ON THE EXISTENCE OF MAXIMAL COHEN-MACAULAY MODULES OVER \( p \)th ROOT EXTENSIONS

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Abstract. Let \( S \) be an unramified regular local ring having mixed characteristic \( p > 0 \) and \( R \) the integral closure of \( S \) in a \( p \)th root extension of its quotient field. We show that \( R \) admits a finite, birational module \( M \) such that \( \text{depth}(M) = \dim(R) \). In other words, \( R \) admits a maximal Cohen-Macaulay module.

1. Introduction

Let \( R \) be a Noetherian local ring. In considering the local homological conjectures over \( R \), one may reduce to the situation where \( R \) is a finite extension of an unramified regular local ring \( S \). Therefore, it is a natural point of departure to assume that \( R \) is the integral closure of \( S \) in a “well-behaved” algebraic extension of its quotient field. Certainly, when \( S \) has mixed characteristic \( p > 0 \), one ought to consider the case that \( R \) is the integral closure of \( S \) in an extension of its quotient field obtained by adjoining the \( p \)th root of an element of \( S \). This was done in [Ko] where it was shown that \( S \) is a direct summand of \( R \), i.e., the Direct Summand Conjecture holds for the extension \( S \subseteq R \). In this note we show that a number of the other local homological conjectures hold for such \( R \) by showing that \( R \) admits a finite, birational module \( M \) satisfying \( \text{depth}(M) = \dim(R) \) (see [H]). In other words, \( R \) admits a maximal Cohen-Macaulay module. Such a module is necessarily free over \( S \). Aside from regularity, one of the crucial points in the mixed characteristic case seems to be that \( S/pS \) is integrally closed. By contrast, using an example from [HM], Roberts has noted that even if \( S \) is a Cohen-Macaulay UFD and \( R \) is the integral closure of \( S \) in a quadratic extension of quotient fields, \( R \) needn’t admit a finite, \( S \)-free module at all (see [R]). For the example in question, \( S \) has mixed characteristic 2, yet \( S/2S \) is not integrally closed.

2. Preliminaries

In this section we will establish our notation and present a few preliminary observations. Throughout, \( S \) will be a Noetherian normal domain with quotient field \( L \). We assume \( \text{char}(L) = 0 \). Fix \( p \in \mathbb{Z} \) to be a prime integer and suppose that either \( p \) is a unit in \( S \) or that \( pS \) is a (proper) prime ideal and \( S/pS \) is integrally closed. Let \( f \in S \) be an element that is not a \( p \)th power and select \( W \) an indeterminate. Write \( F(W) := W^p - f \in S[W] \), a monic irreducible polynomial.
and let $R$ denote the integral closure of $S$ in $K := L(\omega)$, for $\omega$ a root of $F(W)$. Thus $R$ is the integral closure of $S[\omega]$.

Our strategy in this paper is to exploit the fact that $R$ can be realized as $J^{-1}$ for a suitable ideal $J \subseteq S[\omega]$. The study of birational algebras of the form $J^{-1}$ seems to have captured the attention of a number of researchers during the last few years, albeit in notably different contexts (see [EU], [Ka], [KU], [MP] and [V]). Since $J^{-1}$ inherits $S_2$ from $S[\omega]$, this means that in attempting to “construct” $R$, if the candidate is $J^{-1}$ for some $J$, then only the condition $R_1$ must be checked.

The following proposition summarizes some of the conditions relating $R$ to $J^{-1}$ for suitable $J$ that we will call upon in the next section. Parts (i) and (ii) of the proposition were inspired by the main results in [V] and Proposition 3.1 in [KU]. Special cases of part (iii) of the proposition have apparently been known to algebraic geometers for a long while. For some historical comments and fascinating variations, the interested reader should consult [KU].

**Proposition 2.1.** Let $A$ be a Noetherian domain satisfying $S_2$ and assume that $A'$, the integral closure of $A$, is a finite $A$-module.

(i) Suppose $\{P_1, \ldots, P_n\}$ are the height one primes of $A$ for which $A_{P_i}$ is not a DVR. If for each $1 \leq i \leq n$, $\text{rad}(J_i) = P_i$ and $(J_i^{-1})_{P_i} = A_{P_i}$, then $A' = J^{-1}$, for $J := J_1 \cap \cdots \cap J_n$.

