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FATOU'S IDENTITY AND LEBESGUE'S CONVERGENCE THEOREM

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Abstract. The classical Fatou lemma for bounded sequences of nonnegative integrable functions is represented as an equality. A similar result is stated for measure convergent sequences. Neither result requires a uniform integrability assumption. For the latter a converse is proven. Two extensions of Lebesgue's convergence theorem are presented.

1. Introduction

A finite dimensional Fatou-type result for point-valued functions was presented by Schmeidler [16] in 1970 and has received much attention ever since. It was extended successively by Hildenbrand and Mertens (1971), Cesari and Suryanarayana (1978), Artstein (1979), Balder (1984), Page (1991), and Klei and Miyara (1991) in different directions. In this note we present two Fatou-type equalities (Proposition 3 and Theorem 6). Theorem 6 (more precisely, its finite dimensional extension) subsumes all aforementioned results and seems to be the sharpest possible in its kind. Our version of Fatou's lemma illustrates the key role played by the modulus of uniform integrability of the sequence under consideration. It sheds a new light on those results which are given in terms of polar cones: Olech (1987) and Balder and Hess (1995). Finally, we prove two extensions of Lebesgue's convergence theorem (Theorem 8 and Theorem 10).

2. Preliminaries

Throughout this note, $(\Omega, \mathcal{A}, P)$ will be a fixed probability space. We consider only *real* scalars and $L^1(P)$ denotes the Banach space of all classes of *real*-valued Bochner-integrable functions on $(\Omega, \mathcal{A}, P)$; let $L^1_+(P)$ be the nonnegative functions in $L^1(P)$. If (x_n) is a sequence of real numbers, then the *Kuratowski limes superior* $\mathrm{Ls}(x_n)$ is defined as the subset of $\mathbf{R}$ that consists of its real cluster points. For a subset $A \subseteq \mathbf{R}$, $\mathrm{co}(A)$ denotes its convex hull and χ_A its characteristic function. To each sequence (f_n) in $L^1(P)$ we assign the multifunction $\mathrm{Ls}(f_n)$ and the integral $\int \mathrm{Ls}(f_n)dP$ which is understood in the sense of *Aumann* [2]. The *modulus of uniform integrability* $\eta(H)$ of a bounded subset $H \subseteq L^1(P)$ is defined as in [15]: for

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$\varepsilon > 0$, put

$$\eta(H,\varepsilon) = \sup\left\{\int_A |h| dP \colon h \in H, A \in \mathcal{A}, P(A) \leq \varepsilon\right\},$$
$$\eta(H) = \lim_{\varepsilon \to 0^+} \eta(H,\varepsilon).$$

Thus H is uniformly integrable if and only if $\eta(H) = 0$.

3. Results

We start with a Fatou-type lemma proved in [7]. See also [9, Lemma 2.2].

Lemma 1. *Let $f = (f_n)$ be a bounded sequence in $L^1_+(P)$ converging in measure to an element $f_\infty \in L^1_+(P)$. Then the following assertions are equivalent:*

i) *the sequence $(\int f_n dP)$ converges in the real line*;
ii) *$\eta(f') = \eta(f)$ for each subsequence f' of f*;
iii) *$\lim_{n\to+\infty} \int f_n dP = \eta(f') + \int f_\infty dP$ for each subsequence f' of f.*

Lemma 2. *Let $f = (f_n)$ be a bounded measure convergent sequence in $L^1_+(P)$. Denote by $f' = (f'_n)$ a subsequence of f. Then the following assertions are equivalent:*

i) *$\eta(f') = \min\{\eta(\hat{f}) \colon \hat{f}$ subsequence of $f\}$*;
ii) *$\lim_{n\to+\infty} \int f'_n dP = \underline{\lim}_{n\to+\infty} \int f_n dP$.*

