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SERRE'S CONDITION R_k FOR ASSOCIATED GRADED RINGS

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Abstract. A criterion is given for when the associated graded ring of an ideal satisfies Serre's condition R_k. As an application, the integrality and quasi-Gorensteinness of such rings is investigated.

1. Introduction

This note grew out of an attempt to better understand what it means for an ideal I to have an associated graded ring $gr_I(R)$ satisfying Serre's condition R_k. We were inspired by a result of Huneke which essentially says that this condition holds if the local analytic spreads of I are "sufficiently small" ([7]). In our main result we are going to prove that at least for ideals of finite projective dimension, Huneke's upper bounds for the local analytic spreads of I are actually implied by the R_k property of $gr_I(R)$ (Theorem 2.4). Combining both facts one obtains a characterization for when the associated graded ring satisfies R_k (Corollary 2.6). Another immediate consequence of our main theorem is a rather surprising result by Huneke, Simis, and Vasconcelos to the effect that if the associated graded ring of a prime ideal of finite projective dimension is reduced, then it is already a domain ([8]) (Corollary 3.1). We use this fact in turn to investigate the connection between the reducedness and the quasi-Gorenstein property of associated graded rings (Theorem 3.2), a theme that goes back to earlier work by Hochster and by Herzog, Simis, and Vasconcelos ([6], [4], [5]).

2. The property R_k

If S is a positively graded ring, we write $S_+ = \bigoplus_{i>0} S_i$ and denote by $\mathrm{Proj}(S)$ the set of all homogeneous prime ideals of S not containing S_+. We say that S satisfies Serre's condition R_k on a subset X of $\mathrm{Proj}(S)$ if for every $p \in X$ with $\dim S_p \leq k$, S_p or, equivalently, the homogeneous localization $S_{(p)}$ (as in [1, p. 30]) is a regular ring.

Let R be a Noetherian ring, I an R-ideal, and t a variable. We will consider the Rees algebra $\mathcal{R} = R[It]$ and the extended Rees algebra $T = R[It, t^{-1}]$ of I, as well as the associated graded ring $G = gr_I(R)$. Notice that $G \cong \mathcal{R}/I\mathcal{R} \cong T/(t^{-1})$ with t^{-1} a homogeneous T-regular element. If (R, m) is local and $I \neq R$, then the *analytic spread* $\ell(I)$ is defined to be the dimension of $G/mG \cong \mathcal{R}/m\mathcal{R}$. One always

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has $ht\, I \le \ell(I) \le \dim R$. In case R/m is infinite, $\ell(I)$ gives the minimal number of generators of any minimal reduction of I.

The next result, though easy to show, provides the crucial step in the proof of the main theorem.

Proposition 2.1. *Let (R, m) be a Noetherian local ring and let I be an R-ideal with $ht\, I > 0$ and $I \subset m^2$. If $G = gr_I(R)$ satisfies R_k on $\mathrm{Proj}(G) \cap V(mG)$, then $\ell(I) \le \dim R - k - 1$.*

Proof. Write $d = \dim R$. Since $\dim G/mG = \ell(I) \ge ht\, I > 0$, there exists a homogeneous prime ideal q of G with $mG \subset q$ and $\dim G_q \le d - \ell(I) < d$. Thus, $q \in \mathrm{Proj}(G) \cap V(mG)$ with $\dim G_q \le d - \ell(I)$. Now suppose that the asserted inequality does not hold, then $d - \ell(I) \le k$ and hence G_q is regular, say of dimension s.

Let Q be the preimage of q in $\mathcal{R} = R[It]$, and let $x_1, \dots, x_s$ be elements of $Q\mathcal{R}_Q$ whose images in G_q form a regular system of parameters. Then $Q\mathcal{R}_Q = (I, x_1, \dots, x_s)\mathcal{R}_Q$. But $I \subset m^2 \subset Q^2\mathcal{R}_Q$, and hence $Q\mathcal{R}_Q = (x_1, \dots, x_s)\mathcal{R}_Q$ by Nakayama's Lemma. Thus, $\mathcal{R}_Q$ would be a regular local ring of dimension s as well. But then $I\mathcal{R}_Q = 0$, and hence I would be contained in a minimal prime of $\mathcal{R}$, which is impossible since $ht\, I > 0$ ([10, p. 121]). □

To continue we need two lemmas which are essentially well known.