(ii) If $A \neq A'$, then $A' = J^{-1}$, for some height one unmixed ideal $J \subseteq A$. Moreover, if $A$ is Gorenstein in codimension one, then $A' = J^{-1}$ for a unique height one unmixed ideal $J$ satisfying $J \cdot J^{-1} = J = (J^{-1})^{-1}$.

(iii) Suppose that $A = B/(F)$ for $F \in B$ a principal prime and $J \subseteq B$ is a grade two ideal arising as the ideal of $n \times n$ minors of an $(n+1) \times n$ matrix $\phi$. Assume further that $F \in \overline{J}$ and set $J := J/(F)$. Let $\Delta_1, \ldots, \Delta_{n+1}$ denote the signed minors of $\phi$, write $F := b_1 \Delta_1 + \cdots + b_{n+1} \Delta_{n+1}$ and let $\phi'$ denote the $(n+1) \times (n+1)$ matrix obtained by augmenting the column of $b_i$s to $\phi$ (so $F$ is the determinant of $\phi'$). Then $J^{-1}$ can be generated as an $A$-module by $\{\psi_{1,1}/\delta_1, \ldots, \psi_{n+1,n+1}/\delta_{n+1} = 1\}$, where $\psi_{i,i}$ denotes the image of $\frac{\Delta_i}{\delta_i}$ in $A$ of the $(i, i)\text{th}$ cofactor of $\phi$ and $\delta_i$ denotes the image of $\Delta_i$ in $A$ (which we assume to be non-zero). Moreover, $p.d. B(J) = p.d. B(J^{-1}) = 1$.

**Proof.** To prove (i), note that $J^{-1}_Q = A^{-1}_Q$ for all height one primes $Q \subseteq A$. Since $J^{-1}$ and $A'$ are birational and satisfy $S_2$, we obtain $J^{-1} = A'$. For the first statement in (ii), we may, by part (i), consider the case where $A$ is a one-dimensional local ring which is not a DVR. Let $Q$ denote the maximal ideal of $A$. Then $QQ^{-1} \subseteq Q$. Since it always holds that $Q \subseteq QQ^{-1}$, we have $Q = QQ^{-1}$. Therefore $Q^{-1}$ is a finite ring extension properly containing $A$ (since for any ideal $J$, $(JJ^{-1})^{-1}$ is a ring). If $Q^{-1} = A'$, we’re done. If not, then since $Q^{-1}$ inherits $S_2$ from $A$, $Q^{-1}$ contains a height one prime $P$ for which $(Q^{-1})_P$ is not a DVR. Thus $P^{-1}$ is a finite ring extension properly containing $Q^{-1}$. An easy calculation shows that $P^{-1}$, considered over $Q^{-1}$, equals $(QP)^{-1}$, considered over $A$. Iterating this process shows we eventually obtain $A' = J^{-1}$, for some $J \subseteq A$. Now suppose that $A$ is Gorenstein in codimension one. Then $I_Q = (I^{-1})^{-1}_Q$, for all ideals $I \subseteq A$ and all height one primes $Q \subseteq A$. Therefore, $I = (I^{-1})^{-1}$, for all height one, unmixed ideals $I \subseteq A$. In particular, this holds for $J$. Moreover, if $J^{-1} = A' = K^{-1}$, for $K$ height one and unmixed, then $J = K$. Finally, since $J^{-1}$ is a ring, $(J \cdot J^{-1})^{-1} = J = J^{-1}$, so $J \cdot J^{-1} \subseteq (J^{-1})^{-1} = J$. Thus, $J \cdot J^{-1} = J$, as desired. For (iii), the description of
the generators for $J^{-1}$ follows either from [MP], Proposition 3.14 or [KU], Lemma 2.5. For the second part of (iii), see [KU], Proposition 3.1.