Proof. Suppose that i) is true. Let $f_\infty \in L^1_+(P)$ be the limit of the sequence (f_n) and f' a subsequence of f satisfying i). Given any subsequence f'' of f', we apply Lemma 1 to choose a further subsequence $f''' = (f'''_n)$ of f'' such that $\lim_{n\to+\infty} \int f'''_n dP = \eta(f''') + \int f_\infty dP$. We deduce from i) that $\eta(f''') = \eta(f')$. Hence the equality

$$\lim_{n\to+\infty} \int f'_n dP = \eta(f') + \int f_\infty dP.$$

Extract a subsequence $\tilde{f} = (\tilde{f}_n)$ of f satisfying $\lim_{n\to+\infty} \int \tilde{f}_n dP = \underline{\lim}_{n\to+\infty} \int f_n dP$. As seen above, we have

$$\lim_{n\to+\infty} \int \tilde{f}_n dP = \eta(\tilde{f}) + \int f_\infty dP \geq \eta(f') + \int f_\infty dP = \lim_{n\to+\infty} \int f'_n dP.$$

Thus assertion ii) is proved.

Conversely, let $f' = (f'_n)$ be a subsequence of f satisfying ii). Proceeding by contradiction, we suppose that there exists a subsequence $\hat{f} = (\hat{f}_n)$ of f with $\eta(\hat{f}) < \eta(f')$. Choose a subsequence $\hat{\hat{f}} = (\hat{\hat{f}}_n)$ of $\hat{f}$ such that $(\int \hat{\hat{f}}_n dP)$ converges. According to Lemma 1 we have the following relations:

$$\begin{aligned}\lim_{n\to+\infty} \int \hat{\hat{f}}_n dP &= \eta(\hat{\hat{f}}) + \int f_\infty dP < \eta(f') + \int f_\infty dP \\ &= \lim_{n\to+\infty} \int f'_n dP = \underline{\lim}_{n\to+\infty} \int f_n dP.\end{aligned}$$

Clearly, $\lim_{n\to+\infty} \int \hat{\hat{f}}_n dP < \underline{\lim}_{n\to+\infty} \int f_n dP$ is a contradiction. The proof is complete.

From these considerations we may already draw a first Fatou-type identity.

Proposition 3. *Let $f = (f_n)$ be a bounded sequence in $L^1_+(P)$ converging in measure to f_∞. Then the following equality holds:*

$$\underline{\lim}_{n\to+\infty} \int f_n dP = \min\{\eta(\hat{f})\colon \hat{f} \text{ subsequence of } f\} + \int f_\infty dP.$$

Proof. We simply apply Lemma 1 and Lemma 2 to a subsequence (f'_n) of (f_n) for which $\lim_{n\to+\infty} \int f'_n dP = \underline{\lim}_{n\to+\infty} \int f_n dP$.

Remark 4. The preceding result extends immediately to the case where the condition "(f_n) is a bounded sequence in $L^1_+(P)$" is replaced by "(f_n) is a bounded sequence in $L^1(P)$ such that (f_n^-) is uniformly integrable".

Though the converse of Proposition 3 is not true, we have the following result.

Theorem 5. *Let $f = (f_n)$ be a bounded sequence in $L^1_+(P)$. Then the following assertions are equivalent:*

i) $\underline{\lim}_{n\to+\infty} \int f_n dP = \min\{\eta(\hat{f})\colon \hat{f}$ *subsequence of* $f\} + \int \underline{\lim}_{n\to+\infty} f_n dP$;
ii) *there is a subsequence* $\tilde{f} = (\tilde{f}_n)$ *of* (f_n) *which converges in measure to* $\underline{\lim}_{n\to+\infty} f_n$ *and satisfies* $\eta(\tilde{f}) = \min\{\eta(\hat{f})\colon \hat{f}$ *subsequence of* $f\}$;
iii) *each subsequence* $f' = (f'_n)$ *of* (f_n) *for which* $\lim_{n\to+\infty} \int f'_n dP = \underline{\lim}_{n\to+\infty} \int f_n dP$ *converges in measure to* $\underline{\lim}_{n\to+\infty} f_n$ *and satisfies the following equalities*:

$$\eta(f') = \min\{\eta(\hat{f})\colon \hat{f} \text{ subsequence of } f\}, \qquad \underline{\lim}_{n\to+\infty} f'_n = \underline{\lim}_{n\to+\infty} f_n \quad P\text{-a.e.}$$

