NAKAI'S CONJECTURE FOR VARIETIES SMOOTHED BY NORMALIZATION

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Abstract. The notion of D-simplicity is used to give a short proof that varieties whose normalization is smooth satisfy Ishibashi’s extension of Nakai’s conjecture to arbitrary characteristic. This gives a new proof of Nakai’s conjecture for curves and Stanley-Reisner rings.

Introduction

Nakai’s conjecture concerns a very natural question: can differential operators detect singularities on algebraic varieties? On a smooth complex variety, it is well known that the ring of differential operators is generated by derivations. Nakai asked whether the converse holds: if the ring of differential operators is generated by derivations, is the variety smooth? In this paper, the notion of D-simplicity is used to give a short proof that varieties whose normalization is smooth satisfy Ishibashi’s extension [2] of Nakai’s conjecture to arbitrary characteristic. For example, any variety whose irreducible components are smooth satisfies Nakai’s conjecture, giving a proof that Nakai’s conjecture holds for Stanley-Reisner rings. Furthermore, this gives a simple new characteristic-independent proof of Nakai’s conjecture for curves. The argument is quite short and rederives characteristic-dependent results of Mount and Villamayor (characteristic zero) [4] and Ishibashi (prime characteristic) [2].

Definitions and notation

Let $X = \text{Spec}(R)$ be an affine algebraic variety defined over a field $k$ of characteristic zero. When $R$ is regular (that is, when $X$ is smooth), the ring of differential operators $D(R/k)$ (see EGA [1] or [3]) equals $\text{der}(R/k)$, the $R$-subalgebra generated by the derivations (see McConnell and Robson [3, Corollary 15.5.6]). Nakai conjectured that this condition characterizes nonsingularity: $R$ is regular if and only if $\text{der}(R/k) = D(R/k)$. Ishibashi [2] extended Nakai’s conjecture to varieties defined over an arbitrary perfect field.

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A Hasse-Schmidt derivation $\Delta = \{\delta_n\}_{n=0}^{\infty} \subseteq \text{End}_k(R)$ is a collection of $k$-linear endomorphisms of $R$ such that $\delta_0 = id_R$ and

$$\delta_n(ab) = \sum_{i+j=n} \delta_i(a)\delta_j(b).$$

For example, if $R$ has characteristic zero and $d$ is a derivation, $\delta_n = \frac{1}{n!}d^n$ determines a Hasse-Schmidt derivation. Let $HS(R/k)$ be the $R$-algebra generated by the components $\delta_n$ of Hasse-Schmidt derivations on $R$. $HS(R/k)$ is a subalgebra of $D(R/k)$ and if $R$ has characteristic zero, $\text{der}(R/k) = HS(R/k)$. In characteristic zero Grothendieck [1, 16.11.2 and 17.12.4] showed that when $R$ is smooth over $k$, $HS(R/k) = D(R/k)$; a proof in arbitrary characteristic can be found in Traves [9]. Ishibashi's extension of Nakai's conjecture is that $R$ is smooth over $k$ if and only if $HS(R/k) = D(R/k)$. The main result of this paper is that varieties whose normalization is smooth satisfy this extension of Nakai's conjecture.

**Preliminary results**

Let $R$ be a reduced algebra of finite type over a perfect field $k$. For a reduced ring $R$, the normalization $R'$ of $R$ is the integral closure of $R$ in its total ring of quotients, $L = S^{-1}R$, where $S$ is the multiplicative set of nonzerodivisors in $R$. The conductor of $R'$ into $R$ is

$$C = \{c \in R : cR' \subseteq R\}.$$ 

The conductor is an ideal of both $R$ and $R'$.

**Lemma 1.** (1) The conductor is $HS(R/k)$-stable.

(2) Powers of $HS(R/k)$-stable ideals are $HS(R/k)$-stable.

**Proof.** Part (1) is well-known: see Seidenberg [6, Corollary on page 169 and section 5]. To establish (2) it suffices to show that for $a_1, \ldots, a_s$ in the $HS(R/k)$-stable ideal $I$ and $\Delta = \{\delta_n\}$ a Hasse-Schmidt derivation, $\delta_n(a_1 \cdots a_s) \in I^s$. Now

$$\delta_n(a_1 \cdots a_s) = \sum_{i+j=n} \delta_i(a_1)a_j(a_2 \cdots a_s)$$ 

and the claim follows by induction on $s$. \qed

**Lemma 2.** The conductor is not contained in any minimal prime of $R$.

**Proof.** The conductor $C$ equals $\text{Ann}_R(\frac{R'}{R})$. If $C \subseteq P$ with $P$ a minimal prime of $R$, then, since $\text{Supp}(\frac{R'}{R}) = \overline{\text{V}(\text{Ann}(\frac{R'}{R}))}$, $P \in \text{Supp}(\frac{R'}{R})$. Localizing the exact sequence of $R$-modules

$$0 \to R \to R' \to \frac{R'}{R} \to 0$$

at the minimal prime $P$ gives an exact sequence

$$0 \to R_P \to R'_P \to \frac{R'}{R} \to 0.$$ 

But since $P$ is a minimal prime of a reduced ring, $R_P$ is a field and the normalization map $R_P \to R'_P$ is an isomorphism. This forces $(\frac{R'}{R})_P = 0$, a contradiction. So $C$ is not contained in any minimal prime of $R$. \qed

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1 Actually, Ishibashi requires that $k$ be algebraically closed and conjectures $R$ regular, but it is clear that smoothness is the relevant notion.
Lemma 3. If $R$ is a domain that is smooth over $k$, then $R$ is $D(R/k)$-simple.