Returning to our basic set-up, we note that since $S$ is a normal domain, $S[\omega]$ satisfies Serre’s condition $S_2$. Moreover, since $\text{char}(S) = 0$, $R$ is a finite $S$-module. Thus Proposition 2.1 applies. In Section 3 we will identify the ideal $J \subseteq S[\omega]$ for which $J^{-1} = R$. In the meantime, we observe that if $p$ is not a unit in $S$, then there is a unique height one prime in $S[\omega]$ containing $p$. Suppose $p \mid f$. Then $P := (\omega, p)$ is clearly the unique height one prime in $S[\omega]$ containing $p$. Moreover, $S[\omega]_P$ is a DVR if and only if $p^2 \nmid f$. Suppose $p \nmid f$. If $f$ is not a $p$th power modulo $pS$, then $f$ is not a $p$th power over the quotient field of $S/pS$ (since $S/pS$ is integrally closed) and it follows that $F(W)$ is irreducible mod $pS$. Thus $(p, F(W))$ is the unique height two prime in $S[W]$ containing $F(W)$ and $p$, so $pS[\omega]$ is the unique height one prime in $S[\omega]$ containing $p$. If $f \equiv h^p \text{ mod } pS$, then $F(W) \equiv (W - h)^p \text{ mod } pS$ and it follows that $(\omega - h, p)S[\omega]$ is the unique height one prime in $S[\omega]$ containing $p$. Thus, in all cases, there exists a unique height one prime in $S[\omega]$ lying over $pS$.

For the remainder of the paper, we call this prime $P$. Suppose $f = h^p + gp$, so $P = (\omega - h, p)S[\omega]$. Write $\tilde{P} := (W - h, p)S[W]$ for the preimage of $P$ in $S[W]$. Then

$$F(W) = W^p - h^p - gp = (W^{p-1} + \cdots + h^{p-1}) \cdot (W - h) - gp.$$  

In $S[W]$, $W^{p-1} + \cdots + h^{p-1} \equiv ph^{p-1}$ modulo $(W - h)$, so $W^{p-1} + \cdots + h^{p-1} \in \tilde{P}$. Thus, $F(W) \in \tilde{P}^2$ if and only if $p \nmid g$. In other words, in all cases, $F_P$ is not principal if and only if $f = h^p + p^2g$, for some $h, g \in S$.

3. The main result

In this section we will present our main result, Theorem 3.8. Lemmas 3.2 and 3.3 will enable us to describe the ideal $J \subseteq S[\omega]$ for which $R = J^{-1}$. We will then see in the proof of Theorem 3.8 that the module we seek has the form $I^{-1}$, for some ideal $I \subseteq J$.

**Lemma 3.1.** Suppose $p$ is not a unit in $S$, $h \in S/pS$ and $p = 2k + 1$. Set

$$C := \sum_{j=1}^k (-1)^{j+1} \binom{p}{j} (W \cdot h)^j [W^{p-2j} - h^{p-2j}],$$

$$C' := C \cdot (p(W - h))^{-1} \text{ and } \tilde{P} := (p, W - h) \cdot S[W]. \text{ Then } C' \notin \tilde{P}.$$  

**Proof.** Note that since $p$ divides $\binom{p}{j}$ for all $1 \leq j \leq k$, $C'$ is a well-defined element of $S[W]$. Now, $C' \notin \tilde{P}$ if and only if the residue class of $C'$ modulo $W - h$, as an element of $S$, does not belong to $pS$ if and only if $\sum_{j=1}^k (-1)^{j+1} \binom{p}{j} h^{p-1} (p - 2j)$, as an element of $S$, is not divisible by $p$. Since

$$\sum_{j=1}^k (-1)^{j+1} \binom{p}{j} h^{p-1}$$

is divisible by $p$ and $h^{p-1}$ is not divisible by $p$, it is enough to show that

$$\sum_{j=1}^k (-1)^{j+1} \binom{p}{j} \frac{2j}{p}$$
On the other hand, by the discussion in Section 2 that if \( \tau \) is a DVR containing \( \omega \cdot \tau \), then \( \omega \cdot \tau \) is square-free. Therefore, by the standard determinant argument, if \( f \) is square-free, then \( P = R = S[\omega] \) or \( P \) is the only height one prime for which \( S[\omega]_P \) is not a DVR.

**Lemma 3.2.** Suppose \( f \in S \) is square-free and \( S[\omega] \neq R \) (thus \( p \) is not a unit in \( S \)). Then \( R = P^{-1} \). Moreover, \( R \) is a free \( S \)-module.

**Proof.** We first consider the case \( p > 2 \). Since \( S[\omega] \) is not integrally closed, we have \( f = h^p + p^2 g \), for some \( h \) not divisible by \( p \) and \( g \neq 0 \) in \( S \). Thus, \( P = (\omega - h, p)S[\omega] \).