Proof. To prove that i) implies iii), choose a subsequence $f' = (f'_n)$ of (f_n) for which $\lim_{n\to+\infty} \int f'_n dP = \underline{\lim}_{n\to+\infty} \int f_n dP$. By hypothesis and the extended Fatou's lemma [10, Corollaire 4], the following relations hold:

$$\begin{aligned}\min\{\eta(\hat{f})\colon \hat{f} \text{ subsequence of } f\} &+ \int \underline{\lim}_{n\to+\infty} f_n dP \\ = \lim_{n\to+\infty} \int f'_n dP &\geq \eta(f') + \int \underline{\lim}_{n\to+\infty} f'_n dP \\ &\geq \eta(f') + \int \underline{\lim}_{n\to+\infty} f_n dP.\end{aligned}$$

As the preceding terms are obviously equal to each other, we obtain the desired equalities in assertion iii). Let us state that

$$\lim_{n\to+\infty} \int f'_n dP = \eta(f') + \int \underline{\lim}_{n\to+\infty} f'_n dP,$$
$$\eta(f'') = \eta(f') \quad \text{for each subsequence } f'' \text{ of } f'.$$

Now, according to Theorem 3 of [8], this is equivalent to saying that (f'_n) converges in measure to $\underline{\lim}_{n\to+\infty} f'_n$. So assertion iii) is proved.

Evidently, ii) follows from iii). Let $(\tilde{f}_n)$ be as described in assertion ii), which we suppose to be true. During the following reasoning we may assume without loss of generality that the sequence $(\int \tilde{f}_n dP)$ converges in the real line. On the one hand, it follows from Lemma 1 that

$$\begin{aligned}\underline{\lim}_{n\to+\infty} \int f_n dP \leq \lim_{n\to+\infty} \int \tilde{f}_n dP &= \min\{\eta(\hat{f})\colon \hat{f} \text{ subsequence of } f\} \\ &+ \int \underline{\lim}_{n\to+\infty} f_n dP.\end{aligned}$$

On the other hand, the extended Fatou's lemma implies

$$\underline{\lim}_{n\to+\infty}\int f_n dP \geq \min\{\eta(\hat{f})\colon \hat{f} \text{ subsequence of } f\} + \int \underline{\lim}_{n\to+\infty} f_n dP.$$

Thus assertion i) has been proven and the proof is complete.

Theorem 6. *Let $f = (f_n)$ be a bounded sequence in $L^1_+(P)$ for which $(\int f_n dP)$ converges in the real line. Then there exist a function $g_\infty \in L^1_+(P)$ and a subsequence $f' = (f'_n)$ of (f_n) such that for every subsequence $f'' = (f''_n)$ of (f'_n) the following statements hold:*

i) $\lim_{n\to+\infty}\int f_n dP = \eta(f) + \int \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^n f'_k dP$;
ii) $\eta(f) = \eta(f'') = \eta(m(f''))$ *where* $m(f'')$ *denotes the sequence* $(\frac{1}{n}\sum_{k=0}^n f''_k)_n$;
iii) $\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^n f'_k(\omega) \in \operatorname{co}\operatorname{Ls}(f'_n(\omega))$ P*-a.e.*;
iv) $g_\infty(\omega) \in \operatorname{Ls}(f'_n(\omega))$ P*-a.e. and* $\int g_\infty dP = \int \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^n f'_k dP$.

Remarks 7. 1) The proof of Theorem 6 shows that if (f_n) converges in measure to f_∞, then one has $f_\infty(\omega) = \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^n f'_k(\omega)$ P-a.e. and part i) is in accordance with Lemma 1. Observe that this argument needs no appeal to assertion iv).

