PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 8, Pages 2475–2482
S 0002-9939(99)05063-7
Article electronically published on April 15, 1999

EXPONENTIAL MAPS OF SOBOLEV METRICS ON LOOP GROUPS

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(Communicated by Józef Dodziuk)

Abstract. We find conditions for the exponential map on a weak riemannian Hilbert manifold to be a nonlinear Fredholm map of index zero and apply the result to left-invariant Sobolev metrics on loop groups.

1. Introduction

As a preliminary step toward a better understanding of global geometry of a Hilbert riemannian manifold M, one studies singularities of its exponential map. Singular values of exp are the conjugate points in M. In contrast with finite dimensions, two types of conjugate points can occur on a Hilbert manifold depending on whether the differential of the exponential map fails to be one-to-one (a monoconjugate point) or fails to be onto (an epiconjugate point). A simple example of the unit sphere in a Hilbert space with induced metric shows that, in general, conjugate points may have infinite mutiplicity.

In the early sixties Smale introduced the notion of a nonlinear Fredholm map which has played an essential role in the subsequent development of differential topology and geometry of infinite dimensional manifolds (see, for example, [S], [ET], [Q], [T]). The main result of this paper gives a sufficient condition for the exponential map of a possibly weak riemannian metric on a Hilbert manifold M to be a nonlinear Fredholm map of index zero. A different condition in the (strong) riemannian case was given in [Mis] and applied there to the exponential map of the Sobolev H^1 metric on the space of H^1 loops in a finite dimensional manifold. As immediate corollaries we obtain that monoconjugate and epiconjugate points must coincide and have finite multiplicities. Furthermore, the set of points conjugate to a given point in M is of the first Baire category.

Loop groups are among the simplest Hilbert manifolds. Studied extensively for nearly two decades they found many diverse applications particularly in geometry and mathematical physics (see, for instance, [Mic], [PS], [SW], [T] or [U]). One motivation for the results presented here was the paper of Freed [F] which studied the geometry of left-invariant Sobolev H^s metrics on loop groups $L^{s_o}(G) = H^{s_o}(S^1, G)$ with $s_o > 1/2$. Note that for $s < s_o$ these are weak riemannian metrics.

After formulating and proving the main result of this paper in Section 2 we apply it, in Section 3, to the family of H^s metrics on $L^{s_o}(G)$ for $0 < s \leq s_o$. In Section

Received by the editors November 3, 1997.
1991 *Mathematics Subject Classification.* Primary 58B20; Secondary 22E65.
Key words and phrases. Exponential map, Fredholm operators, loop groups.

4 we construct an example in which the exponential map of the L^2 metric ($s = 0$) fails to be Fredholm.

2. Exponential map on Hilbert manifolds

We begin with a few definitions and well known facts. Our basic references are [F], [K] and [P]. Let M be a manifold modelled on a Hilbert space H. Recall that a weak riemannian metric on M is a smooth assignment to each $p \in M$ of a continuous, positive definite, symmetric bilinear form $p \to \langle \cdot, \cdot \rangle(p)$ on the tangent space T_pM. Observe that $T_pM \simeq H$ may not be complete as a metric space under the distance induced by $\langle \cdot, \cdot \rangle(p)$. Consequently, in contrast with the riemannian case, the existence of a smooth Levi-Civita connection ∇ associated with a weak riemannian metric is not immediately guaranteed. If, however, such a connection can be found it is necessarily unique and has all the usual properties. For example, it defines parallel translation τ_t, curvature tensor R, geodesics and a smooth exponential map.

For any $p \in M$, the exponential map $\exp_p : T_pM \to M$ is a local diffeomorphism in a neighbourhood of the origin in T_pM by the inverse function theorem. The differential $d\exp_p$ can be computed using the Jacobi equation as follows. If $\gamma(t) = \exp_p(tv)$ is the geodesic starting at p in the direction v, then for any $t \geq 0$ and any $w \in T_pM$ we have

$$d\exp_p(tv)(tw) = Y(t), \tag{2.1}$$

where Y is the solution of the following initial value problem for the Jacobi equation along γ

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma} Y + R(Y, \dot\gamma)\dot\gamma = 0 \tag{2.2}$$

with

$$Y(0) = 0, \qquad \nabla_{\dot\gamma} Y(0) = w.$$

Next, let E_1, E_2 and E_3 be Banach spaces. A bounded linear operator $T : E_1 \to E_2$ is called Fredholm if it has closed range and its kernel and co-kernel ($\operatorname{coker} T = E_2/T(E_1)$) are finite dimensional. The index of T is the number

$$\operatorname{ind} T = \dim\ker T - \dim\operatorname{coker} T. \tag{2.3}$$

One proves that if $S : E_2 \to E_3$ is another Fredholm operator, then the composition $S \circ T$ is again Fredholm with $\operatorname{ind}(S \circ T) = \operatorname{ind} S + \operatorname{ind} T$. Furthermore, any linear operator on a Banach space of the form $\mathrm{id} + K$, where K is compact, is Fredholm of index zero.