It follows from the proof and statement of Proposition 2.1 that \( P^{-1} \) is a ring and that \( P^{-1} \) is generated as an \( S[\omega] \)-module by \( \{1, \tau\} \), for

\[
\tau = \frac{1}{p} \sum_{j=1}^{P} \omega^{p-j} h^{j-1} = \frac{g \cdot p}{\omega - h}.
\]

Therefore \( P^{-1} = S[\omega, \tau] \). If we show that \( S[\omega, \tau] \) satisfies \( R_1 \), then \( S[\omega, \tau] = R \), since \( P^{-1} \) satisfies \( S_2 \) (as an \( S[\omega] \)-module and as a ring). Since \( f \) is square-free, it suffices to show that \( P_Q^{-1} \) is a DVR for each height one \( Q \subseteq P^{-1} \) containing \( p \). To do this, we find an equation satisfied by \( \tau \) over \( S[\omega] \). On the one hand,

\[
(\omega - h) \cdot \tau = 0 \cdot (w - h) + g \cdot p.
\]

On the other hand,

\[
p \cdot \tau = (\omega - h)p^{p-2} \cdot (\omega - h) + c' \cdot p,
\]

where \( c' \) denotes the image of \( S[W] \) containing \( C' \in S[W] \), defined in Lemma 3.1. Therefore, by the standard determinant argument, \( \tau \) satisfies

\[l(T) := T^2 - c'T - g(\omega - h)p^{p-2}\]

over \( S[\omega] \). Now, let \( \pi : S[W, T] \to S[\omega, \tau] \) denote the canonical map and set \( H := ker(\pi) \) and let \( Q \subseteq S[\omega, \tau] \) be any height one prime containing \( p \). Then \( Q \) corresponds to a height three prime \( Q' \subseteq S[W, T] \) containing \( p \) and \( H \). Since \( P \subseteq Q \) and \( H \subseteq Q' \), \( W - h \) and \( T^2 - c'T - g(W - h)p^{p-2} \) belong to \( Q' \). Therefore, \( Q' = (p, W - h, T) \) or \( Q' = (p, W - h, T - C') \). Suppose \( Q' = (p, W - h, T) \). Then \( Q = (p, \omega - h, \tau)S[\omega, \tau] \). We have

\[
\tau^2 - c' \tau - g(\omega - h)p^{p-2} = 0 \quad \text{and} \quad p(\tau - c') = (\omega - h)p^{p-1}.
\]

By Lemma 3.1, \( c' \not\in Q \), so \( \tau - c' \not\in Q \), and it follows that \( Q_Q = (\omega - h)Q \). Now suppose \( Q' = (p, W - h, T - C') \). Then \( Q = (p, \omega - h, \tau - c')S[\omega, \tau] \). Since

\[
\tau^2 - c' \tau - g(\omega - h)p^{p-2} = 0 \quad \text{and} \quad (\omega - h) \cdot \tau = g \cdot p,
\]

it follows that \( Q_Q = (p)Q \) (since \( \tau \not\in Q \), by Lemma 3.1). Thus, in either case, \( Q_Q \) is principal, so \( R = S[\omega, \tau] = P^{-1} \).
The proof is similar if $p = 2$ and $f = h^2 + 4g$, with $2 \nmid h$. One notes that $P^{-1} = S[\omega, \tau] = S[\tau]$, for $\tau := \frac{h+\omega}{2}$ and that $\tau$ satisfies $l(T) := T^2 - hT - g$. To show $R = S[\tau]$, one uses the fact that $l(T)$ and $l'(T)$ are relatively prime over the quotient field of $S/2S$.

To see that $R$ is a free $S$-module, we first note that $R$ is clearly generated as an $S$-module by the set $\{1, \omega, \ldots, \omega^{p-1}, \tau, \tau \omega, \ldots, \tau \omega^{p-1}\}$. However, $\tau \omega = pg\cdot 1 + h \cdot \tau$. This implies that $\tau \omega^i$ belongs to the $S$-module generated by $\{1, \omega, \ldots, \omega^{p-1}, \tau\}$, for all $1 \leq i \leq p - 1$. Moreover, since

$$\omega^{p-1} = -h^{p-1} \cdot 1 - h^{p-2} \cdot \omega - \cdots - h \cdot \omega^{p-2} + p \cdot \tau,$$

we may dispose of $\omega^{p-1}$ as well. Thus, $R$ is generated as an $S$-module by the set $\{1, \omega, \ldots, \omega^{p-2}, \tau\}$. Since these elements are clearly linearly independent over $S$, $R$ is a free $S$-module.

