VARIATIONAL PRINCIPLES FOR AVERAGE EXIT TIME MOMENTS FOR DIFFUSIONS IN EUCLIDEAN SPACE

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Abstract. Let \( D \) be a smoothly bounded domain in Euclidean space and let \( X_t \) be a diffusion in Euclidean space. For a class of diffusions, we develop variational principles which characterize the average of the moments of the exit time from \( D \) of a particle driven by \( X_t \), where the average is taken over all starting points in \( D \).

1. Introduction

In this note we study diffusions on \( \mathbb{R}^d \) and properties of their corresponding exit times from smoothly bounded, connected, open domains in \( \mathbb{R}^d \) with compact closure. We will denote by \( X_t \) a diffusion in \( \mathbb{R}^d \) with corresponding generator \( L \) a uniformly elliptic operator of divergence form. We will write \( Lf = \text{div}(af) \) where the coefficient matrix \( a = a_{ij}(x) \) is smooth and symmetric.

Let \( \tau = \tau(\omega) = \inf\{t \geq 0 : X_t(\omega) \notin D\} \) be the first exit time of \( X_t \) from \( D \).

We study the average \( k \)th moment of the exit time for a particle driven by \( X_t \), starting in \( D \):

\[ \mathcal{E}_k = \mathcal{E}_k(D) = \int_D E_x(\tau^k)dx \]

where \( E_x \) denotes expectation under the measure \( P_x \) satisfying \( P_x\{X_0 = x\} = 1 \), for all \( x \in \mathbb{R}^d \). Note that \( \mathcal{E}_k \) is invariant under Euclidean motions.

We give a variational characterization of \( \mathcal{E}_k \) for each positive integer value of \( k \) in the following theorem:

**Theorem 1.1.** Let \( X_t \) be a diffusion on \( \mathbb{R}^d \) with generator \( L \) a uniformly elliptic operator of divergence form, \( Lf = \text{div}(af) \), where the coefficient matrix \( a \) is smooth and symmetric. Let \( D \) be a smoothly bounded open domain in \( \mathbb{R}^d \) with compact closure, \( D \). Define \( \mathcal{E}_k \) as above and let \( \mathcal{F}_k \) be defined by

\[ \mathcal{F}_k = \left\{ f \in C^\infty(\bar{D}); \int_D f(x)dx \neq 0, f = Lf = \cdots = L^{k-1}f = 0 \text{ on } \partial D \right\}. \]
Let \( \lfloor \frac{k}{2} \rfloor \) be the greatest integer of \( \frac{k}{2} \). Then, for \( k \) even,

\[
\mathcal{E}_k = k! \sup_{f \in \mathcal{F}} \frac{(\int_D f)^2}{\int_D |L^{\lfloor \frac{k}{2} \rfloor} f|^2}
\]

and for \( k \) odd,

\[
\mathcal{E}_k = k! \sup_{f \in \mathcal{F}} \frac{(\int_D f)^2}{\int_D |\nabla L^{\lfloor \frac{k}{2} \rfloor} f|_L^2}
\]

where \( \langle \nabla f, \nabla g \rangle_L = \sum_{i,j} a_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \) is the inner product associated with \( L \).

The proof of Theorem 1.1 is an application of the generalized Dynkin formula [AK] (cf. also [P1]), followed by an explicit computation. That smooth minimizers for the variational principles cited in Theorem 1.1 exist is explicit in our computations.

Our study of the sequence \( \{\mathcal{E}_k\} \) is largely motivated by the now classic work in spectral analysis concerning to what extent a smoothly bounded domain in Euclidean space is determined by its Dirichlet spectrum. More precisely, when the diffusion is standard Brownian motion with generator \( L = \frac{1}{2} \Delta \), we are interested in studying to what extent the sequence \( \{\mathcal{E}_k\} \) determines the geometry of the underlying domain. There are a number of preliminary results in this direction. For example, in [KMM] the authors prove that among domains of a fixed volume, each element of the sequence is maximized if and only if the underlying domain is a ball of the appropriate volume.

For the case \( k = 1 \), it is known that the functional \( \mathcal{E}_1 \) computes the torsional rigidity of a domain. The St. Venant torsion problem, a problem with a long and distinguished history, is to determine those domains of a given volume which maximize torsional rigidity. The problem was settled by Polya (cf. [P2]) who proved that among domains of a fixed volume, the torsional rigidity is maximized by a ball. This result can be recovered using (1.2) and properties of the quotient given in (1.2) under symmetric rearrangement (cf. also [KM1]).