A smooth map between Banach manifolds $f : M_1 \to M_2$ is called Fredholm if for each $p \in M_1$ the derivative $df(p) : T_pM_1 \to T_{f(p)}M_2$ is a Fredholm operator. If M_1 is connected, then $\operatorname{ind}(df(p))$ is independent of p and one defines the index of f by setting $\operatorname{ind}(f) = \operatorname{ind}(df(p))$ (see [S]).

Our main result in this section is the following:

Theorem 1. *Let M be a connected differentiable manifold modelled on a Hilbert space H. Let $\langle \cdot, \cdot \rangle$ be a weak riemannian metric on M which has a smooth Levi-Civita connection ∇. Suppose that for any $p \in M$ and any $w \in T_pM$ the curvature operator $v \to R_w(v) = R(v, w)w$ is compact. Then the exponential map at p is a nonlinear Fredholm map of index zero.*

Proof. Let $v \in T_pM \simeq H$ and let γ be a geodesic in M starting at p with velocity v. It suffices to show that the linear map

$$d\exp_p(v) : T_pM \to T_qM, \qquad q = \gamma(1)$$

is a bounded Fredholm operator of index zero. Recall from (2.1) and (2.2) that $d\exp_p$ can be determined from the solutions to the Jacobi equation on $\gamma(t) = \exp_p(tv)$. Let $\tau_t : T_pM \to T_{\gamma_t}M$ be the isomorphism of the tangent spaces given by parallel translation along γ determined by ∇. Since τ_t and $\nabla_{\dot\gamma}$ commute, we can rewrite (2.2) as an initial value problem on $T_pM \simeq H$

$$\frac{d^2}{dt^2}u(t) + A(t)u(t) = 0, \tag{2.4}$$

$$u(0) = 0, \qquad \frac{du}{dt}(0) = w,$$

where $Y(t) = \tau_t \circ u(t)$ and $A(t) = \tau_t^{-1} \circ R_{\dot\gamma} \circ \tau_t$. Since a compact linear map is bounded and since the dependence $t \to A(t)$ is smooth, it follows from the standard existence and uniqueness theorem for linear differential equations in Hilbert spaces that the Cauchy problem (2.4) is globally well-posed and the evolution operator $U(t) : H \to H$, assigning to each $w \in H$ the unique solution $U(t)(w) = u(t)$, is bounded in the H norm.

Next, rewrite (2.4) as an integral equation

$$u(t) = tw - \int_0^t \int_0^s A(r)u(r)\,drds = tw - \int_0^t \int_0^s A(r)U(r)(w)\,drds \tag{2.5}$$

and let w_n be a bounded sequence in H. Since a composition of a bounded operator with a compact operator is compact the sequence $y_n(r) = A(r)(U(r)w_n)$ contains, for each r, a converging subsequence in H. Call it $y_{n_k}(r)$. Clearly, $y_{n_k}(r)$ depends continuously on r. Set

$$u_{n_k} = \int_0^1 \int_0^s y_{n_k}(r)\,drds$$

and observe that

$$\|u_{n_k} - u_{n_l}\|_H \le \int_0^1 \int_0^s \|y_{n_k}(r) - y_{n_k}(r)\|_H drds \le \sup_{[0,1]} \|y_{n_k}(r) - y_{n_k}(r)\|_H \to 0$$

as n_k, $n_l \to \infty$. Thus, the sequence u_{n_k} is Cauchy and therefore converges in H. This implies that the operator defined by the integral in (2.5) is also compact and consequently,

$$\tau_1^{-1} \circ d\exp_p(v) = U(1) = \mathrm{id} + K,$$

where $K : H \to H$ is compact. Since τ_t is an isomorphism it follows that $d\exp_p$ is Fredholm of index zero. □

Fredholmness of the exponential map has some simple but pleasant consequences for conjugate points in M (see also [Mis]).

