CUT-POINT SPACES

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Abstract. The notion of a cut-point space is introduced as a connected topological space without any non-cut point. It is shown that a cut-point space is infinite. The non-cut point existence theorem is proved for general (not necessarily $T_1$) topological spaces to show that a cut-point space is non-compact. Also, the class of irreducible cut-point spaces is studied and it is shown that this class (up to homeomorphism) has exactly one member: the Khalimsky line.

1. Introduction

The real line $\mathbb{R}$ is a source of intuition in topology. Many other familiar topological spaces can be obtained from $\mathbb{R}$ by topological constructions. It has the following properties:

(a) it is connected but the removal of any one of its points leaves it disconnected;
(b) it is metrizable;
(c) its topology can be generated by a linear ordering.

Conversely, it can be proved that every topological space with the above properties is homeomorphic to $\mathbb{R}$. Conditions (b) and (c) are too strong. They impose structures on the topological space, so this characterization of $\mathbb{R}$ seems somehow extrinsic.

In this paper we study the topological spaces that satisfy condition (a), and call them cut-point spaces. In section 2, a cut-point space is defined again formally and some examples are given. In section 3, it is shown that every cut-point space has an infinite number of closed points. Also, it is proved that every cut-point space is non-compact. To prove the latter, we need the most general form of the non-cut point existence theorem. The special case of this theorem for metric topological spaces is proved in [4]. A proof of the theorem for $T_1$ topological spaces can be found in [1] (see also [5]). In Section 4, an irreducible cut-point space is defined naturally as a cut-point space whose proper subsets are not cut-point spaces. It is shown that an irreducible cut-point space is necessarily homeomorphic to the Khalimsky line (see Example 2.5 for the definition of the Khalimsky line). This result may also be viewed as a straightforward characterization of the Khalimsky line. Objects in $n$-dimensional digital images have sometimes been regarded as subspaces of the product of $n$ copies of the Khalimsky line [2], [3].

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2. DEFINITIONS AND EXAMPLES

2.1. Definition. Let $X$ be a nonempty connected topological space. A point $x$ in $X$ is said to be a cut point of $X$ if $X \setminus \{x\}$ is a disconnected subset of $X$. A nonempty connected topological space $X$ is said to be a cut-point space if every $x$ in $X$ is a cut point of $X$.

In the following three examples, $\mathbb{R}^2$ is the Euclidean plane with the standard topology.

2.2. Example. The union of $n$ straight lines in $\mathbb{R}^2$ is a cut-point space if and only if either all of them are concurrent or exactly $n - 1$ of them are parallel.

2.3. Example. Let $X_1 = \{(x, y) \in \mathbb{R}^2 : x \leq 0 \text{ and } |y| = 1\}$ and let $X_2 = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y = \sin \frac{x}{2}\}$. Define $X = X_1 \cup X_2$. Then $X$ is a cut-point space. For each $x \in X$, $X \setminus \{x\}$ has exactly two components.

A “connected ordered topological space” (COTS) is a connected topological space $X$ with the following property: if $Y$ is a three-point subset of $X$, there is a $y$ in $X$ such that $Y$ meets two connected components of $X \setminus \{y\}$ (see [2]). Put $Y = \{(0, -1), (1, \sin 1), (0, 1)\}$ in Example 2.3 to see that $X$ is not a COTS.

2.4. Example. Let $X_0 = \{(x, 0) \in \mathbb{R}^2 : x \leq 0\} \cup \{(x, 1) \in \mathbb{R}^2 : x > 0\}$ and let for each positive integer $n$, $Y_n = \{\left(\frac{1}{n}, y\right) \in \mathbb{R}^2 : 0 < y \leq 1\}$. Define $X = X_0 \cup \bigcup_{n=1}^{\infty} Y_n$.

Then $X$ is a cut-point space.

