VARIATIONAL PRINCIPLES FOR AVERAGE EXIT TIME MOMENTS FOR DIFFUSIONS IN EUCLIDEAN SPACE

KIMBERLY K. J. KINATEDER AND PATRICK MCDONALD

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Abstract. Let $D$ be a smoothly bounded domain in Euclidean space and let $X_t$ be a diffusion in Euclidean space. For a class of diffusions, we develop variational principles which characterize the average of the moments of the exit time from $D$ of a particle driven by $X_t$, where the average is taken over all starting points in $D$.

1. Introduction

In this note we study diffusions on $\mathbb{R}^d$ and properties of their corresponding exit times from smoothly bounded, connected, open domains in $\mathbb{R}^d$ with compact closure. We will denote by $X_t$ a diffusion in $\mathbb{R}^d$ with corresponding generator $L$ a uniformly elliptic operator of divergence form. We will write $Lf = \text{div}(a\nabla f)$ where the coefficient matrix $a = a_{ij}(x)$ is smooth and symmetric.

Let $\tau = \tau(\omega) = \inf\{t \geq 0 : X_t(\omega) \notin D\}$ be the first exit time of $X_t$ from $D(S)$.

We study the average $k$th moment of the exit time for a particle driven by $X_t$, starting in $D$:

$$E_k = E_k(D) = \int_D E_x(\tau^k)dx$$

where $E_x$ denotes expectation under the measure $P_x$ satisfying $P_x\{X_0 = x\} = 1$, for all $x \in \mathbb{R}^d$. Note that $E_k$ is invariant under Euclidean motions.

We give a variational characterization of $E_k$ for each positive integer value of $k$ in the following theorem:

**Theorem 1.1.** Let $X_t$ be a diffusion on $\mathbb{R}^d$ with generator $L$ a uniformly elliptic operator of divergence form, $Lf = \text{div}(a\nabla f)$, where the coefficient matrix $a$ is smooth and symmetric. Let $D$ be a smoothly bounded open domain in $\mathbb{R}^d$ with compact closure, $D$. Define $E_k$ as above and let $F_k$ be defined by

$$F_k = \left\{ f \in C^\infty(\bar{D}) ; \int_D f(x)dx \neq 0, f = Lf = \cdots = L^{k-1}f = 0 \text{ on } \partial D \right\}.$$
Let \( \lfloor \frac{k}{2} \rfloor \) be the greatest integer of \( \frac{k}{2} \). Then, for \( k \) even,

\[
\mathcal{E}_k = k! \sup_{f \in \mathcal{F}_k} \frac{(\int_D f)^2}{\int_D |L^{\lfloor \frac{k}{2} \rfloor} f|^2}
\]

and for \( k \) odd,

\[
\mathcal{E}_k = k! \sup_{f \in \mathcal{F}_k} \frac{(\int_D f)^2}{\int_D |\nabla L^{\lfloor \frac{k}{2} \rfloor} f|^2}
\]

where \( \langle \nabla f, \nabla g \rangle_L = \sum_{i,j} a_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \) is the inner product associated with \( L \).

The proof of Theorem 1.1 is an application of the generalized Dynkin formula [AK] (cf. also [P1]), followed by an explicit computation. That smooth minimizers for the variational principles cited in Theorem 1.1 exist is explicit in our computations.

Our study of the sequence \( \{\mathcal{E}_k\} \) is largely motivated by the now classic work in spectral analysis concerning to what extent a smoothly bounded domain in Euclidean space is determined by its Dirichlet spectrum. More precisely, when the diffusion is standard Brownian motion with generator \( L = \frac{1}{2} \Delta \), we are interested in studying to what extent the sequence \( \{\mathcal{E}_k\} \) determines the geometry of the underlying domain. There are a number of preliminary results in this direction. For example, in [KMM] the authors prove that among domains of a fixed volume, each element of the sequence is maximized if and only if the underlying domain is a ball of the appropriate volume.