**Lemma 3.3.** Suppose $f = \lambda \alpha^e$, with $\alpha \in S$ a prime element, $\lambda$ a unit in $S$ and $2 \leq e < p$. If $p$ is not a unit in $S$, assume $a = p$. Then there exist integers $1 \leq s_1 < s_2 < \cdots < s_{e-1} < p$ satisfying

(i) $s_{e-1} \leq p - s_i$, $1 \leq i \leq e - 1$.

(ii) $R = J^{-1}$ for $J := (\omega^{s_{e-1}}, \omega^{s_{e-2}}a, \ldots, \omega^{s_1}, \alpha, e - 2, \alpha e - 1)S[\omega]$.

**Proof.** We begin by noting that either condition in the hypothesis implies that $Q := (\omega, \alpha)S[\omega]$ is the only height one prime for which $S[\omega]/Q$ is not a DVR. Now, since $p$ and $e$ are relatively prime, we can find positive integers $u$ and $v$ such that $1 = u \cdot p + (-v) \cdot e$. If we set $\tau := \frac{\omega}{2^v}$, then $\tau e = \lambda^{-v} \omega$ and $\tau p = \lambda^{-v} \alpha$. It follows that $S[\omega, \tau] = S[\tau] = R$, since either $p$ is a unit and $\alpha$ is square-free or $p$ is not a unit and $(\tau, p)S[\tau] = \tau S[\tau]$. Thus, $\{1, \tau, \ldots, \tau^{e-1}\}$ generate $R$ as an $S[\omega]$-module.

Moreover, $e$ and $\tau$ are relatively prime, the set $\{u\}$, for $1 \leq e - 1$, when reduced mod $e$, equals the set $\{1\}$. This will enable us to replace the generators $\{1, \tau, \ldots, \tau^{e-1}\}$ by $\{1, \frac{\lambda_1}{\omega^2}, \ldots, \frac{\lambda_{e-1}}{\omega^{e-1}}\}$. To elaborate, given $1 \leq i \leq e - 1$, there is a unique $1 \leq j_i \leq e - 1$ such that $u_j = i$ ( mod $e$). Write $u_{j_i} = t_i e + i$, $t_i \geq 0$. Then

$$(1 + ve)_{j_i} = pu_{j_i} = t_iep + ip,$$

so $(ve)_{j_i} + j_i = (t_iep) + ip$. If we write $ip = s_i e + r$, with $0 \leq r < e$, then uniqueness of the euclidean algorithm gives $v_{j_i} = t_i p + s_i$ and $r = j_i$. Thus, $\tau^{j_i} = \frac{\alpha^{s_i}}{\omega^{s_i}} = \frac{\alpha^r}{\omega^r}$ and $ip = s_i e + j_i$. For $i = e - 1$, this yields $s_{e-1} < p$. Moreover, $p = (s_{e-1} - s_i) e + (j_{e-1} - j_i)$, so $s_{e-1} + s_i > 0$. Similarly, $ep = (s_{e-1} + s_i) e + (j_{e-1} + j_i)$, so $s_{e-1} + s_i \leq p$. Thus, $s_1, \ldots, s_{e-1}$ have the required numerical properties.

We now have $\{1, \tau, \ldots, \tau^{e-1}\} = \{1, \frac{\lambda_1}{\omega^2}, \ldots, \frac{\lambda_{e-1}}{\omega^{e-1}}\}$. Multiplying by appropriate powers of $\lambda$ allows us to use $\{1, \frac{\lambda_1}{\omega^2}, \ldots, \frac{\lambda_{e-1}}{\omega^{e-1}}\}$ as a generating set for $R$ over $S[\omega]$. In Proposition 2.1 take $A := S[\omega]$, $B := S[W]$, $F := F(W)$ and $J$ the ideal of $(e - 1) \times (e - 1)$ signed minors of the $e \times (e - 1)$ matrix