Proof. Since $k$ is perfect, $R$ is regular. In the characteristic zero case, it is well known that $R$ is $D(R/k)$-simple (see McConnell and Robson [3, Theorem 15.3.8 and Corollary 15.5.6]). For the prime characteristic case, note that since $R$ is strongly F-regular and since $D$-domains are F-finite. Smith [7, Theorem 2.2] has shown that strongly F-regular F-finite $R$ is regular. In the characteristic zero case, it is well known that $R$ is $D(R/k)$-simple.

1. Nakai’s conjecture

Theorem 4. Let $R$ be a reduced $k$-algebra of finite type and let $R'$ be its integral closure in its total ring of quotients. If $R'$ is a product of $D$-simple rings and $HS(R/k) = D(R/k)$, then $R$ is normal.

Proof. Note that $R'$ is isomorphic to $R_1 \times \cdots \times R_t$, where $R_i$ is the normalization of $R/k$ with the $\{P_i\}$ ranging over the minimal primes of $R$. By Lemma 1 (1), $C$ is $HS(R/k)$-stable and by Lemma 1 (2), $C^2$ is also $HS(R/k)$-stable. Thus, $C^2$ is $D(R/k)$-stable.

Assume that $C^2 \neq C$. Since $C$ is not contained in any of the minimal primes $P_i$ of $R$, there are elements $c_i \in C \setminus P_i$ with $c_j \notin C^2$. To see this, take $x \in C \setminus C^2$ and note that $x \neq 0 \Rightarrow x \notin \bigcap P_i \Rightarrow x$ is not in some $P_j$. Set $c_j = x$ and pick the other $c_i \in C \setminus P_i$. Let $c = (c_1, \ldots, c_t) \in R'$, after identifying $R'$ with the product in the first paragraph. Then $c \in C \setminus C^2$ and $c$ is nonzero in each component. By the D-simplicity of each of the $R_i$, there is an operator $\theta = (\theta_1, \ldots, \theta_t) \in D(R_1) \times \cdots \times D(R_t)$, such that $\theta(c^2) = 1$. If each $\theta_i \in D(R_i)$ is an operator of order $\leq n_i$, then $\theta = (\theta_1, \ldots, \theta_t)$ maps $R'$ to itself and $\theta$ is a differential operator of order $\leq n = \max(n_i)$. Thus, $\theta \in D(R')$ and $c \theta \in D(R)$. Now $(c \theta)(c^2) = c \notin C^2$, contradicting the fact that $C^2$ is $D(R/k)$-stable. This forces $C^2 = C$.

In fact, $C = R$. Indeed, if $C$ is contained in a maximal ideal $m$, then $C_m = C_m \subset mC_m \subset C_m$, so $mC_m = C_m$.

Now Nakayama’s lemma forces $C_m = 0$. But then $C$ must consist of zero divisors, contradicting Lemma 2. So $C = R$ and $R$ is normal.

Theorem 5. If $HS(R/k) = D(R/k)$ and the normalization $R'$ of $R$ is smooth over $k$, then $R$ is smooth over $k$.

Proof. Since the normalization $R'$ of $R$ is a product of smooth domains, $R'$ is a product of $D$-simple rings (by Lemma 3). The result now follows from Theorem 4.

This theorem says that the ring of differential operators of a singular complex variety whose normalization is smooth is not generated by derivations. Thus, $D(R/C)$ is complicated even for very mild singularities (those that can be resolved by normalization). We expect that $D(R/C)$ will become more complicated as the singularities become worse. This provides further evidence for Nakai’s conjecture.

The theorem shows that Nakai’s conjecture holds for reduced varieties smoothed by normalization; for example, curves.

Corollary 6. Let $R$ be a reduced $k$-algebra of finite type, where $k$ is a perfect field. If $R$ is 1-dimensional and $HS(R/k) = D(R/k)$, then $R$ is smooth over $k$. In
particular, if \( R \) is a domain of characteristic 0 and \( \text{der}(R/k) = D(R/k) \), then \( R \) is regular.

Theorem 5 can also be used to recover a result due to Schreiner [5] (also, see Traves [8]): Nakai’s conjecture holds for Stanley-Reisner rings (that is, for subvarieties of \( \mathbb{A}^N_k \) which are the union of coordinate subspaces). More generally, the theorem implies that Nakai’s conjecture holds for varieties all of whose components are smooth, as remarked by Lazarsfeld.

**Corollary 7.** Let \( R \) be a reduced \( k \)-algebra of finite type, where \( k \) is a perfect field. If \( HS(R/k) = D(R/k) \) and if \( R_P \) is smooth over \( k \) for each minimal prime \( P \) of \( R \), then \( R \) is smooth over \( k \). In particular, if \( R \) is a Stanley-Reisner ring and \( HS(R/k) = D(R/k) \), then \( R \) is a polynomial ring.

Both Corollary 6 and Corollary 7 follow immediately from Theorem 5: just observe that the normalization of \( X = \text{Spec}(R) \) is isomorphic to the disjoint union of the normalization of the components of \( X \), each of which is smooth over \( k \).

**References**


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