2. Basic results and definitions

Let \( (\Omega, \mathcal{B}) \) be a measurable space and \( \{P_x\}_{x \in \mathbb{R}^d} \) a family of probability measures on \( (\Omega, \mathcal{B}) \). Let \( \{X_t\}_{t \geq 0} \) denote a \( d \)-dimensional diffusion with generator \( L \), a uniformly elliptic operator in divergence form and for which \( P_x\{X_0 = x\} = 1 \), for \( x \in \mathbb{R}^d \).

Let \( D \) be a smoothly bounded, connected, open domain with compact closure. As in the introduction, we define the first exit time for a particle driven by \( X_t \) from \( D \) by \( \tau = \tau(\omega) = \inf \{t : X_t(\omega) \notin D\} \). For each \( x \in \mathbb{R}^d \), we will denote the expected value of a random variable \( Y \) under the probability measure \( P_x \) by \( E_x(Y) \).

There is a useful relationship between the solution of a certain Poisson problem on the domain \( D \) and the expected value of the \( k \)th power of the first exit time of a particle driven by \( X_t \) from \( D \) starting at \( x \in D \). Suppose \( u_k \) solves the problem

\[
L^k u_k + (-1)^{k-1} k! = 0 \text{ on } D,
\]

\[
u_k = Lu_k = \cdots L^{k-1} u_k = 0 \text{ on } \partial D.
\]
Note that $u_k$ can be defined inductively by
\begin{align}
Lu_1 + 1 &= 0 \text{ on } D, \\
        u_1 &= 0 \text{ on } \partial D \tag{2.1}
\end{align}
and
\begin{align}
Lu_k + ku_{k-1} &= 0 \text{ on } D, \\
        u_k &= 0 \text{ on } \partial D. \tag{2.2}
\end{align}

Using the generalized Dynkin formula [H] (cf. also [AK] and [P1]) we have
\begin{align*}
E_x[u_k(X_0)] - E_x[u_k(X_\tau)] &= \sum_{j=1}^{k-1} \frac{(-1)^j}{j!} E_x[\tau^j L^j u_k(X_\tau)] \\
&\quad + \frac{(-1)^k}{(k-1)!} E_x \left[ \int_0^\tau s^{k-1} L^k u_k(X_s) ds \right].
\end{align*}
Using the definition of $u_k$ and $\tau$, this gives $u_k(x) = E_x[\tau^k]$ and $\mathcal{E}_k$ can be expressed in terms of $u_k$ by $\mathcal{E}_k(D) = \int_D u_k(x) dx$.

We will need a number of integral formulae involving the function $u_1$ and the geometry of the diffusion $L$. To ease notation in the sequel we define, for $\alpha$ and $\beta$ tangent vectors at $x \in D$, a scalar product, \( \langle \alpha, \beta \rangle_L \), by
\begin{equation}
\langle \alpha, \beta \rangle_L = \alpha^T a(x) \beta \tag{2.3}
\end{equation}
where $\alpha^T$ denotes the transpose of $\alpha$.

Let $u_1$ be as defined in (2.1) and let $f \in \mathcal{F}_k$. Let $\nu$ be the outward pointing unit normal vector to $\partial D$. By the Divergence Theorem,
\begin{equation}
\int_D f L u_1 - u_1 L f = \int_{\partial D} f \langle \nabla u_1, \nu \rangle_L - u_1 \langle \nabla f, \nu \rangle_L = 0. \tag{2.4}
\end{equation}
We conclude
\begin{equation}
\int_D f = - \int_D u_1 L f.
\end{equation}
If $X$ is a vectorfield on $D$ and $f \in \mathcal{F}_k$, then $\text{div}(f X) = f \text{div}(X) + \langle \nabla f, X \rangle$ where $\langle \alpha, \beta \rangle$ is the standard scalar product. By the Divergence Theorem,
\begin{equation}
\int_D \text{div}(f X) = \int_{\partial D} f \langle X, \nu \rangle = 0
\end{equation}
and we conclude that
\begin{equation}
\int_D f \text{div}(X) = - \int_D \langle \nabla f, X \rangle.
\end{equation}
In particular, if $u_1$ is as defined in (2.1) and $X = a_{ij}(x) \nabla u_1$, then
\begin{equation}
\int_D f = \int_D \langle \nabla u_1, \nabla f \rangle_L. \tag{2.5}
\end{equation}
3. Variational characterizations

Throughout this section let $D$ be as above and let $F_k$ be given as in Theorem 1.1. We begin with a lemma which generalizes (2.4) and (2.5).