Lemma 2.2. *Let R be a Noetherian ring, I an R-ideal, $G = gr_I(R)$, x a superficial element for I, $x' = x + I^2 \in [G]_1$, and $\bar{I} = I/(x) \subset \bar{R} = R/(x)$. Then the natural epimorphism of graded G-modules*

$$G/(x') \twoheadrightarrow gr_{\bar{I}}(\bar{R})$$

is an isomorphism on $\mathrm{Proj}(G)$.

Proof. By the Artin-Rees Lemma and the definition of superficial elements ([11, p. 72]), there is an integer k so that for $n >> 0$,

$$I^n \cap (x) = I^n \cap xI^{n-k} = x((I^n : (x)) \cap I^{n-k}) = xI^{n-1}.$$

Thus, $\ker(G/(x') \twoheadrightarrow gr_{\bar{I}}(\bar{R}))$ is concentrated in finitely many degrees, and hence is annihilated by some power of G_+. □

Lemma 2.3 ([11, 27.5], [9, p. 130])**.** *Let (R, m) be a Noetherian local ring, let $x \notin m^2$ be an R-regular element, and let M be a finitely generated R-module annihilated by x. If M has finite projective dimension over R, then M has finite projective dimension over $R/(x)$.*

Proof. By passing to a sufficiently high syzygy module of M over $\bar{R} = R/(x)$, we may assume that $\mathrm{proj\,dim}_R\, M = 1$. Now $M \cong F/H$, where $H \subset F$ are free R-modules of rank n with $xF \subset H \subset mF$. There exists a basis $\{e_1, \dots, e_n\}$ of F so that $\{xe_1, \dots, xe_k\}$ form part of a basis of H and $\{xe_{k+1}, \dots, xe_n\} \subset mH \subset m^2F$, for some integer k with $0 \le k \le n$. Now $k = n$ since otherwise $x \in m^2$. Hence, $H = xF$, which means that M is a free $\bar{R}$-module. □

We are now ready to prove the main result.

Theorem 2.4. *Let R be a Noetherian ring, let I be an R-ideal of finite projective dimension, and write $G = gr_I(R)$. If G satisfies R_k on $\mathrm{Proj}(G)$, then $\ell(I_p) \le \max\{ht\, I_p, \dim R_p - k - 1\}$ for every $p \in V(I)$.*

Proof. Localizing at $p \in V(I)$ we may assume that (R, m) is local. We are going to prove by induction on $g = \text{grade } I$ that $\ell(I) \leq \max\{ht\, I, \dim R - k - 1\}$.

If $g = 0$, then $I = 0$ since $\text{proj}\dim_R R/I < \infty$, and the assertion follows. So let $g > 0$. By Proposition 2.1 we may assume that $I \not\subset m^2$. Choose generators $f_1, \ldots, f_n$ of I, let $Z_1, \ldots, Z_n$ be variables, and write $\tilde{R} = R(Z_1, \ldots, Z_n)$, $\tilde{m} = m\tilde{R}$, $\tilde{I} = I\tilde{R}$, and $\tilde{G} = G \otimes_R \tilde{R}$. Further, set $x = \sum_{i=1}^n Z_i f_i$, $x' = x + \tilde{I}^2 \in [\tilde{G}]_1$, and $\bar{I} = \tilde{I}/(x) \subset \bar{R} = \tilde{R}/(x)$. Since $I \not\subset m^2$ and $g > 0$, it follows that $x \notin \tilde{m}^2$ is $\tilde{R}$-regular, x being a generic element for I. Thus, $\text{proj}\dim_{\bar{R}} \bar{R}/\bar{I} < \infty$ by Lemma 2.3. On the other hand, x' is a generic element for G_+. Therefore, x is a superficial element of $\tilde{I}$, and $\tilde{G}/(x')$ still satisfies R_k on $\text{Proj}(\tilde{G}/(x'))$ since for every $q \in \text{Proj}(\tilde{G}/(x'))$, $(\tilde{G}/(x'))_q$ is the localization of a polynomial ring over G. Hence, by Lemma 2.2, $gr_{\bar{I}}(\bar{R})$ satisfies R_k on $\text{Proj}(gr_{\bar{I}}(\bar{R}))$. Now the induction hypothesis yields $\ell(\bar{I}) \leq \max\{ht\, \bar{I}, \dim \bar{R} - k - 1\} = \max\{ht\, I, \dim R - k - 1\} - 1$.