A connected topological space is said to have the “connected intersection property” if the intersection of every two connected subsets of it is connected. In Example 2.4, let $X_1 = X_0 \cup \bigcup_{n=1}^{\infty} Y_{2n-1}$ and $X_2 = X_0 \cup \bigcup_{n=1}^{\infty} Y_{2n}$. Since $X_1 \cap X_2 = X_0$ is not connected, $X$ does not possess the connected intersection property. Example 2.4 is a slightly modified version of an example in [6].

2.5. Example (The Khalimsky line). Let $\mathbb{Z}$ be the set of integers and let

$$B = \{\{2i - 1, 2i, 2i + 1\} : i \in \mathbb{Z}\} \cup \{\{2i + 1\} : i \in \mathbb{Z}\}.$$ 

Then $B$ is a base for a topology on $\mathbb{Z}$. The set $\mathbb{Z}$ with this topology is a cut-point space and is called the Khalimsky line. Each point in $\mathbb{Z}$ has a smallest open neighborhood and the base $B$ is the collection of all such neighborhoods. It can be easily seen that the Khalimsky line is irreducible in the sense that no proper subset of it is a cut-point space.

3. TOPOLOGICAL PROPERTIES OF CUT-POINT SPACES

Theorem 3.2 is the key theorem of this section. The main theorem of this section is Theorem 3.9 which implies the non-compactness of cut-point spaces. Notation 3.1 is adopted from [5].

3.1. Notation. Let $Y$ be a topological space. We write $Y = A \mid B$ to mean $A$ and $B$ are two nonempty subsets of $Y$ such that $Y = A \cup B$ and $A \cap B = \hat{A} \cap B = \emptyset$. 
3.2. Theorem. Let $X$ be a connected topological space, and let $x$ be a cut point of $X$ such that $X \setminus \{x\} = A \cap B$. Then $\{x\}$ is open or closed. If $\{x\}$ is open, then $A$ and $B$ are closed; if $\{x\}$ is closed, then $A$ and $B$ are open.

Proof. Since $A$ is both open and closed in $X \setminus \{x\}$, there is an open subset $V$ of $X$ such that $A = V \cap (X \setminus \{x\}) = V \setminus \{x\}$, and there is a closed subset $F$ of $X$ such that $A = F \cap (X \setminus \{x\}) = F \setminus \{x\}$. Thus $A = V \setminus \{x\} = F \setminus \{x\}$. Since the assumption $V = F$ contradicts the connectedness of $X$, we have $\{x\} = V \setminus F$ or $\{x\} = F \setminus V$. If $\{x\} = V \setminus F$, then $\{x\}$ is open and $A = F$ is closed. If $\{x\} = F \setminus V$, then $\{x\}$ is closed and $A = V$ is open. □

3.3. Corollary. Let $X$ be a connected topological space, and let $Y$ be the subset of all cut points of $X$. Then the following statements are obviously true.

(a) Every nonempty connected subset of $Y$ that is not a singleton, contains at least one closed point.

(b) If $x \in Y$ is open, then every limit point of $\{x\}$ in $Y$ is a closed point.

3.4. Lemma. Let $X$ be a connected topological space, and let $x$ be a cut point of it. If $X \setminus \{x\} = A \cap B$, then $A \cup \{x\}$ is connected.

Proof. If $A \cup \{x\}$ is not connected, then there are subsets $C$ and $D$ of $X$ such that $A \cup \{x\} = C \cap D$. Without loss of generality, we may assume that $x \in C$. Then $D \subseteq A$. Since $(B \cup C) \cap D = (B \cap D) \cup (C \setminus D) = B \cap D \subseteq B \cap A = \emptyset$, $(B \cup C) \cap D = \emptyset$. Since $(B \cup C) \cap D = (B \cap D) \cup (C \setminus D) = (B \cap D) \subseteq B \cap A = \emptyset$, $(B \cup C) \cap D = \emptyset$. Therefore $X = (B \cup C) \cap D$. This contradicts the connectedness of $X$. □

3.5. Lemma. Let $X$ be a connected topological space and let $x$ be a cut point of it. If $X \setminus \{x\} = A \cap B$ and if every point of $A$ is a cut point of $X$, then $A$ contains at least one closed point.