**Lemma 3.1.** Let $u_n$ be as defined by (2.2), let $k$ be a positive integer, and let $f \in F_k$. If $k = 2n$, then

$$\int_D f = \frac{(-1)^n}{n!} \int_D u_n L^n f.$$  

(3.1)

If $k = 2n + 1$, then

$$\int_D f = \frac{(-1)^n}{(n+1)!} \int_D \langle \nabla u_{n+1}, \nabla L^n f \rangle_L$$

(3.2)

where the scalar product is as given in (2.3).

**Proof.** Suppose that $k = 2n$ and for $0 \leq l \leq n - 1$, define

$$P_l = (L^l u_n) (L^{n-l} f) - (L^{l+1} u_n) (L^{n-l-1} f).$$

Then

$$\sum_{l=0}^{n-1} P_l = u_n L^n f - f L^n u_n.$$  

(3.3)

Let $\nu$ be the outward pointing unit normal vector along $\partial D$. By the Divergence Theorem and the fact that $L^l u_n = 0$ on $\partial D$, for $l = 0, \ldots, n - 1$,

$$\int_D P_l = \int_{\partial D} (L^l u_n) \langle \nabla L^{n-l-1} f, \nu \rangle_L - (L^{n-l-1} f) \langle \nabla L^l u_n, \nu \rangle_L = 0.$$  

(3.4)

Combining (3.3) and (3.4) and using that $L^n u_n = (-1)^n n!$, we have established (3.1).

Suppose $k = 2n + 1$ and for $0 \leq l \leq n$, define

$$R_l = (L^l u_{n+1}) (L^{n+1-l} f) - (L^{l+1} u_{n+1}) (L^{n+1-l-1} f).$$

Then

$$\sum_{l=0}^{n} R_l = u_{n+1} L^{n+1} f - f L^{n+1} u_{n+1}.$$  

As above, we use the Divergence Theorem to see that

$$\int_D R_l = \int_{\partial D} (L^l u_{n+1}) \langle \nabla L^{n-l-1} f, \nu \rangle_L - (L^{n-l-1} f) \langle \nabla L^l u_{n+1}, \nu \rangle_L = 0.$$  

Since $L^{n+1} u_{n+1} = (-1)^{n+1} (n+1)!$, we conclude

$$\int_D f = \frac{(-1)^{n+1}}{(n+1)!} \int_D u_{n+1} L(L^n f).$$

If $X$ is the vectorfield given by $X = a \nabla (L^n f)$, then following the argument used to establish (2.5),

$$\int_D u_{n+1} L(L^n f) = \int_D u_{n+1} \text{div}(X)$$

$$= - \int_D \langle \nabla u_{n+1}, \nabla L^n f \rangle_L$$

and we have established (3.2).
We now prove Theorem 1.1. Suppose \( k = 2n \) and, for \( f \in \mathcal{F}_k \), consider the quotient

\[
Q_k(f) = \frac{\left( \int_D f \right)^2}{\int_D |L^n f|^2}.
\]

From (3.1)

\[
Q_k(f) = \frac{\left( \frac{1}{n!} \right)^2 \left( \int_D u_n L^n f \right)^2}{\int_D |L^n f|^2}.
\]

Let \( \mathcal{G}_k = \{ g \in \mathcal{F}_n : g = L^n f \text{ for some } f \in \mathcal{F}_k \} \). Let \( \mathcal{H}_k \) be the completion of \( \mathcal{G}_k \) in the Hilbert space, \( L^2 \), of square integrable functions on \( D \). If we denote the inner product of \( g \) and \( h \in L^2 \) by \( \langle g, h \rangle \) and by \( \| g \| \) the \( L^2 \) norm of \( g \), then we can view \( Q_k \) as a map \( Q_k : \mathcal{G}_k \subset \mathcal{H}_k \to \mathbb{R} \),

\[
Q_k(g) = \left( \frac{1}{n!} \right)^2 \left( \frac{\langle u_n, g \rangle}{\| g \|} \right)^2.
\]