Corollary 2. *With the hypotheses of Theorem 1,*

1. monoconjugate and epiconjugate points in M coincide,

2. for any $p \in M$ the set of points conjugate to p along any geodesic is of the first Baire category.

Proof. The second statement follows at once from Theorem 1 and the Sard-Smale theorem (see [S]). To prove the first statement, let γ be a geodesic in M with $\gamma(0) = p$, $\gamma(1) = q$ and $\dot\gamma(0) = v$. Then $d\exp_p(v)$ is Fredholm of index zero. In particular, it has closed range and so there is an orthogonal splitting

$$T_qM = \operatorname{ran}(d\exp_p(v)) \oplus_H \operatorname{coker}(d\exp_p(v)).$$

Suppose that q is epiconjugate to p along γ. Then $\operatorname{ran}(d\exp_p(v))$ is a proper subspace of T_qM and therefore, $\operatorname{coker}(d\exp_p(v))$ is nonempty. Since $\operatorname{ind}(d\exp_p(v)) = 0$, it follows from (2.3) that $\ker(d\exp_p(v))$ must also be nonempty. Thus, q epiconjugate to p implies q monoconjugate to p. The reverse implication is shown analogously. □

Remark 1. Theorem 1 also enables one to develop a version of Morse theory on M. We will not pursue this in the present article.

3. H^s METRICS ON LOOP GROUPS

We apply the results of the previous section to loop groups. Recall that the set $L^{s_o}(G) = H^{s_o}(S^1, G)$ of Sobolev H^{s_o} maps from the unit circle S^1 into a compact, connected Lie group G has a structure of a Hilbert Lie group for $s_o > 1/2$ (see [FU], also [PS]). Its Lie algebra is the set $L^{s_o}(\mathfrak{g}) = H^{s_o}(S^1, \mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of G. Group multiplication in $L^{s_o}(G)$ and the Lie algebra commutator are defined pointwise.

Loop groups carry a natural family of invariant Sobolev metrics. Pick an inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ given by minus the Killing form on $\mathfrak{g}$ and for any $s \geq 0$ define one on $L^{s_o}(\mathfrak{g})$ by

$$\langle v, w\rangle_{H^s} = \int_{S^1} \langle (1+\Delta)^s v(x), w(x)\rangle_{\mathfrak{g}}\, dx, \tag{3.6}$$

where v and w are in $L^{s_o}(\mathfrak{g})$ and $\Delta = -\frac{d^2}{dx^2}$ is the Laplacian. Using the diffeomorphisms $L_\sigma : L^{s_o}(G) \to L^{s_o}(G)$ (given pointwise by left translations on G, $L_\sigma(\gamma) = \sigma\cdot\gamma$) define the corresponding left-invariant metric on the group by

$$\langle X, Y\rangle_\sigma = \langle L_{\sigma^{-1}*\sigma}(X), L_{\sigma^{-1}*\sigma}(Y)\rangle_{H^s}, \tag{3.7}$$

where X and Y are in $T_\sigma L^{s_o}(G)$ and $\sigma \in L^{s_o}(G)$.

Note that for $s < s_o$ the expression in (3.7) is only a weak riemannian metric on $L^{s_o}(G)$. Nevertheless, one readily determines that it has a Levi-Civita connection given explicitly by the following standard formula (see [A] or [F]). For any v, $w \in L^{s_o}(\mathfrak{g})$ let $X(\sigma) = L_{\sigma*e}(w)$ and $Y(\sigma) = L_{\sigma*e}(v)$ be two left-invariant vector fields on $L^{s_o}(G)$ and set, for any $s \geq 0$,

$$\begin{aligned}(\nabla^s_X Y)(\sigma) &= \frac{1}{2}\Big([X,Y] - \operatorname{ad}^{s*}_X Y - \operatorname{ad}^{s*}_Y X\Big)(\sigma) \\ &= \frac{1}{2} L_{\sigma*e}\Big(\operatorname{ad}_w v + (1+\Delta)^{-s}\operatorname{ad}_w(1+\Delta)^s v + (1+\Delta)^{-s}\operatorname{ad}_v(1+\Delta)^s w\Big),\end{aligned} \tag{3.8}$$

where

$$\operatorname{ad}^{s*}_w v = -(1+\Delta)^{-s}\operatorname{ad}_w(1+\Delta)^s v$$

is the operator adjoint to $\operatorname{ad}_w v = [w, v]$ with respect to the inner product (3.6). In particular, it is clear from this formula that $(\nabla^s_X Y)_\sigma$ is left-invariant and smooth as a function of σ.