Proof. Suppose that $A$ consists exclusively of open points. Since, by Lemma 3.4, $A \cup \{x\}$ is connected, $\{x\}$ is closed and hence (by Theorem 3.2) $A \cup \{x\}$ is closed too. Thus, for every $y \in A$, $\overline{\{y\}} \subseteq A \cup \{x\}$, and therefore, by Corollary 3.3 (b), $x$ is the only possible limit point of $\{y\}$. As $\{y\}$ has a limit point (since $\{y\}$ is open and $X$ is connected), $x$ is a limit point of $\{y\}$. This implies that $\{x, y\}$ is connected for any $y \in A$. Let $y_0 \in A$. Since $B \cup \{x\}$ is connected by Lemma 3.4,

$$X \setminus \{y_0\} = \bigcup_{y \in A, y \neq y_0} \{x, y\} \cup (B \cup \{x\})$$

is connected too. This contradicts the fact that $y_0$ is a cut point of $X$. □

3.6. Lemma. Let $X$ be a connected topological space, and let $x$ and $y$ be two cut points of it such that $X \setminus \{x\} = A \cap B$ and $X \setminus \{y\} = C \cap D$. If $x \in C$ and $y \in A$, then $D \subseteq A$ and $B \subseteq C$.

Proof. Since $D \cup \{y\}$ is connected by Lemma 3.4, and since $D \cup \{y\} \subseteq X \setminus \{x\}$, we have $D \cup \{y\} \subseteq A$ or $D \cup \{y\} \subseteq B$. Since $y \in A$, the second inclusion is not true. Hence $D \subseteq A$. A similar argument shows that $B \subseteq C$. □

In the next theorem, we show that a finite topological space cannot be a cut-point space.
3.7. **Theorem.** Let $X$ be a cut-point space. Then the set of closed points of $X$ is infinite.

**Proof.** By mathematical induction, we construct a sequence $x_1, x_2, \ldots$ of distinct closed points in $X$. Define $C_0 = X$. By Corollary 3.3 (a), there exists a closed point $x_1$ in $C_0$. Since $x_1$ is a cut point of $X$, there are open subsets $C_1$ and $D_1$ of $X$ such that $X \setminus \{x_1\} = C_1 \cup D_1$. Now, suppose that the distinct closed points $x_1, x_2, \ldots, x_n$ in $X$ and the open subsets $C_1, \ldots, C_n, D_1, \ldots, D_n$ of $X$ are chosen such that $X \setminus \{x_i\} = C_i \cup D_i$ for each $i, 1 \leq i \leq n$. According to Lemma 3.5, there is a closed point $x_{n+1} \in C_n$. There are open subsets $C_{n+1}$ and $D_{n+1}$ of $X$ such that $X \setminus \{x_{n+1}\} = C_{n+1} \cup D_{n+1}$. By interchanging $C_{n+1}$ and $D_{n+1}$, if necessary, we may assume that $x_n \in D_{n+1}$. Thus, by Lemma 3.6, $C_n \supseteq C_{n+1}$. Since $x_i \notin C_i, x_i \notin C_n$ for any $i, 1 \leq i \leq n$. The fact that $x_{n+1} \in C_n$ implies that $x_{n+1}$ is different from $x_1, \ldots, x_n$.

3.8. **Corollary.** Let $X$ be a cut-point space. Then $|X| = \infty$.

Of course, Theorem 3.7 is a generalization of Corollary 3.8. Using the Hausdorff Maximal Principle, we prove another generalization of Corollary 3.8 in the following theorem.

3.9. **Theorem.** Let $X$ be a compact connected topological space with more than one point. Then $X$ has at least two non-cut points.