2) The Fatou-type results frequently found in the literature assert that $\lim_{n\to+\infty}\int f_n dP \geq \int g_\infty dP$, where g_∞ denotes an integrable (non-constructive) selector of $\operatorname{Ls}(f_n)$. Additional uniform integrability assumptions are introduced to obtain equality. Theorem 6 tells us why.

3) Olech [12] and Balder and Hess [4] formulated a Fatou-type inclusion in terms of polar cones. The need for the introduction of the so-called *correction term* [4] is illustrated in part i) by the presence of the term $\eta(f)$. If $\eta(f) > 0$, and under the assumptions of Theorem 6, then this *correction term* is as big as the cone $\mathbf{R}_+$. In this case, the single-value version of Theorem 3.2 in [4] only asserts that $\lim_{n\to+\infty}\int f_n dP \in \mathbf{R}_+ + \int \operatorname{Ls}(f_n) dP$.

4) Extracting a suitable subsequence of (f_n) yields an expression for $\underline{\lim}_{n\to+\infty}\int f_n dP$ similar to the one that represents $\lim_{n\to+\infty}\int f_n dP$.

5) It suffices to choose suitable notations to extend the preceding result to the following framework: (f_n) is a bounded sequence in $L^1(\mathbf{R})$ for which $(\int f_n dP)$ converges and such that (f_n^-) is uniformly integrable. Another extension concerns integrable functions with values in $\mathbf{R}^m$ or $\mathbf{R}^m_+$.

6) Only the proof of assertion iv) requires the Lyapunov theorem, a result which fails in infinite dimensions.

Proof of Theorem 6. By virtue of Rosenthal's subsequence splitting lemma ([15] and [9, Lemma 2.1]) we choose a subsequence $f' = (f'_n)$ of (f_n) and a sequence (A_n) of pairwise disjoint measurable sets such that

$$(\chi_{\Omega\setminus A_n}\cdot f'_n) \text{ converges weakly and } \lim_{n\to+\infty}\int_{A_n} f'_n dP = \eta(f).$$

Let f_∞ be the weak limit of the sequence $(\chi_{\Omega\setminus A_n}\cdot f'_n)$. We know, e.g., from [10, Proposition 1] that $f_\infty(\omega) \in \operatorname{co}\operatorname{Ls}(f'_n(\omega))$ P-a.e. We apply Komlós' theorem [11] to the sequence (f'_n). There exist $f_* \in L^1(\mathbf{R})$ and a subsequence, still denoted by (f'_n), such that for each further subsequence $f'' = (f''_n)$ of (f'_n) one has

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^n f''_k(\omega) = f_*(\omega) \quad P\text{-a.e.}$$

It follows that $f_*(\omega) = f_\infty(\omega)$ P-a.e. Applying Lemma 1 to the sequence $m(f'')$, we obtain

$$\lim_{n\to+\infty} \int f_n dP = \eta(m(f'')) + \int f_\infty dP.$$

On account of the subsequence splitting lemma we have

$$\lim_{n\to+\infty} \int f_n dP = \eta(f) + \int f_\infty dP.$$

Hence the relations

$$\eta(f) \geq \eta(f'') \geq \eta(m(f'')) = \eta(f).$$

It only remains to prove part iv). We have seen that $f_\infty \in \int \operatorname{co}\operatorname{Ls}(f'_n)dP$. If $(\Omega, \mathcal{A}, P)$ is atomless, then the Lyapunov-Richter theorem [14] asserts that

$$\int \operatorname{co}\operatorname{Ls}(f'_n)dP = \int \operatorname{Ls}(f'_n)dP.$$

The general case is studied by splitting $(\Omega, \mathcal{A}, P)$ into an atomless part and a purely atomic part with at most countably many atoms.

We have the following strengthening of Lebesgue's theorem.