$$\phi = \begin{pmatrix}
-a & 0 & \ldots & 0 & 0 \\
W^{\alpha_{e} - 1} & -a & \ldots & 0 & 0 \\
0 & W^{\alpha_{e} - 2} & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & W^{\alpha_2} & -a \\
0 & 0 & \ldots & 0 & W^{\alpha_1}
\end{pmatrix}$$
with \( \alpha_1 + \alpha_2 + \cdots + \alpha_t = s_i \), for \( 1 \leq i \leq e - 1 \). To obtain \( \phi' \), we augment \( \phi \) by the column whose transpose is \( (W^p, 0, \ldots, 0, (-1)^{e-1}\lambda a) \) (so \( \det(\phi') = F(W) \)). Then \( J^{-1} \) is generated as an \( S[\omega] \)-module by \( \{1, \frac{\lambda a_1}{\omega}, \ldots, \frac{\lambda a_{e-1}}{\omega} \} \). Thus, \( R = S[\omega, \tau] = J^{-1} \) for \( J = (\omega^{s-e}, \ldots, \omega^{s-e}) \), as desired.

For a proof of the next lemma, see [Ka], Lemma 4.1.

**Lemma 3.4.** In \( S[W] \) consider the ideals \( H := (W^{e_k}, W^{e_k-1}a_1, \ldots, W^{e_k}a_k) \) and \( K := (W^f, W^{f_{k-1}}b_1, \ldots, W^{f_{k-1}}b_1, b_1) \), where

1. \( e_k > e_{k-1} > \cdots > e_1 \) and \( f_t > f_{t-1} > \cdots > f_1 \).
2. \( a_1 | a_2 | \cdots | a_k \) and \( b_1 | b_2 | \cdots | b_1 \).
3. Each \( a_i \) and \( b_j \) is a product of prime elements.
4. For all \( i \) and \( j \), \( a_i \) and \( b_j \) have no prime factor in common.

Then there exist integers \( g_1 > \cdots > g_1 \) and products of primes \( c_1 | c_2 | \cdots | c_8 \) such that \( H \cap K = (W^{g_1}, W^{g_1-1}c_1, \ldots, W^{g_1}c_{e-1}, c_8) \). Moreover, \( H, K \) and \( H \cap K \) are all grade two perfect ideals.

**Lemma 3.5.** Let \( A \) be a domain and \( I \subseteq J \) ideals such that \( J^{-1} \) is a ring. Then \( I^{-1} \) is a \( J^{-1} \)-module if and only if \( I^{-1} = (I \cdot J^{-1})^{-1} \). In particular, if \( x \in J \) and \( x \cdot J^{-1} \subseteq J \), then \( (x \cdot J^{-1})^{-1} \) is a \( J^{-1} \)-module.

**Proof.** We first observe \( (I \cdot J^{-1})^{-1} \) is always a \( J^{-1} \)-module. Indeed, \( y \in (I \cdot J^{-1})^{-1} \) implies \( I \cdot J^{-1}y \subseteq R \). Thus \( J^{-1}y = J^{-1}y \subseteq I^{-1} \), so \( (I \cdot J^{-1})y \subseteq R \) and \( J^{-1}y \subseteq (I \cdot J^{-1})^{-1} \). Therefore, \( (I \cdot J^{-1})^{-1} \) is a \( J^{-1} \)-module and the first statement follows easily from this. For the second statement, we note that if \( x \cdot J^{-1} \subseteq J \), then for \( I := x \cdot J^{-1} \), \( I \cdot J^{-1} = x \cdot J^{-1}J^{-1} = x \cdot J^{-1} = I \). Thus, \( I^{-1} = (I \cdot J^{-1})^{-1} \), so \( I^{-1} \) is a \( J^{-1} \)-module by the first statement.

**Remark 3.6.** Proposition 2.2 in [Ko] states that \( R \) is a free \( S \)-module, if \( S \) is an unramified regular local ring and \( p \mid f \). The proof shows that \( R \) is a free \( S \)-module just under the assumption that \( f \) can be written as a product of primes and \( S/pS \) is a domain. In [Ko], Proposition 1.5, it is shown that if \( S \) is a UFD, then there exists a free \( S \)-module \( F \subseteq R \) such that \( pR \) is contained in \( F \). Thus, if \( p \) is a unit in \( S \), then \( R \) is also a free \( S \)-module. Finally, if \( f \) is square-free, \( R \) is a free \( S \)-module by Lemma 3.2. We record these facts in a common setting in the following proposition. For a version of the proposition for \( p^r \)th root extensions, see [Ka], Theorem 4.2.