For the case \( k = 1 \), it is known that the functional \( \mathcal{E}_1 \) computes the torsional rigidity of a domain. The St. Venant torsion problem, a problem with a long and distinguished history, is to determine those domains of a given volume which maximize torsional rigidity. The problem was settled by Polya (cf. [P2]) who proved that among domains of a fixed volume, the torsional rigidity is maximized by a ball. This result can be recovered using (1.2) and properties of the quotient given in (1.2) under symmetric rearrangement (cf. also [KM1]).
Note that \( u_k \) can be defined inductively by
\[
Lu_1 + 1 = 0 \text{ on } D,
\]
\[
u_1 = 0 \text{ on } \partial D
\]
and
\[
Lu_k + k u_{k-1} = 0 \text{ on } D,
\]
\[
u_k = 0 \text{ on } \partial D.
\]
Using the generalized Dynkin formula [H] (cf. also [AK] and [P1]) we have
\[

\begin{align*}
E_x[u_k(X_0)] - E_x[u_k(X_\tau)] &= \sum_{j=1}^{k-1} \frac{(-1)^j}{j!} E_x[\tau^j L u_k(X_\tau)] \\
&\quad + \frac{(-1)^k}{(k-1)!} E_x\left[ \int_0^\tau s^{k-1} L^k u_k(X_s) ds \right].
\end{align*}
\]
Using the definition of \( u_k \) and \( \tau \), this gives \( u_k(x) = E_x[\tau^k] \) and \( \mathcal{E}_k \) can be expressed in terms of \( u_k \) by \( \mathcal{E}_k(D) = \int_D u_k(x) dx \).

We will need a number of integral formulae involving the function \( u_1 \) and the geometry of the diffusion \( L \). To ease notation in the sequel we define, for \( \alpha \) and \( \beta \) tangent vectors at \( x \in D \), a scalar product, \( \langle \alpha, \beta \rangle_L \), by
\[
\langle \alpha, \beta \rangle_L = \alpha^T a(x) \beta
\]
where \( \alpha^T \) denotes the transpose of \( \alpha \).

Let \( u_1 \) be as defined in (2.1) and let \( f \in \mathcal{F}_k \). Let \( \nu \) be the outward pointing unit normal vector to \( \partial D \). By the Divergence Theorem,
\[
\int_D f Lu_1 - u_1 Lf = \int_{\partial D} f \langle \nabla u_1, \nu \rangle_L - u_1 \langle \nabla f, \nu \rangle_L = 0.
\]
We conclude
\[
\int_D f = - \int_D u_1 Lf.
\]
If \( X \) is a vector field on \( D \) and \( f \in \mathcal{F}_k \), then \( \text{div}(fX) = f \text{div}(X) + \langle \nabla f, X \rangle \) where \( \langle \alpha, \beta \rangle \) is the standard scalar product. By the Divergence Theorem,
\[
\int_D \text{div}(fX) = \int_{\partial D} f \langle X, \nu \rangle = 0
\]
and we conclude that
\[
\int_D f \text{div}(X) = - \int_D \langle \nabla f, X \rangle.
\]
In particular, if \( u_1 \) is as defined in (2.1) and \( X = a_{ij}(x) \nabla u_1 \), then
\[
\int_D f = \int_D \langle \nabla u_1, \nabla f \rangle_L.
\]
3. Variational characterizations

Throughout this section let $D$ be as above and let $F_k$ be given as in Theorem 1.1. We begin with a lemma which generalizes (2.4) and (2.5).

**Lemma 3.1.** Let $u_n$ be as defined by (2.2), let $k$ be a positive integer, and let $f \in F_k$. If $k = 2n$, then

$$(3.1) \quad \int_D f = \frac{(-1)^n}{n!} \int_D u_n L^n f.$$ 

If $k = 2n + 1$, then

$$(3.2) \quad \int_D f = \frac{(-1)^n}{(n + 1)!} \int_D \langle \nabla u_{n+1}, \nabla L^n f \rangle_L$$

where the scalar product is as given in (2.3).