It remains to show the inequality $\ell(I) - 1 \leq \ell(\bar{I})$. Indeed, writing $K = \tilde{R}/\tilde{m}$ and using the convention $\dim \varnothing = -1$, one concludes from Lemma 2.2 that

$$\begin{aligned} \ell(\bar{I}) &= \dim \text{Proj}(gr_{\bar{I}}(\bar{R}) \otimes_{\tilde{R}} K) + 1 = \dim \text{Proj}(\tilde{G}/(x') \otimes_{\tilde{R}} K) + 1 \\ &= \dim(\tilde{G} \otimes_{\tilde{R}} K)/(x') \geq \ell(I) - 1. \end{aligned}$$

□

The assumption of I having finite projective dimension cannot be deleted in the above theorem even if the ideal is generically a complete intersection. This can be seen by taking $R = k[X, Y, U, V]/(XU - YV)$, k a field, and $I = (X, Y)R$; notice that $gr_I(R) \cong R$ satisfies R_2, whereas $\ell(I_m) = 2$, with m denoting the irrelevant maximal ideal.

The inequalities that appear in Theorem 2.4 have an easy dimension theoretic interpretation (see also [7, the proof of Proposition 2.1]).

Remark 2.5. Let R be a Noetherian ring that is locally equidimensional and universally catenary, let I be an R-ideal, and write $G = gr_I(R)$. The following are equivalent:

(a) $\ell(I_p) \leq \dim R_p - k - 1$ for every $p \in V(I)$ with $\dim(R/I)_p \geq k + 1$.
(b) For every $q \in \text{Spec}(G)$ with $\dim G_q \leq k$ one has $\dim(R/I)_p \leq k$, where p is the contraction of q.

Proof. Localizing at $p \in V(I)$ we may assume that (R, m) is local with $\dim R/I \geq k + 1$. It suffices to show that $\ell(I) \leq \dim R - k - 1$ if and only if $\dim G_q \geq k + 1$ for every $q \in V(mG)$. This amounts to proving that $\dim G/mG \leq \dim G - k - 1$ if and only if $ht\, mG \geq k + 1$. But indeed, $\dim G = ht\, mG + \dim G/mG$, because $G \cong T/(t^{-1})$ with t^{-1} a homogeneous regular element and $T = R[It, t^{-1}]$ a graded ring that has a unique maximal homogeneous ideal and is equidimensional and catenary ([10, pp. 121–122]). □

In [7], Huneke proved the following result: Let R be a homomorphic image of a regular domain, let I be a prime ideal that is generically a complete intersection, and assume that either R is a Cohen-Macaulay Nagata domain or that $G = gr_I(R)$ is Cohen-Macaulay. Further, suppose that $\ell(I_p) \leq \max\{htI, \dim R_p - k - 1\}$ for every $p \in V(I)$. Then G satisfies R_k if and only if R/I does.