Theorem 8. *Let (f_n) be a bounded and uniformly integrable sequence in $L^1_+(P)$ such that $(f_n(\omega))$ has at most one real cluster point P-a.e. Then (f_n) converges in norm to $\underline{\lim}_{\,n\to+\infty} f_n$.*

Proof. We start showing that the sequence $(\int f_n dP)$ converges. Let f' be any subsequence of f. By Théorème 3 of [10] there exists a further subsequence $f'' = (f''_n)$ of f' such that

$$\lim_{n\to+\infty} \int f''_n dP \in \int \operatorname{Ls}(f''_n)dP.$$

For P-a.e. $\omega \in \Omega$ one has

$$\underline{\lim}_{\,n\to+\infty} f''_n(\omega) \in \operatorname{Ls}(f''_n(\omega)) \subseteq \operatorname{Ls}(f_n(\omega)) = \{\underline{\lim}_{\,n\to+\infty} f_n(\omega)\}.$$

It follows that $\lim_{n\to+\infty} \int f''_n dP = \int \underline{\lim}_{\,n\to+\infty} f_n dP$ and also $\lim_{n\to+\infty} \int f_n dP = \int \underline{\lim}_{\,n\to+\infty} f_n dP$. The latter equality implies the desired result [10, Théorème 5].

Remark 9. One easily defines a bounded uniformly integrable sequence of functions $f_n \colon [0,1] \to \mathbf{R}_+$ such that, for every $\omega \in [0,1]$, $\operatorname{Ls}(f_n(\omega)) = \{0\}$ and $\overline{\lim}_{\,n\to+\infty} f_n(\omega) = +\infty$. Consequently, $(f_n(\omega))$ does not converge for any ω.

We conclude with an improvement of Theorem 8.

Theorem 10. *Let $f = (f_n)$ be a bounded sequence in $L^1_+(P)$ such that $(f_n(\omega))$ has at most one real cluster point P-a.e. Then the following statements hold:*

i) *(f_n) converges in measure to $\underline{\lim}_{\,n\to+\infty} f_n$;*

ii) $\underline{\lim}_{\,n\to+\infty} \int f_n dP = \min\{\eta(\hat{f}) \colon \hat{f}$ *subsequence of* $f\} + \int \underline{\lim}_{\,n\to+\infty} f_n dP$.

If in addition $(\int f_n dP)$ converges, then $\lim_{n\to+\infty} \int f_n dP = \eta(f) + \int \underline{\lim}_{\,n\to+\infty} f_n dP$.

Proof. To prove i), it is sufficient to extract from each subsequence of f a further subsequence which converges in measure to $\underline{\lim}_{\,n\to+\infty} f_n$. Without loss of generality we can restrict ourselves to demonstrate that the sequence f itself has a subsequence that converges in measure to $\underline{\lim}_{\,n\to+\infty} f_n$. On account of Rosenthal's subsequence splitting lemma [15] there exist a subsequence $f' = (f'_n)$ of f and a sequence (A_n) of pairwise disjoint measurable sets such that $(\chi_{\Omega\setminus A_n} \cdot f'_n)$ converges weakly. Clearly, for P-a.e. $\omega \in \Omega$,

$$\underline{\lim}_{n\to+\infty} f'_n(\omega) \in \mathrm{Ls}(f'_n(\omega)) \subseteq \mathrm{Ls}(f_n(\omega)) = \{\underline{\lim}_{n\to+\infty} f_n(\omega)\}\,.$$

Consequently, $\mathrm{Ls}(f'_n(\omega)) = \{\underline{\lim}_{n\to+\infty} f_n(\omega)\} = \{\underline{\lim}_{n\to+\infty} f'_n(\omega)\}$ P-a.e. Note that $\mathrm{Ls}(f'_n(\omega)) = \mathrm{Ls}(\chi_{\Omega\setminus A_n} \cdot f'_n(\omega))$ for each $\omega \in \Omega$. Therefore Theorem 8 applies to the sequence $(\chi_{\Omega\setminus A_n} \cdot f'_n)$ and asserts that it converges in norm to $\underline{\lim}_{n\to+\infty} f_n$. We finish the proof of part i) by observing that $(\chi_{A_n} \cdot f'_n)$ converges in measure to zero. An appeal to Lemma 1 and Proposition 3 completes the proof.

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