We now turn to the curvature tensor R^s of (3.8). By left invariance it is sufficient to examine it at the identity in $L^{s_o}(G)$

$$R^s : L^{s_o}(\mathfrak{g}) \times L^{s_o}(\mathfrak{g}) \times L^{s_o}(\mathfrak{g}) \to L^{s_o}(\mathfrak{g}), \tag{3.9}$$

$$R^s(w,v)u = \nabla^s_w \nabla^s_v u - \nabla^s_v \nabla^s_w u - \nabla^s_{[w,v]} u.$$

Proposition 3. *For any $s \geq 0$ the curvature tensor of ∇^s is a trilinear operator bounded in the H^{s_o} norm.*

Proof. From (3.8) observe that if w and v are in $H^{s_o}(S^1, \mathfrak{g})$, then so is $\nabla^s_w v$. In fact, since $s_o > n/2$ and since ad_w acts like a multiplication operator in H^{s_o}, using the Sobolev lemma and the fact that H^{s_o} is a Schauder ring, gives

$$\begin{aligned}\|\nabla^s_w v\|_{H^{s_o}} &\leq \frac{1}{2}\|\mathrm{ad}_w v\|_{H^{s_o}} + \frac{1}{2}\|(1+\Delta)^{-s}\mathrm{ad}_w(1+\Delta)^s v\|_{H^{s_o}} \\ &\quad + \frac{1}{2}\|(1+\Delta)^{-s}\mathrm{ad}_v(1+\Delta)^s w\|_{H^{s_o}} \\ &\leq Const\|w\|_{H^{s_o}}\|v\|_{H^{s_o}}.\end{aligned}$$

The desired bound on the curvature R^s now follows at once from this and the definition (3.9). □

More can be said if the inequality is strict. The following result is just a small extension of the arguments of Freed (see [F] pp. 230-231).

Proposition 4. *For any $v \in L^{s_o}(\mathfrak{g})$ and any $s > 0$ the curvature operator $R^s_v = R^s(\cdot, v)v : L^{s_o}(\mathfrak{g}) \to L^{s_o}(\mathfrak{g})$ is compact.*

Proof. First suppose that v is smooth. Clearly, the operator $w \to \mathrm{ad}_v w = [v, w]$ is pseudodifferential of order zero and therefore, using standard properties of pseudodifferential operators (see [P]), from (3.8) so is $w \to \nabla^s_v w$. On the other hand, rewriting (3.8) as

$$\nabla^s_w v = \frac{1}{2}\Big((1+\Delta)^{-s}[\mathrm{ad}_v, (1+\Delta)^s]w + (1+\Delta)^{-s}\mathrm{ad}_w(1+\Delta)^s v\Big)$$

shows that $w \to \nabla^s_w v$ is of order $\max(-1, -2s)$. From these and (3.9) it follows that the curvature operator $w \to R^s_v(w)$ is pseudodifferential of order $\max(-1, -2s) < 0$ and therefore, compact by the Rellich lemma.

Assume $v \in H^{s_o}(S^1, \mathfrak{g})$. Pick a sequence v_k in $C^\infty(S^1, \mathfrak{g})$ converging to v in the H^{s_o} norm. For any $w \in H^{s_o}(S^1, \mathfrak{g})$, using Proposition 3 and linearity

$$\begin{aligned}\|R^s(w,v)v - R^s(w,v_k)v_k\|_{H^{s_o}} &\leq \|R^s(w, v - v_k)v\|_{H^{s_o}} + \|R^s(w, v_k)(v - v_k)\|_{H^{s_o}} \\ &\leq Const\|w\|_{H^{s_o}}\|v\|_{H^{s_o}}\|v - v_k\|_{H^{s_o}}\end{aligned}$$

for sufficiently large k. Thus, $R^s_{v_k} \to R^s_v$ in the operator norm. Since for each k, $R^s_{v_k}$ is compact and since compact operators form a closed subset in the space of all bounded linear operators, the limit R^s_v is also compact. □

Combining Proposition 4 with Theorem 1 we obtain

Theorem 5. *For any $s > 0$, the exponential map of the H^s metric (3.7) on $L^{s_o}(G)$ is a nonlinear Fredholm map of index zero.*

Remark 2. All the results, in particular Theorem 5, given in this section remain valid (with the same proofs) for a more general class of Hilbert Lie groups, namely, $H^{s_o}(M,G)$, where M is an arbitrary compact, n-dimensional manifold and $s_o > n/2$.