Combining his method of proof with Theorem 2.4 one obtains the following characterization:

Corollary 2.6. *Let R be a Cohen-Macaulay ring and let I be an R-ideal of finite projective dimension that is a complete intersection locally at each of its minimal primes. Then $gr_I(R)$ satisfies R_k if and only if R/I satisfies R_k and $\ell(I_p) \leq \max\{ht\, I_p, \dim R_p - k - 1\}$ for every $p \in V(I)$.*

Proof. By Theorem 2.4 we may assume that in either case,

$$\ell(I_p) \leq \max\{ht\, I_p, \dim R_p - k - 1\}$$

for every $p \in V(I)$. It remains to show that $G = gr_I(R)$ satisfies R_k if and only if R/I has this property. By Remark 2.5, every prime q of G with $\dim G_q \leq k$ contracts to a prime $p \in V(I)$ with $\dim(R/I)_p \leq k$. Thus, localizing at $p \in V(I)$, we may assume that R is a local ring with $\dim R/I \leq k$. Now by our assumption on the local analytic spreads, $\ell(I) = ht\, I$. Hence, I is a complete intersection, because R is Cohen-Macaulay and I is a complete intersection locally at each of its minimal primes ([2]). Therefore, G is a polynomial ring over R/I and thus, satisfies R_k if and only if R/I does. ☐

3. Integrality and Gorensteinness

Another immediate consequence of our main result is the following theorem of Huneke, Simis, and Vasconcelos about the minimal primes of associated graded rings ([8, 1.2]):

Corollary 3.1. *Let R be a Noetherian ring that is locally equidimensional and universally catenary, let I be an R-ideal of finite projective dimension that is a complete intersection locally at each of its minimal primes, and write $G = gr_I(R)$. If G satisfies R_0 on $\mathrm{Proj}(G)$, then the natural map $\mathrm{Spec}(G) \to \mathrm{Spec}(R/I)$ yields a one-to-one correspondence between the sets of minimal primes $\mathrm{Min}(G)$ and $\mathrm{Min}(R/I)$.*

Proof. By Theorem 2.4 and Remark 2.5, every minimal prime of G contracts to a minimal prime of R/I, thus giving a well-defined map $\mathrm{Min}(G) \to \mathrm{Min}(R/I)$. This map is bijective since for every $p \in \mathrm{Min}(R/I)$, $G \otimes_R k(p)$ is a polynomial ring over the residue field $k(p)$. ☐

Corollary 3.1 implies that if I is an ideal of finite projective dimension in a ring R as above and if the associated graded ring $G = gr_I(R)$ is reduced, then G is torsionfree as a module over R/I; in particular, G is a domain in case I is prime ([8, 1.1]).

We now turn to the question of whether the reducedness of the associated graded ring implies that this ring or the extended Rees algebra is (quasi)-Gorenstein (cf. also [6], [4], [5]). Recall that a Noetherian ring S is said to be quasi-Gorenstein if for every maximal ideal m of S, S_m is the canonical module of S_m. In case S happens to be a Noetherian graded ring with a unique maximal homogeneous ideal, then S is quasi-Gorenstein if and only if the graded canonical module ω_S of S exists and is of the form $\omega_S \cong S(-n)$ for some integer n (see [1, Section 3.6] for information about graded canonical modules). The next result has been shown by Herzog, Simis, and Vasconcelos under the additional assumptions that R is a normal Cohen-Macaulay ring, that $ht\, I \geq 2$, and that the Rees algebra of I is Cohen-Macaulay ([5, 4.2.3]). In their setting, the quasi-Gorensteinness of the extended Rees algebra simply means that the associated graded ring is Gorenstein.

Theorem 3.2. *Let R be a quasi-Gorenstein ring that is an epimorphic image of a local Gorenstein ring. Let I be a proper R-ideal of finite projective dimension and assume that $gr_I(R)$ is reduced. Then $R[It,t^{-1}]$ is quasi-Gorenstein if and only if I is unmixed.*