**Proposition 3.7.** In addition to our standing hypotheses, assume that \( S \) is a UFD. Then \( R \) is a free \( S \)-module in each of the following cases:

1. \( p \) is a unit in \( S \).
2. \( p \) is not a unit and either \( p \mid f \) or \( f = \text{square-free} \).

We are now ready for our theorem.

**Theorem 3.8.** Assume that \( S \) is a regular local ring. Then there exists a finite, birational \( R \)-module \( M \) satisfying \( \text{depth}_S(M) = \text{dim}(R) \). In other words, \( M \) is a maximal Cohen-Macaulay module for \( R \).

**Proof.** By Proposition 3.7, \( R \) is a free \( S \)-module, and therefore Cohen-Macaulay, unless we assume that \( p \) is not a unit in \( S \), \( p \mid f \) and \( f \) is not square-free. In particular, we may assume that \( f \) is not a unit in \( S \). Factor \( f \) as a unit \( \lambda \) times prime elements \( a_i \), say \( f = \lambda a_1^e_1 \cdots a_r^e_r \). We may assume that for \( 1 \leq i \leq r, 1 < e_i < p \), if \( 1 \leq i \leq t \) and \( e_i = 1 \), if \( t < i \leq r \). Set \( Q_i := (\omega, a_i)S[\omega] \) for \( 1 \leq i \leq t \). For
each 1 \leq i \leq t choose s(i, 1) < \cdots < s(i, e_i - 1) satisfying the conclusion of Lemma 3.3 over S[\omega]_Q, and set J_i := (\omega^{s(i,1)-1}, \omega^{s(i,2)-2}a_i, \ldots, \omega^{s(i,e_i-2)}a_i^{e_i-2}, a_i^{e_i-1})S[\omega]. Thus, R_Q = (J_i^{-1})_Q, for all i. We now have two cases to consider. Suppose first that f is not a \(p\)th power modulo \(p^2 S\). We will show that \(R\) is Cohen-Macaulay. By our discussion in section two, \(Q_1, \ldots, Q_t\) are exactly the height one primes \(Q \subseteq S[\omega]\) for which \(S[\omega]_Q\) is not a DVR, so by Proposition 2.1 and Lemma 3.3, \(R = J^{-1}\) for \(J := J_1 \cap \cdots \cap J_t\). Set \(B := S[W/(W,N)]\) for \(N\), the maximal ideal of \(S\) and use “tilde” to denote pre-images in \(B\). By Lemma 3.4, \(\tilde{J} \subseteq B\) is a grade two perfect ideal. Therefore, \(p.d.B(J) = p.d.B(J^{-1}) = 1\), by Proposition 2.1(iii). Thus, \(\text{depth}_B(J^{-1}) = \dim(B) - 1\), so \(\text{depth}_S(R) = \dim(R)\), which is what we want.

Suppose that \(f\) is a \(p\)th power modulo \(p^2 S\). Write \(f = h^p + p^2 g\), for \(h, g \in S, p \nmid h\). Then \(P = (\omega - h, p)\). Moreover, \(P\) and \(Q_1, \ldots, Q_t\) are the height one primes \(Q \subseteq S[\omega]\) for which \(S[\omega]_Q\) is not a DVR. By Proposition 2.1 and Lemma 3.2, \(R = J^{-1}\), for \(J := J_1 \cap \cdots \cap J_t \cap P\). Now, as in the proof of Lemma 3.3, \(J_i^{-1}\) is generated as an \(S[\omega]\)-module by the set \(\{1, \frac{\lambda_i}{\omega^{s(i,1)}}, \ldots, \frac{\lambda_i^{e_i-1}}{\omega^{s(i,e_i-1)}}\}\), where, for each \(i\), \(\lambda_i := \prod_{\underline{a} \neq e_i} 1\). Thus \(K_i = (\omega^{p-s(i,1)}, \omega^{p-s(i,2)}a_i, \ldots, a_i^{e_i-1})S[\omega]\), for \(K_i := a_i^{e_i-1}J_i^{-1}\) and \(1 \leq i \leq t\). By Lemma 3.3, \(K_i \subseteq J_i\), so upon setting \(I := K_1 \cap \cdots \cap K_t \cap P\), it follows from Lemma 3.5 that \(I^{-1}\) is a \(J^{-1}\)-module (since this holds locally for every height one prime in \(S[\omega]\)). Taking \(M := I^{-1}\), we will show that \(M\) is the required module. For this, we claim that \(\tilde{I} \subseteq B\) is a grade two perfect ideal. If the claim holds, \(1 = p.d.B(I) = p.d.B(I^{-1}) = p.d.B(M)\). Thus \(\text{depth}_B(M) = \dim(B) - 1\), so \(\text{depth}_S(M) = \dim(R)\), which is what we want.