Clearly, the domain of \( Q_k \) can be extended to nonzero elements of \( \mathcal{H}_k \) and \( Q_k(c g) = Q_k(g) \) for every nonzero scalar \( c \). It follows that \( Q_k \) is maximized when \( g \in \mathcal{H}_k \) is in the direction of \( u_n \in \mathcal{H}_k \). If \( g = c u_n \) we have \( L^n(c' u_{2n}) = g \), and computing \( Q_k(c u_n) \) we see that

\[
\sup_{g \in \mathcal{H}_k} Q_k(g) = Q_k(c u_n)
\]

\[
= \frac{\left( \int_D u_{2n} \right)^2}{\int_D |L^n u_{2n}|^2}
\]

where we have applied (3.1) of Lemma 3.1 to the numerator. Note that \( (L^n u_{2n})^2 = (-1)^n (2n)! u_n L u_{2n} \). Applying Lemma 3.1 to the denominator we obtain

\[
\sup_{g \in \mathcal{H}_k} Q_k(g) = \frac{\left( \int_D u_{2n} \right)^2}{(2n)! \int_D u_{2n}}
\]

\[
= \frac{1}{k!} \mathcal{E}_k(D)
\]

which establishes (1.1) of Theorem 1.1.

The proof of (1.2) of Theorem 1.1 is similar. Suppose \( k = 2n+1 \) and, for \( f \in \mathcal{F}_k \), consider the quotient

\[
\tilde{Q}_k(f) = \frac{\left( \int_D f \right)^2}{\int_D |\nabla L^n f|^2}.
\]

From (3.2) of Lemma 3.1,

\[
\tilde{Q}_k(f) = \frac{\left( \frac{1}{(n+1)!} \right)^2 \left( \int_D \langle \nabla u_{n+1}, \nabla L^n f \rangle_L \right)^2}{\int_D |\nabla L^n f|^2}.
\]

Let \( C^\infty(\bar{D}, \mathbb{R}^d) \) be the space of smooth vectorfields on \( \bar{D} \). Let

\[
\hat{\mathcal{G}}_k = \{ X \in C^\infty(\bar{D}, \mathbb{R}^d) : X = \nabla g \text{ for some } g \in \mathcal{F}_{n+1} \text{ with } g = L^n f \text{ for some } f \in \mathcal{F}_k \}.
\]
Let $\tilde{H}_k$ be the completion of $\tilde{G}_k$ in the space of vectorfields square integrable with respect to the inner product $\langle \alpha, \beta \rangle_L$. We can view $\tilde{Q}_k$ as a map $\tilde{Q}_k : \tilde{G}_k \subset \tilde{H}_k \to \mathbb{R}$, 

$$\tilde{Q}_k(g) = \left( \frac{1}{(n+1)!} \right)^2 \left( \frac{\langle \nabla u_{n+1}, g \rangle_L}{\|g\|_L} \right)^2.$$ 

It is clear that the domain of $\tilde{Q}_k$ extends to nonzero vectors in the space $\tilde{H}_k$ and that for all nonzero scalars $c$, $\tilde{Q}_k(cg) = \tilde{Q}_k(g)$. It follows that $\tilde{Q}_k$ is maximized when $g = c\nabla u_{n+1}$ where $c$ is some nonzero constant. Computing $\tilde{Q}_k(\nabla u_{n+1})$ we see that 

$$\sup_{g \in \tilde{H}_k} \tilde{Q}_k(g) = \tilde{Q}_k(c\nabla u_{n+1})$$ 

$$= \frac{\left( \int_D u_{2n+1} \right)^2}{\|\nabla L^n u_{2n+1}\|_L^2}$$ 

where we have used (3.2) on the numerator.

Note that $\|\nabla L^n u_{2n+1}\|_L^2 = \left( \frac{(-1)^n}{n+1} \right)^2 (2n+1) \langle \nabla u_{n+1}, \nabla L^n u_{2n+1} \rangle_L$. Applying (3.2) of Lemma 3.1 to the denominator we obtain 

$$\sup_{g \in \tilde{H}_k} \tilde{Q}_k(g) = \frac{\left( \int_D u_{2n+1} \right)^2}{(2n+1)! \int_D u_{2n+1}}$$ 

$$= \frac{1}{k!} \mathcal{E}_k(D)$$ 

which establishes (1.2) of Theorem 1.1.

REFERENCES


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