**Proof.** Suppose that $X$ has at most one non-cut point. Let $x_0$ be a cut point of $X$ and let $X \setminus \{x_0\} = A_0 | B_0$. Since $X$ has at most one non-cut point, either $A_0$ or $B_0$ (without loss of generality assume $A_0$) exclusively consists of cut points. By Lemma 3.5, $A_0$ contains some closed cut point of $X$, say $x$. Let $X \setminus \{x\} = A | B$ and without loss of generality assume that $x_0 \in B$. Then by Lemma 3.6, $A \subseteq A_0$. Define $S = \{U : U$ is an open subset of $X, U \supseteq B, U \cap X$ is a singleton, and $U \neq X\}$. Since $B$ is open and $B = B \cup \{x\}, B \in S$. For each $U_\alpha \in S$ and $U_\beta \in S$, write $U_\alpha \leq U_\beta$ if $U_\alpha = U_\beta$, or if $U_\alpha \subseteq U_\beta$. $(S, \leq)$ is clearly a partially ordered set, and by the Hausdorff Maximal Principle there is a maximal chain $C$ in $S$. Let $U_\alpha \in S$, and let $(x_\alpha) = U_\alpha \setminus U_\alpha$. Since $X \setminus \{x_\alpha\} = U_\alpha | (X \setminus U_\alpha)$, by Lemma 3.5 there is a closed point $y \in X \setminus U_\alpha \subseteq A$. Let $X \setminus \{y\} = C | D$. Since $U_\alpha$ is connected by Lemma 3.4, $U_\alpha \subseteq C$ or $U_\alpha \subseteq D$, i.e. $U_\alpha \subset C$ or $U_\alpha \subset D$. Since $U_\alpha$ was arbitrary in $S, S$ (and consequently $C$) does not have a maximal element. Thus $\bigcup_{U \in C} U = \bigcup_{U \in C} \bar{U}$. Write $V = \bigcup_{U \in C} U$. Since $\bar{U}$ is connected for each $U \in S, V$ is connected too. We claim that $V = X$. Suppose otherwise. Then $X \setminus V$ is a nonempty closed subset of $X$. Since $X \setminus V \subseteq A$, every point in $X \setminus V$ is a cut point of $X$ and is either open or closed by Theorem 3.2. As $X \setminus V$ is not open (it is closed and $X$ is connected), the points of $X \setminus V$ cannot all be open, and so there is a closed cut point $x'$ in $X \setminus V$. Let $X \setminus \{x'\} = G | H$. Since $V$ is connected, $V \subseteq G$ or $V \subseteq H$. Assume (without loss of generality) that $V \subseteq G$. Since $G \in S, U \subseteq G$ for any $U \in C$. Since $C$ does not have a maximal element, $G \notin C$. This contradicts the maximality of the chain $C$. Hence $V = X$, and therefore $C$ is an infinite open covering of $X$. Since $C$ is a chain without a maximal element, there is no finite subcovering of $C$ for $X$. This contradicts the compactness of $X$.

3.10. **Corollary.** Let $X$ be a cut-point space. Then $X$ is non-compact.
4. Irreducible cut-point spaces and characterization of the Khalimsky line

In this section, we define an irreducible cut-point space and we show that it is necessarily homeomorphic to the Khalimsky line (see Example 2.5).

4.1. Definition. A cut-point space is said to be an irreducible cut-point space if no proper subset of it (with the subspace topology) is a cut-point space.

4.2. Lemma. Let $X$ be a cut-point space, let $x \in X$, and let $X \setminus \{x\} = A \cup B$. If $A$ is not connected, then $A \cup \{x\}$ is a cut-point space.

Proof. Put $Y = A \cup \{x\}$. Clearly $x$ is a cut point of $Y$. Let $y$ be an arbitrary point in $A$. Since $X \setminus \{y\} = (Y \setminus \{y\}) \cup (B \cup \{x\})$ is not connected, and since $x \in (Y \setminus \{y\}) \cap (B \cup \{x\})$, either $Y \setminus \{y\}$ or $B \cup \{x\}$ is disconnected. By Lemma 3.4, $B \cup \{x\}$ is connected. Thus $Y \setminus \{y\}$ is disconnected. \hfill \qed

4.3. Corollary. If $X$ is an irreducible cut-point space, then, for every $x \in X$, $X \setminus \{x\}$ has exactly two components.