They also hold if we replace $L^{s_o}(G)$ with its subgroup of based loops $\Omega^{s_o}(G)$, in which case (3.6) would also be replaced with

$$\langle v, w\rangle_{H^s} = \int_{S^1} \langle \Delta^s v(x), w(x)\rangle_{\mathfrak{g}}\, dx.$$

4. L^2 metric on loop groups

Our purpose here is to show that, in general, one cannot relax the assumptions of Theorem 5 of the preceding section to allow the L^2 metric (s=0).

Consider the group $G = SU(2)$ of 2×2 unitary matrices of determinant 1. The corresponding Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ is the linear space of 2×2 trace free, skew-hermitian matrices. Its standard basis

$$e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

satisfies the commutation relations

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = 2e_1, \quad [e_1, e_3] = -2e_2.$$

The L^2 inner product on the Lie algebra $L^{s_o}(\mathfrak{su}(2))$ is

(4.10)
$$\langle v, w\rangle_{L^2} = \int_{S^1} -\frac{1}{2}\mathrm{tr}\,(v(x)w(x))\, dx = \int_{S^1} \sum_{i=1}^{3} v^i(x) w^i(x)\, dx,$$

where $v(x) = \sum_i v^i(x) e_i$ and $w(x) = \sum_i w^i(x) e_i$. It extends by (3.7) to a left-invariant L^2 metric on the group $L^{s_o}(SU(2))$.

Observe that for any $v \in L^{s_o}(\mathfrak{su}(2))$ the operator $\mathrm{ad}_v = [v, \cdot]$ is skew adjoint

$$\mathrm{ad}_v^{o*} = -\mathrm{ad}_v$$

with respect to (4.10). This implies that the L^2 metric is in fact bi-invariant and so $L^{s_o}(SU(2))$ is a symmetric space. In this case from (3.8) we obtain a simple expression for the Levi-Civita connection

$$\nabla^o_w v = \frac{1}{2}[w, v] \tag{4.11}$$

and therefore, from (3.9), for the curvature tensor

$$R^o(w, v)u = -\frac{1}{4}[[w, v], u]. \tag{4.12}$$

Consequently, any one-parameter subgroup γ of $L^{s_o}(SU(2))$ is necessarily a geodesic of the L^2 metric; furthermore, if X, Y and Z are parallel vector fields along γ, then $R^o(X, Y)Z$ is also parallel along γ (useful references are [Mil1] and [Mil2]).

In the Lie algebra $L(\mathfrak{su}(2))$ pick the vector

$$v(x) = \frac{1}{\sqrt{2\pi}} e_2$$

and let γ be the one-parameter subgroup of $L^{s_o}(SU(2))$ generated by v. Consider the curvature operator

$$R^o_v = R^o(\cdot, v)v : L^{s_o}(\mathfrak{su}(2)) \to L^{s_o}(\mathfrak{su}(2))$$

and observe that for each $k \in Z$

$$w_k(x) = \frac{1}{\sqrt{\pi}} \sin kx \cdot e_1$$

is an eigenvector of R^o_v corresponding to the eigenvalue $\lambda = \frac{1}{2\pi}$.

Let $f(t)$ be a smooth function. Extend each w_k to a vector field along γ by parallel translation and define

$$t \to Y_k(t)(x) = f(t)\tau_t(w_k(x)).$$

Compute

$$\begin{aligned}\nabla^o_{\dot\gamma}\nabla^o_{\dot\gamma} Y_k(t) + R^o(Y_k(t), \dot\gamma)\dot\gamma &= f''(t)\tau_t(w_k) + f(t)\tau_t(R^o(w_k, v)v) \\ &= \left(f''(t) + \frac{1}{2\pi} f(t)\right)\tau_t(w_k).\end{aligned}$$

Therefore, if $f(t) = \sin(\frac{t}{\sqrt{2\pi}})$, each $Y_k(t)$ is a Jacobi field along γ vanishing at $t = 0$ and $t = \pi\sqrt{2\pi}$. Since $Y_k(t)$ are clearly linearly independent, the kernel of $d\exp_e(\pi\sqrt{2\pi})v$ is infinite dimensional and consequently, $d\exp_e$ cannot be Fredholm.

Acknowledgements

I thank Professors David Ebin and Jerrold Marsden for many useful conversations on the subject.

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