Proof. Write $G = gr_I(R)$, $T = R[It,t^{-1}]$, and $\omega = \omega_T$. Notice that T is a graded ring with a unique maximal homogeneous ideal and that ω is a graded T-module. Furthermore, the T-regular element t^{-1} is homogeneous of degree -1, and $T/(t^{-1}) \cong G$. This yields a homogeneous embedding $\omega \otimes_T G \hookrightarrow \omega_G(1)$, which is an isomorphism whenever T or, equivalently, G is Cohen-Macaulay. The local ring R is universally catenary, and, being quasi-Gorenstein, it is S_2, hence equidimensional ([3, 2.4.1]). Thus, T and G are equidimensional, and therefore ω and ω_G localize. Furthermore, if p is a minimal prime of I of height g, then $IR_p = pR_p$ is a complete intersection, since I is reduced and has finite projective dimension. Hence, $G \otimes_R R_p$ is a polynomial ring in g variables over a field. Thus, $\omega_{G\otimes_R R_p} \cong G \otimes_R R_p(-g)$ and then, by the graded Nakayama Lemma, $\omega \otimes_R R_p \cong \omega_{T\otimes_R R_p} \cong T \otimes_R R_p(-g+1)$ is generated in degree $g-1$.

Now if T is quasi-Gorenstein, then $\omega \cong T(-n)$ for some integer n. Hence, for every minimal prime p of I of height g, $T \otimes_R R_p(-g+1) \cong \omega \otimes_R R_p \cong T \otimes_R R_p(-n)$. But the maximal homogeneous ideal of T is a maximal ideal; thus $-g+1 = -n$ ([1, 1.5.16]), showing that $ht\, p$ does not depend on the choice of p.

Conversely, assume that I is unmixed of height g. Notice that $T_{t^{-1}} \cong R[t,t^{-1}]$ is a graded quasi-Gorenstein ring with a unique maximal homogeneous ideal, and thus $\omega \otimes_T T_{t^{-1}} \cong \omega_{R[t,t^{-1}]} \cong R[t,t^{-1}](-1) \cong R[t,t^{-1}]$. We make the identifications $\omega \subset \omega \otimes_T T_{t^{-1}} = R[t,t^{-1}]$. Hence, $[\omega]_i = Rt^i$ for every integer $i << 0$, and whenever $[\omega]_i = Rt^i$, then $[\omega]_i$ generates $\omega \otimes_T T_{t^{-1}}$ as a module over $T_{t^{-1}}$.

Now write $A = R/I$ and $K = \mathrm{Quot}(A)$. By Corollary 3.1 and the remark following it, G is torsionfree over A, hence ω_G has the same property, and thus $\omega_G \hookrightarrow \omega_G \otimes_A K \cong \omega_{G\otimes_A K}$. By the above, $\omega_{G\otimes_A K}$ is concentrated in degrees $\geq g$, and therefore the same holds for $\omega \otimes_T G(-1) \hookrightarrow \omega_G$. Hence, $[\omega]_{i-1} = [\omega]_i t^{-1}$ whenever $i \leq g-1$, showing that $[\omega]_{g-1} = Rt^{g-1}$ generates $\omega \otimes_T T_{t^{-1}}$ as a module over $T_{t^{-1}}$.

We claim that $[\omega]_{g-1} = Rt^{g-1}$ generates ω as a T-module. Indeed, as a quasi-Gorenstein ring, $T_{t^{-1}}$ satisfies S_2. On the other hand, $T/(t^{-1}) \cong G$ is S_1 by reducedness. Therefore, T satisfies S_2 ([1, 2.2.33]). Consequently, it suffices to check the equality $[\omega]_{g-1}T = \omega$ locally at every $q \in \mathrm{Spec}(T)$ with $\dim T_q \leq 1$. Since $[\omega]_{g-1}T_{t^{-1}} = \omega \otimes_T T_{t^{-1}}$, we may assume that q contains t^{-1}. But then $q/(t^{-1})$ is a minimal prime of G, and so by Corollary 3.1, $p = q \cap R$ is a minimal prime of I. Thus, $\omega \otimes_R R_p$ is generated in degree $g-1$, which yields the asserted equality $[\omega]_{g-1}T_q = \omega \otimes_T T_q$. □

As before, Corollary 3.1 and Theorem 3.2 do not hold without the assumption of I having finite projective dimension ([8, Example 1.2] and [4, 1.2]).

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