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CUT-POINT SPACES

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Abstract. The notion of a cut-point space is introduced as a connected topological space without any non-cut point. It is shown that a cut-point space is infinite. The non-cut point existence theorem is proved for general (not necessarily T_1) topological spaces to show that a cut-point space is non-compact. Also, the class of irreducible cut-point spaces is studied and it is shown that this class (up to homeomorphism) has exactly one member: the Khalimsky line.

1. Introduction

The real line $\mathbf{R}$ is a source of intuition in topology. Many other familiar topological spaces can be obtained from $\mathbf{R}$ by topological constructions. It has the following properties:
(a) it is connected but the removal of any one of its points leaves it disconnected;
(b) it is metrizable;
(c) its topology can be generated by a linear ordering.
Conversely, it can be proved that every topological space with the above properties is homeomorphic to $\mathbf{R}$. Conditions (b) and (c) are too strong. They impose structures on the topological space, so this characterization of $\mathbf{R}$ seems somehow extrinsic.

In this paper we study the topological spaces that satisfy condition (a), and call them cut-point spaces. In section 2, a cut-point space is defined again formally and some examples are given. In section 3, it is shown that every cut-point space has an infinite number of closed points. Also, it is proved that every cut-point space is non-compact. To prove the latter, we need the most general form of the non-cut point existence theorem. The special case of this theorem for metric topological spaces is proved in [4]. A proof of the theorem for T_1 topological spaces can be found in [1] (see also [5]). In Section 4, an irreducible cut-point space is defined naturally as a cut-point space whose proper subsets are not cut-point spaces. It is shown that an irreducible cut-point space is necessarily homeomorphic to the Khalimsky line (see Example 2.5 for the definition of the Khalimsky line). This result may also be viewed as a straightforward characterization of the Khalimsky line. Objects in n-dimensional digital images have sometimes been regarded as subspaces of the product of n copies of the Khalimsky line [2], [3].

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Remark. Let X be a topological space and let $Y \subseteq X$. Everywhere in this paper, the topology of Y is the subspace topology. A point $x \in X$ is said to be closed (resp. open) if $\{x\}$ is a closed (resp. open) subset of X.

2. Definitions and examples

2.1. Definition. Let X be a nonempty connected topological space. A point x in X is said to be a *cut point* of X if $X \backslash \{x\}$ is a disconnected subset of X. A nonempty connected topological space X is said to be a *cut-point space* if every x in X is a cut point of X.

In the following three examples, $\mathbf{R}^2$ is the Euclidean plane with the standard topology.

2.2. Example. The union of n straight lines in $\mathbf{R}^2$ is a cut-point space if and only if either all of them are concurrent or exactly $n-1$ of them are parallel.

2.3. Example. Let $X_1 = \{(x, y) \in \mathbf{R}^2 : x \leq 0 \text{ and } |y| = 1\}$ and let $X_2 = \{(x, y) \in \mathbf{R}^2 : x > 0 \text{ and } y = \sin \frac{1}{x}\}$. Define $X = X_1 \cup X_2$. Then X is a cut-point space. For each $x \in X$, $X \backslash \{x\}$ has exactly two components.

A "connected ordered topological space" (COTS) is a connected topological space X with the following property: if Y is a three-point subset of X, there is a y in Y such that Y meets two connected components of $X \backslash \{y\}$ (see [2]). Put $Y = \{(0, -1), (1, \sin 1), (0, 1)\}$ in Example 2.3 to see that X is not a COTS.

2.4. Example. Let $X_0 = \{(x, 0) \in \mathbf{R}^2 : x \leq 0\} \cup \{(x, 1) \in \mathbf{R}^2 : x > 0\}$ and let for each positive integer n, $Y_n = \{(\frac{1}{n}, y) \in \mathbf{R}^2 : 0 < y \leq 1\}$. Define $X = X_0 \cup (\bigcup_{n=1}^{\infty} Y_n)$. Then X is a cut-point space.

