.1) and Lemma 3.2 imply that \[ m \] is a multiplier on \[ L^p(\mathbb{R}^d). \] Hence \[ T \] satisfies (ii). Lastly, if \[ T \] were of weak type \( (r,r) \) on \[ L^r(\mathbb{R}^d) \] for \( 1 < r < p \), then \[ T \] would be bounded on \[ L^q(\mathbb{R}^d) \] since \( r < q < p \). Hence \[ m \] would be a multiplier on \[ L^q(\mathbb{R}^d). \] By deLeeuw’s theorem \[ [L, \text{Proposition 3.3}] \] the restriction \[ \Phi \] of \[ m \] to \[ \mathbb{Z}^d \] would then be a multiplier on \[ L^q(\mathbb{T}^d), \] but this contradicts our earlier observation that \[ \Phi \] is not a multiplier on \[ L^q(\mathbb{T}^d). \] Thus \[ T \] satisfies (iii).

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