Proof. Let $X \setminus \{x\} = A \cup B$. Since $X$ is irreducible, $A \cup \{x\}$ and $B \cup \{x\}$ are not cut-point spaces. Thus, by Lemma 4.2, $A$ and $B$ are connected. \hfill \qed

4.4. Lemma. Let $X$ be an irreducible cut-point space, let $x \in X$ and let $X \setminus \{x\} = A \cup B$. Then there are exactly two points $y \in A$ and $z \in B$ such that $\{x, y\}$ and $\{x, z\}$ are connected. Furthermore if $x$ is closed then $y$ and $z$ are open, and if $x$ is open then $y$ and $z$ are closed.

Proof. Since, by Corollary 4.3, $A$ is connected and since $X$ is an irreducible cut-point space, $A$ has a non-cut point $y$; i.e. $A \setminus \{y\}$ is connected. We claim that $y$ is the unique point in $A$ such that $\{x, y\}$ is connected. First we prove that if $\{x, y'\}$ is connected for some $y' \in A$, then $y' = y$. Let $y'$ be a point in $A$ such that $\{x, y'\}$ is connected. Suppose that $y' \neq y$. Since, by Lemma 3.4, $B \cup \{x\}$ is connected, and since $X \setminus \{y\} = (A \setminus \{y\}) \cup (B \cup \{x\})$, the connectedness of $\{x, y'\}$ implies the connectedness of $X \setminus \{y\}$ (a contradiction). To prove that $\{x, y\}$ is connected we consider two cases.

(1) $x$ is closed. In this case, $A$ (open and) not closed but $A \cup \{x\}$ is closed (both by Theorem 3.2). Thus $x$ is a limit point of $A$. On the other hand, since $X \setminus \{y\} = (A \setminus \{y\}) \cup (B \cup \{x\})$ is not connected, $x$ is not a limit point of $A \setminus \{y\}$. Hence $x$ is a limit point of $\{y\}$.

(2) $x$ is open. In this case, $A$ (closed and) not open but $A \cup \{x\}$ is open (both by Theorem 3.2). Thus there is a point $y'$ in $A$ which is not an interior point of $A$. Since $y'$ is an interior point of $A \cup \{x\}$, $y'$ is a limit point of $\{x\}$. Hence $\{x, y'\}$ is connected. Since, as we proved above, $y' = y$, $\{x, y\}$ is connected. A similar argument shows that there is a unique point $z$ in $B$ such that $\{x, z\}$ is connected. The last statement of the lemma is implied by Theorem 3.2 and the connectedness of $\{x, y\}$ and $\{x, z\}$. \hfill \qed

4.5. Theorem. A topological space $X$ is an irreducible cut-point space if and only if $X$ is homeomorphic to the “Khalimsky line”.