A connected topological space is said to have the "connected intersection property" if the intersection of every two connected subsets of it is connected. In Example 2.4, let $X_1 = X_0 \cup (\bigcup_{n=1}^{\infty} Y_{2n-1})$ and $X_2 = X_0 \cup (\bigcup_{n=1}^{\infty} Y_{2n})$. Since $X_1 \cap X_2 = X_0$ is not connected, X does not possess the connected intersection property. Example 2.4 is a slightly modified version of an example in [6].

2.5. Example (The Khalimsky line)**.** Let $\mathbf{Z}$ be the set of integers and let

$$\mathcal{B} = \{\{2i-1, 2i, 2i+1\} : i \in \mathbf{Z}\} \cup \{\{2i+1\} : i \in \mathbf{Z}\}.$$

Then $\mathcal{B}$ is a base for a topology on $\mathbf{Z}$. The set $\mathbf{Z}$ with this topology is a cut-point space and is called the *Khalimsky line*. Each point in $\mathbf{Z}$ has a smallest open neighborhood and the base $\mathcal{B}$ is the collection of all such neighborhoods. It can be easily seen that the Khalimsky line is irreducible in the sense that no proper subset of it is a cut-point space.

3. Topological properties of cut-point spaces

Theorem 3.2 is the key theorem of this section. The main theorem of this section is Theorem 3.9 which implies the non-compactness of cut-point spaces. Notation 3.1 is adopted from [5].

3.1. *Notation.* Let Y be a topological space. We write $Y = A|B$ to mean A and B are two nonempty subsets of Y such that $Y = A \cup B$ and $A \cap \bar{B} = \bar{A} \cap B = \varnothing$.

3.2. Theorem. *Let X be a connected topological space, and let x be a cut point of X such that $X\backslash\{x\} = A|B$. Then $\{x\}$ is open or closed. If $\{x\}$ is open, then A and B are closed; if $\{x\}$ is closed, then A and B are open.*

Proof. Since A is both open and closed in $X\backslash\{x\}$, there is an open subset V of X such that $A = V \cap (X\backslash\{x\}) = V\backslash\{x\}$, and there is a closed subset F of X such that $A = F \cap (X\backslash\{x\}) = F\backslash\{x\}$. Thus $A = V\backslash\{x\} = F\backslash\{x\}$. Since the assumption $V = F$ contradicts the connectedness of X, we have $\{x\} = V\backslash F$ or $\{x\} = F\backslash V$. If $\{x\} = V\backslash F$, then $\{x\}$ is open and $A = F$ is closed. If $\{x\} = F\backslash V$, then $\{x\}$ is closed and $A = V$ is open. □

3.3. Corollary. *Let X be a connected topological space, and let Y be the subset of all cut points of X. Then the following statements are obviously true.*

(a) *Every nonempty connected subset of Y that is not a singleton, contains at least one closed point.*
(b) *If $x \in Y$ is open, then every limit point of $\{x\}$ in Y is a closed point.*

3.4. Lemma. *Let X be a connected topological space, and let x be a cut point of it. If $X\backslash\{x\} = A|B$, then $A \cup \{x\}$ is connected.*

Proof. If $A \cup \{x\}$ is not connected, then there are subsets C and D of X such that $A \cup \{x\} = C|D$. Without loss of generality, we may assume that $x \in C$. Then $D \subseteq A$. Since $\overline{(B \cup C)} \cap D = (\bar{B} \cap D) \cup (\bar{C} \cap D) = \bar{B} \cap D \subseteq \bar{B} \cap A = \varnothing$, $\overline{(B \cup C)} \cap D = \varnothing$. Since $(B \cup C) \cap \bar{D} = (B \cap \bar{D}) \cup (C \cap \bar{D}) = (B \cap \bar{D}) \subseteq B \cap \bar{A} = \varnothing$, $(B \cup C) \cap \bar{D} = \varnothing$. Therefore $X = (B \cup C)|D$. This contradicts the connectedness of X. □