**Proof.** Suppose that $k = 2n$ and for $0 \leq l \leq n - 1$, define

$$P_l = (L^l u_n)(L^{n-l} f) - (L^{l+1} u_n)(L^{n-(l+1)} f).$$

Then

$$(3.3) \quad \sum_{l=0}^{n-1} P_l = u_n L^n f - f L^n u_n.$$

Let $\nu$ be the outward pointing unit normal vector along $\partial D$. By the Divergence Theorem and the fact that $L^l u_n = 0$ on $\partial D$, for $l = 0, \ldots, n - 1$,

$$(3.4) \quad \int_D P_l = \int_{\partial D} (L^l u_n) \langle \nabla L^{n-l-1} f, \nu \rangle_L - (L^{n-(l+1)} f) \langle \nabla L^l u_n, \nu \rangle_L = 0.$$ 

Combining (3.3) and (3.4) and using that $L^n u_n = (-1)^n n!$, we have established (3.1).

Suppose $k = 2n + 1$ and for $0 \leq l \leq n$, define

$$R_l = (L^l u_{n+1})(L^{n+1-l} f) - (L^{l+1} u_{n+1})(L^{n+1-(l+1)} f).$$

Then

$$(3.5) \quad \sum_{l=0}^{n} R_l = u_{n+1} L^{n+1} f - f L^{n+1} u_{n+1}.$$ 

As above, we use the Divergence Theorem to see that

$$\int_D R_l = \int_{\partial D} (L^l u_{n+1}) \langle \nabla L^{n-l-1} f, \nu \rangle_L - (L^{n-l} f) \langle \nabla L^l u_{n+1}, \nu \rangle_L = 0.$$ 

Since $L^{n+1} u_{n+1} = (-1)^{n+1}(n + 1)!$, we conclude

$$\int_D f = \frac{(-1)^{n+1}}{(n + 1)!} \int_D u_{n+1} L(L^n f).$$

If $X$ is the vectorfield given by $X = a \nabla (L^n f)$, then following the argument used to establish (2.5),

$$\int_D u_{n+1} L(L^n f) = \int_D u_{n+1} \text{div}(X)$$

$$= -\int_D \langle \nabla u_{n+1}, \nabla L^n f \rangle_L$$

and we have established (3.2). \qed
We now prove Theorem 1.1. Suppose $k = 2n$ and, for $f \in \mathcal{F}_k$, consider the quotient

\[ Q_k(f) = \frac{\left( \int_D f \right)^2}{\int_D |L^n f|^2}. \]

From (3.1)

\[ Q_k(f) = \left( \frac{1}{n!} \right)^2 \frac{\left( \int_D u_n L^n f \right)^2}{\int_D |L^n f|^2}. \]

Let \( G_k = \{ g \in \mathcal{F}_n : g = L^n f \text{ for some } f \in \mathcal{F}_k \} \). Let \( \mathcal{H}_k \) be the completion of \( G_k \) in the Hilbert space, \( L^2 \), of square integrable functions on \( D \). If we denote the inner product of \( g \) and \( h \) by \( \langle g, h \rangle \) and by \( \|g\| \) the \( L^2 \) norm of \( g \), then we can view \( Q_k \) as a map \( Q_k : G_k \subset \mathcal{H}_k \rightarrow \mathbb{R} \),

\[ Q_k(g) = \left( \frac{1}{n!} \right)^2 \frac{\left( \langle u_n, g \rangle \right)^2}{\|g\|}. \]