Proof. It can be easily seen that the Khalimsky line is an irreducible cut-point space. Let $X$ be an irreducible cut-point space. By mathematical induction, we find a subset $Y$ of $X$ that is homeomorphic to the Khalimsky line, and then, by
irreducibility of $X$, we conclude that $X = Y$. Let $x_0$ be a closed point in $X$, and let $X \setminus \{x_0\} = A_0|B_0$. By Lemma 4.4 there are points $x_{-1}$ in $A_0$ and $x_1$ in $B_0$ such that $\{x_{-1}, x_0\}$ and $\{x_0, x_1\}$ are connected. Define $Y_1 = \{x_{-1}, x_0, x_1\}$. Let $A_1$ be the component of $X \setminus \{x_1\}$ that contains $x_0$ and let $B_1$ be the other component of $X \setminus \{x_1\}$. Let $B_{-1}$ be the component of $X \setminus \{x_{-1}\}$ that contains $x_0$ and let $A_{-1}$ be the other component of $X \setminus \{x_{-1}\}$. Assume that for an arbitrary positive integer $n$, the subset $Y_n = \{x_i : i \in \mathbb{Z} \text{ and } -n \leq i \leq n\}$ of $X$ (with $2n + 1$ points) is chosen such that for each $i$ and $j$ which satisfy $-n \leq i, j \leq n$ and $|i - j| = 1$, \{x_i, x_j\} is connected. Moreover, assume that for each nonzero $i$, $-n \leq i \leq n$, the components $A_i$ and $B_i$ of $X \setminus \{x_i\}$ are chosen such that $x_0 \in A_i$ if $i$ is positive, and $x_0 \in B_i$ if $i$ is negative. Since $Y_n \setminus \{x_{-n}\} = \bigcup_{-n < i < j \leq n, j = i+1} \{x_i, x_j\}$ is connected, it is a subset of $A_{-n}$ or $B_{-n}$, and since $x_0 \notin A_{-n}$, $Y_n \setminus \{x_{-n}\} \subseteq B_{-n}$. By Lemma 4.4 there is a unique point $x_{-n-1}$ in $A_{-n}$ such that $\{x_{-n-1}, x_{-n}\}$ is connected. Since $(Y_n \cup \{x_{-n-1}\}) \setminus \{x_n\} = \bigcup_{-n-1 \leq i < j \leq n, j = i+1} \{x_i, x_j\}$ is connected, it is a subset of $A_n$ or $B_n$, and since $x_0 \notin B_n$, $(Y_n \cup \{x_{-n-1}\}) \setminus \{x_n\} \subseteq A_n$. By Lemma 4.4 there is a unique point $x_{n+1}$ in $B_n$ such that $\{x_n, x_{n+1}\}$ is connected. Thus we obtain a subset $Y_{n+1} = \{x_i : i \in \mathbb{Z} \text{ and } -(n+1) \leq i \leq n+1\}$ of $X$ (with $2n + 3$ points) such that for each $i$ and $j$ which satisfy $-(n+1) \leq i, j \leq n+1$ and $|i - j| = 1$, \{x_i, x_j\} is connected. To complete the induction step, we define the subsets $A_{n-1}$, $B_{n-1}$, $A_{n+1}$, $B_{n+1}$ of $X$ such that $X \setminus \{x_{n-1}\} = A_{n-1}|B_{n-1}$, $X \setminus \{x_{n+1}\} = A_{n+1}|B_{n+1}$, $x_0 \in B_{n-1}$ and $x_0 \in A_{n+1}$. Put

$$Y = \bigcup_{n=1}^{\infty} Y_n = \{x_i : i \in \mathbb{Z}\}.$$ 

It can be easily seen that for each integer $i$, $Y \cap A_i = \{x_j : j < i\}$ and $Y \cap B_i = \{x_j : j > i\}$. Since $x_0$ is closed, (by iterated application of Lemma 4.4) $x_n$ is closed if $n$ is even, and $x_n$ is open if $n$ is odd. Clearly, for each $i \in \mathbb{Z}$, the smallest open neighborhood of $x_{2i+1}$ in $Y$ is $\{x_{2i+1}\}$. Since for each $i \in \mathbb{Z}$, $x_{2i}$ is a limit point of $\{x_{2i-1}\}$ and $\{x_{2i+1}\}$, every open neighborhood of $x_{2i}$ in $X$ (and hence in $Y$) contains $x_{2i-1}$ and $x_{2i+1}$. On the other hand, since $x_{2i-2}$ and $x_{2i+2}$ are closed, $B_{2i-2}$ and $A_{2i+2}$ are open in $X$. Thus $\{x_{2i-1}, x_{2i}, x_{2i+1}\} = (Y \cap B_{2i-2}) \cap (Y \cap A_{2i+2})$ is the smallest open neighborhood of $x_{2i}$ in $Y$. Hence

$$B' = \{(x_{2i-1}, x_{2i}, x_{2i+1}) : i \in \mathbb{Z}\} \cup \{(x_{2i+1}) : i \in \mathbb{Z}\}$$

is a base for the topology of $Y$. Comparing this base with the base of the Khalimsky line in Example 2.5, we see that $Y$ is homeomorphic to the Khalimsky line.

References


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