3.5. Lemma. *Let X be a connected topological space and let x be a cut point of it. If $X\backslash\{x\} = A|B$ and if every point of A is a cut point of X, then A contains at least one closed point.*

Proof. Suppose that A consists exclusively of open points. Since, by Lemma 3.4, $A \cup \{x\}$ is connected, $\{x\}$ is closed and hence (by Theorem 3.2) $A \cup \{x\}$ is closed too. Thus, for every $y \in A$, $\overline{\{y\}} \subseteq A \cup \{x\}$, and therefore, by Corollary 3.3 (b), x is the only possible limit point of $\{y\}$. As $\{y\}$ has a limit point (since $\{y\}$ is open and X is connected), x is a limit point of $\{y\}$. This implies that $\{x, y\}$ is connected for any $y \in A$. Let $y_0 \in A$. Since $B \cup \{x\}$ is connected by Lemma 3.4,

$$X\backslash\{y_0\} = \bigcup_{y \in A, y \neq y_0} \{x, y\} \cup (B \cup \{x\})$$

is connected too. This contradicts the fact that y_0 is a cut point of X. □

3.6. Lemma. *Let X be a connected topological space, and let x and y be two cut points of it such that $X\backslash\{x\} = A|B$ and $X\backslash\{y\} = C|D$. If $x \in C$ and $y \in A$, then $D \subseteq A$ and $B \subseteq C$.*

Proof. Since $D \cup \{y\}$ is connected by Lemma 3.4, and since $D \cup \{y\} \subseteq X\backslash\{x\}$, we have $D \cup \{y\} \subseteq A$ or $D \cup \{y\} \subseteq B$. Since $y \in A$, the second inclusion is not true. Hence $D \subseteq A$. A similar argument shows that $B \subseteq C$. □

In the next theorem, we show that a finite topological space cannot be a cut-point space.

3.7. Theorem. *Let X be a cut-point space. Then the set of closed points of X is infinite.*

Proof. By mathematical induction, we construct a sequence $x_1, x_2, \cdots$ of distinct closed points in X. Define $C_0 = X$. By Corollary 3.3 (a), there exists a closed point x_1 in C_0. Since x_1 is a cut point of X, there are open subsets C_1 and D_1 of X such that $X \backslash \{x_1\} = C_1 | D_1$. Now, suppose that the distinct closed points $x_1, x_2, \cdots, x_n$ in X and the open subsets $C_1, \cdots, C_n, D_1, \cdots, D_n$ of X are chosen such that $X \backslash \{x_i\} = C_i | D_i$, $x_i \in C_{i-1}$ and $C_{i-1} \supseteq C_i$ for each $i, 1 \leq i \leq n$. According to Lemma 3.5, there is a closed point $x_{n+1} \in C_n$. There are open subsets C_{n+1} and D_{n+1} of X such that $X \backslash \{x_{n+1}\} = C_{n+1} | D_{n+1}$. By interchanging C_{n+1} and D_{n+1}, if necessary, we may asuume that $x_n \in D_{n+1}$. Thus, by Lemma 3.6, $C_n \supseteq C_{n+1}$. Since $x_i \notin C_i$, $x_i \notin C_n$ for any $i, 1 \leq i \leq n$. The fact that $x_{n+1} \in C_n$ implies that x_{n+1} is different from $x_1, \cdots, x_n$. □

3.8. Corollary. *Let X be a cut-point space. Then $|X| = \infty$.*

Of course, Theorem 3.7 is a generalization of Corollary 3.8. Using the Hausdorff Maximal Principle, we prove another generalization of Corollary 3.8 in the following theorem.