Clearly, the domain of \( Q_k \) can be extended to nonzero elements of \( \mathcal{H}_k \) and \( Q_k(cg) = Q_k(g) \) for every nonzero scalar \( c \). It follows that \( Q_k \) is maximized when \( g \in \mathcal{H}_k \) is in the direction of \( u_n \in \mathcal{H}_k \). If \( g = cu_n \) we have \( L^n(c'u_{2n}) = g \), and computing \( Q_k(cu_n) \) we see that

\[ \sup_{g \in \mathcal{H}_k} Q_k(g) = Q_k(cu_n) = \frac{\left( \int_D u_{2n} \right)^2}{\int_D (L^nu_{2n})^2} \]

where we have applied (3.1) of Lemma 3.1 to the numerator. Note that \( (L^nu_{2n})^2 = \frac{(-1)^n}{n!}(2n)!u_n Lu_{2n} \). Applying Lemma 3.1 to the denominator we obtain

\[ \sup_{g \in \mathcal{H}_k} Q_k(g) = \frac{\left( \int_D u_{2n} \right)^2}{(2n)! \int_D u_{2n}} = \frac{1}{k!} E_k(D) \]

which establishes (1.1) of Theorem 1.1.

The proof of (1.2) of Theorem 1.1 is similar. Suppose \( k = 2n+1 \) and, for \( f \in \mathcal{F}_k \), consider the quotient

\[ \tilde{Q}_k(f) = \frac{\left( \int_D f \right)^2}{\int_D |\nabla L^n f|^2}. \]

From (3.2) of Lemma 3.1,

\[ \tilde{Q}_k(f) = \left( \frac{1}{(n+1)!} \right)^2 \frac{\left( \int_D \langle \nabla u_{n+1}, \nabla L^n f \rangle \right)^2}{\int_D |\nabla L^n f|^2}. \]

Let \( C^\infty(\bar{D}, \mathbb{R}^d) \) be the space of smooth vectorfields on \( \bar{D} \). Let

\[ \tilde{G}_k = \{ X \in C^\infty(\bar{D}, \mathbb{R}^d) : X = \nabla g \text{ for some } g \in \mathcal{F}_{n+1} \} \]

with \( g = L^n f \) for some \( f \in \mathcal{F}_k \).
Let $\tilde{\mathcal{H}}_k$ be the completion of $\tilde{\mathcal{G}}_k$ in the space of vector fields square integrable with respect to the inner product $\langle \alpha, \beta \rangle_L$. We can view $\tilde{Q}_k$ as a map $\tilde{Q}_k : \tilde{\mathcal{G}}_k \subset \tilde{\mathcal{H}}_k \to \mathbb{R}$, where

$$\tilde{Q}_k(g) = \left( \frac{1}{(n+1)!} \right)^2 \left( \frac{\langle \nabla u_{n+1}, g \rangle_L}{\|g\|_L} \right)^2.$$

It is clear that the domain of $\tilde{Q}_k$ extends to nonzero vectors in the space $\tilde{\mathcal{H}}_k$ and that for all nonzero scalars $c$, $\tilde{Q}_k(cg) = \tilde{Q}_k(g)$. It follows that $\tilde{Q}_k$ is maximized when $g = c\nabla u_{n+1}$ where $c$ is some nonzero constant. Computing $\tilde{Q}_k(\nabla u_{n+1})$ we see that

$$\sup_{g \in \tilde{\mathcal{H}}_k} \tilde{Q}_k(g) = \tilde{Q}_k(c\nabla u_{n+1})$$

$$= \frac{\left( \int_D u_{2n+1} \right)^2}{\int_D \|\nabla L^n u_{n+1}\|_L^2}$$

where we have used (3.2) on the numerator.

Note that $\|\nabla L^n u_{n+1}\|_L^2 = \left( \frac{-1}{n+1} \right)^n (2n+1)! \langle \nabla u_{n+1}, \nabla L^n u_{2n+1} \rangle_L$. Applying (3.2) of Lemma 3.1 to the denominator we obtain

$$\sup_{g \in \tilde{\mathcal{H}}_k} \tilde{Q}_k(g) = \frac{\left( \int_D u_{2n+1} \right)^2}{(2n+1)! \int_D u_{2n+1}}$$

$$= \frac{1}{k!} E_k(D)$$

which establishes (1.2) of Theorem 1.1.

REFERENCES


Department of Mathematics, Wright State University, Dayton, Ohio 45435

Department of Mathematics, New College of the University of South Florida, Sarasota, Florida 34243

E-mail address: pmacdonal@virtu.sar.usf.edu