3.9. Theorem. *Let X be a compact connected topological space with more than one point. Then X has at least two non-cut points.*

Proof. Suppose that X has at most one non-cut point. Let x_0 be a cut point of X and let $X \backslash \{x_0\} = A_0 | B_0$. Since X has at most one non-cut point, either A_0 or B_0 (without loss of generality assume A_0) exclusively consists of cut points. By Lemma 3.5, A_0 contains some closed cut point of X, say x. Let $X \backslash \{x\} = A | B$ and without loss of generality assume that $x_0 \in B$. Then by Lemma 3.6, $A \subseteq A_0$. Define $\mathcal{S} = \{U : U$ is an open subset of $X, U \supseteq B, \bar{U} \backslash U$ is a singleton, and $\bar{U} \neq X\}$. Since B is open and $\bar{B} = B \cup \{x\}$, $B \in \mathcal{S}$. For each $U_\alpha \in \mathcal{S}$ and $U_\beta \in \mathcal{S}$, write $U_\alpha \leq U_\beta$ if $U_\alpha = U_\beta$, or if $\bar{U}_\alpha \subseteq U_\beta$. $(\mathcal{S}, \leq)$ is clearly a partially ordered set, and by the Hausdorff Maximal Principle there is a maximal chain $\mathcal{C}$ in $\mathcal{S}$. Let $U_\alpha \in \mathcal{S}$, and let $\{x_\alpha\} = \bar{U}_\alpha \backslash U_\alpha$. Since $X \backslash \{x_\alpha\} = U_\alpha | (X \backslash \bar{U}_\alpha)$, by Lemma 3.5 there is a closed point $y \in X \backslash \bar{U}_\alpha \subseteq A$. Let $X \backslash \{y\} = C | D$. Since $\bar{U}_\alpha$ is connected by Lemma 3.4, $\bar{U}_\alpha \subseteq C$ or $\bar{U}_\alpha \subseteq D$, i.e. $U_\alpha < C$ or $U_\alpha < D$. Since U_α was arbitrary in $\mathcal{S}, \mathcal{S}$ (and consequently $\mathcal{C}$) does not have a maximal element. Thus $\bigcup_{U \in \mathcal{C}} U = \bigcup_{U \in \mathcal{C}} \bar{U}$. Write $V = \bigcup_{U \in \mathcal{C}} U$. Since $\bar{U}$ is connected for each $U \in \mathcal{S}$, V is connected too. We claim that $V = X$. Suppose otherwise. Then $X \backslash V$ is a nonempty closed subset of X. Since $X \backslash V \subseteq A$, every point in $X \backslash V$ is a cut point of X and is either open or closed by Theorem 3.2. As $X \backslash V$ is not open (it is closed and X is connected), the points of $X \backslash V$ cannot all be open, and so there is a closed cut point x' in $X \backslash V$. Let $X \backslash \{x'\} = G | H$. Since V is connected, $V \subseteq G$ or $V \subseteq H$. Assume (without loss of generality) that $V \subseteq G$. Since $G \in \mathcal{S}, U \leq G$ for any $U \in \mathcal{C}$. Since $\mathcal{C}$ does not have a maximal element, $G \notin \mathcal{C}$. This contradicts the maximality of the chain $\mathcal{C}$. Hence $V = X$, and therefore $\mathcal{C}$ is an infinite open covering of X. Since $\mathcal{C}$ is a chain without a maximal element, there is no finite subcovering of $\mathcal{C}$ for X. This contradicts the compactness of X. □

3.10. Corollary. *Let X be a cut-point space. Then X is non-compact.*

4. Irreducible cut-point spaces and characterization of the Khalimsky line

In this section, we define an irreducible cut-point space and we show that it is necessarily homeomorphic to the Khalimsky line (see Example 2.5).

4.1. Definition. A cut-point space is said to be an *irreducible cut-point space* if no proper subset of it (with the subspace topology) is a cut-point space.

4.2. Lemma. *Let X be a cut-point space, let $x \in X$, and let $X \backslash \{x\} = A|B$. If A is not connected, then $A \cup \{x\}$ is a cut-point space.*

Proof. Put $Y = A \cup \{x\}$. Clearly x is a cut point of Y. Let y be an arbitrary point in A. Since $X\backslash\{y\} = (Y\backslash\{y\}) \cup (B \cup \{x\})$ is not connected, and since $x \in (Y\backslash\{y\}) \cap (B \cup \{x\})$, either $Y\backslash\{y\}$ or $B \cup \{x\}$ is disconnected. By Lemma 3.4, $B \cup \{x\}$ is connected. Thus $Y\backslash\{y\}$ is disconnected. $\square$

4.3. Corollary. *If X is an irreducible cut-point space, then, for every $x \in X$, $X\backslash\{x\}$ has exactly two components.*

Proof. Let $X\backslash\{x\} = A|B$. Since X is irreducible, $A \cup \{x\}$ and $B \cup \{x\}$ are not cut-point spaces. Thus, by Lemma 4.2, A and B are connected. $\square$

4.4. Lemma. *Let X be an irreducible cut-point space, let $x \in X$ and let $X\backslash\{x\} = A|B$. Then there are exactly two points $y \in A$ and $z \in B$ such that $\{x, y\}$ and $\{x, z\}$ are connected. Furthermore if x is closed then y and z are open, and if x is open then y and z are closed.*

Proof. Since, by Corollary 4.3, A is connected and since X is an irreducible cut-point space, A has a non-cut point y; i.e. $A\backslash\{y\}$ is connected. We claim that y is the unique point in A such that $\{x, y\}$ is connected. First we prove that if $\{x, y'\}$ is connected for some $y' \in A$, then $y' = y$. Let y' be a point in A such that $\{x, y'\}$ is connected. Suppose that $y' \neq y$. Since, by Lemma 3.4, $B \cup \{x\}$ is connected, and since $X\backslash\{y\} = (A\backslash\{y\}) \cup (B \cup \{x\})$, the connectedness of $\{x, y'\}$ implies the connectedness of $X\backslash\{y\}$ (a contradiction). To prove that $\{x, y\}$ is connected we consider two cases.

(1) x is closed. In this case, A is (open and) not closed but $A \cup \{x\}$ is closed (both by Theorem 3.2). Thus x is a limit point of A. On the other hand, since $X\backslash\{y\} = (A\backslash\{y\}) \cup (B \cup \{x\})$ is not connected, x is not a limit point of $A\backslash\{y\}$. Hence x is a limit point of $\{y\}$.

(2) x is open. In this case, A is (closed and) not open but $A \cup \{x\}$ is open (both by Theorem 3.2). Thus there is a point y' in A which is not an interior point of A. Since y' is an interior point of $A \cup \{x\}$, y' is a limit point of $\{x\}$. Hence $\{x, y'\}$ is connected. Since, as we proved above, $y' = y$, $\{x, y\}$ is connected.

A similar argument shows that there is a unique point z in B such that $\{x, z\}$ is connected. The last statement of the lemma is implied by Theorem 3.2 and the connectedness of $\{x, y\}$ and $\{x, z\}$. $\square$

4.5. Theorem. *A topological space X is an irreducible cut-point space if and only if X is homeomorphic to the "Khalimsky line".*

Proof. It can be easily seen that the Khalimsky line is an irreducible cut-point space. Let X be an irreducible cut-point space. By mathematical induction, we find a subset Y of X that is homeomorphic to the Khalimsky line, and then, by irreducibility of X, we conclude that $X = Y$. Let x_0 be a closed point in X, and let $X\backslash\{x_0\} = A_0|B_0$. By Lemma 4.4 there are points x_{-1} in A_0 and x_1 in B_0 such that $\{x_{-1}, x_0\}$ and $\{x_0, x_1\}$ are connected. Define $Y_1 = \{x_{-1}, x_0, x_1\}$. Let A_1 be the component of $X\backslash\{x_1\}$ that contains x_0 and let B_1 be the other component of $X\backslash\{x_1\}$. Let B_{-1} be the component of $X\backslash\{x_{-1}\}$ that contains x_0 and let A_{-1} be the other component of $X\backslash\{x_{-1}\}$. Assume that for an arbitrary positive integer n, the subset $Y_n = \{x_i : i \in \mathbf{Z} \text{ and } -n \leq i \leq n\}$ of X (with $2n+1$ points) is chosen such that for each i and j which satisfy $-n \leq i, j \leq n$ and $|i-j| = 1$, $\{x_i, x_j\}$ is connected. Moreover, assume that for each nonzero $i, -n \leq i \leq n$, the components A_i and B_i of $X\backslash\{x_i\}$ are chosen such that $x_0 \in A_i$ if i is positive, and $x_0 \in B_i$ if i is negative. Since $Y_n\backslash\{x_{-n}\} = \displaystyle\bigcup_{-n<i<j\leq n, j=i+1} \{x_i, x_j\}$ is connected, it is a subset of A_{-n} or B_{-n}, and since $x_0 \notin A_{-n}$, $Y_n\backslash\{x_{-n}\} \subseteq B_{-n}$. By Lemma 4.4 there is a unique point x_{-n-1} in A_{-n} such that $\{x_{-n-1}, x_{-n}\}$ is connected. Since $(Y_n \cup \{x_{-n-1}\})\backslash\{x_n\} = \displaystyle\bigcup_{-n-1\leq i<j<n, j=i+1} \{x_i, x_j\}$ is connected, it is a subset of A_n or B_n, and since $x_0 \notin B_n$, $(Y_n \cup \{x_{-n-1}\})\backslash\{x_n\} \subseteq A_n$. By Lemma 4.4 there is a unique point x_{n+1} in B_n such that $\{x_n, x_{n+1}\}$ is connected. Thus we obtain a subset $Y_{n+1} = \{x_i : i \in \mathbf{Z} \text{ and } -(n+1) \leq i \leq n+1\}$ of X (with $2n+3$ points) such that for each i and j which satisfy $-(n+1) \leq i, j \leq n+1$ and $|i-j| = 1, \{x_i, x_j\}$ is connected. To complete the induction step, we define the subsets A_{-n-1}, B_{-n-1}, A_{n+1}, B_{n+1} of X such that $X\backslash\{x_{-n-1}\} = A_{-n-1}|B_{-n-1}$, $X\backslash\{x_{n+1}\} = A_{n+1}|B_{n+1}$, $x_0 \in B_{-n-1}$ and $x_0 \in A_{n+1}$. Put

$$Y = \bigcup_{n=1}^{\infty} Y_n = \{x_i : i \in \mathbf{Z}\}.$$

It can be easily seen that for each integer i, $Y \cap A_i = \{x_j : j < i\}$ and $Y \cap B_i = \{x_j : j > i\}$. Since x_0 is closed, (by iterated application of Lemma 4.4) x_n is closed if n is even, and x_n is open if n is odd. Clearly, for each $i \in \mathbf{Z}$, the smallest open neighborhood of x_{2i+1} in Y is $\{x_{2i+1}\}$. Since for each $i \in \mathbf{Z}$, x_{2i} is a limit point of $\{x_{2i-1}\}$ and $\{x_{2i+1}\}$, every open neighborhood of x_{2i} in X (and hence in Y) contains x_{2i-1} and x_{2i+1}. On the other hand, since x_{2i-2} and x_{2i+2} are closed, B_{2i-2} and A_{2i+2} are open in X. Thus $\{x_{2i-1}, x_{2i}, x_{2i+1}\} = (Y \cap B_{2i-2}) \cap (Y \cap A_{2i+2})$ is the smallest open neighborhood of x_{2i} in Y. Hence

$$\mathcal{B}' = \{\{x_{2i-1}, x_{2i}, x_{2i+1}\} : i \in \mathbf{Z}\} \cup \{\{x_{2i+1}\} : i \in \mathbf{Z}\}$$

is a base for the topology of Y. Comparing this base with the base of the Khalimsky line in Example 2.5, we see that Y is homeomorphic to the Khalimsky line. $\square$

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