To prove the claim, we set \(\hat{K} := \tilde{K}_1 \cap \cdots \cap \tilde{K}_t\) and consider the short exact sequence

\[
0 \longrightarrow B/\tilde{I} \longrightarrow B/\tilde{K} \oplus B/\tilde{P} \longrightarrow B/(\tilde{K} + \tilde{P}) \longrightarrow 0.
\]

Since \(\hat{K}\) is a grade two perfect ideal (by Lemma 3.4), the Depth Lemma and the Auslander-Buchsbaum formula imply that \(\tilde{I}\) is a grade two perfect ideal, once we show \(\text{depth}(B/(\tilde{K} + \tilde{P})) = \dim(B) - 3\). Set \(a := a_i^{e_i-1} \cdots a_i^{e_i-1}\). We now argue that \(\hat{K} + \hat{P} = (a, p, W - h)\). If we can show this, clearly \(\text{depth}(B/(\hat{K} + \hat{P})) = \dim(B) - 3\) and we will have verified the claim. Take \(k \in \hat{K}\) and consider its image \(\bar{k} \in S[\omega]\). Select \(Q \subseteq S[\omega]\), a height one prime. If \(Q = Q_i\), for some \(1 \leq i \leq t\), then \(k \in (a_i^{e_i-1}J_i)Q_i = aR_Q\). If \(Q \neq Q_i\) for any \(1 \leq i \leq t\), then clearly \(k \in aRQ = R_Q\). It follows that \(k \in aR \cap S[\omega]\). In other words, \(k\) is integral over the principal ideal \(aS[\omega]\). Therefore, the image of \(k\) in \(S[\omega]/(\omega - h, p) = S/pS\) is integral over the principal ideal generated by the image of \(a\). Since \(S/pS\) is integrally closed, the image of \(k\) in \(S/pS\) is a multiple of the image of \(a\). Therefore, \(\hat{k} \in (a, p, W - h)\) in \(S[W]\). It follows that \(\hat{K} \subseteq (a, p, W - h)\). Since \(a \in \hat{K}\), we obtain \(\hat{K} + \hat{P} = (a, p, W - h)\), which is what we want. This completes the proof of Theorem 3.8.

**Remark 3.9.** Of course if \(S\) is an unramified regular local ring, \(S\) fulfills our standing hypotheses, so Theorem 3.8 applies. However, the theorem also applies to certain ramified regular local rings. For instance, take \(T\) to be the ring \(\mathbb{Z}[X_1, \ldots, X_d]\) localized at \((p, X_1, \ldots, X_d)\) and let \(H \in \mathbb{Z}[X_1, \ldots, X_d]\) be any polynomial in \((X_1, \ldots, X_d)^2\) for which \(\mathbb{Z}[X_1, \ldots, X_d]/(H)\) is an integrally closed domain. If we set \(S := T/(p - H)\), then \(S\) is a ramified regular local ring and \(S/pS\) is an integrally closed domain.
We close with an example where $R$ is not a free $S$-module, yet $R$ admits a finite, birational module which is a free $S$-module. The example is an unramified variation of Koh’s Example (2.4).

**Example 3.10.** Let $S$ be an unramified regular local ring having mixed characteristic 3 and take $x, y \in S$ such that $3, x, y$ form part of a regular system of parameters. Set $a := xy^3 + 9$, $b := x^3y^3 + 9$ and $f := ab^2$, so $\omega^3 = f = ab^2 = h^3 + 9g$, for $h = x^3y^2$. From Lemmas 3.2 and 3.3 it follows that $R = (Q \cap P)^{-1}$ for $Q := (\omega, b)$ and $P := (\omega - h, 3)$. Set $J := Q \cap P$. We first show that $R$ is not a free $S$-module. Suppose to the contrary that $J^{-1}$ is free over $S$. As in the proof of Theorem 3.8, set $B := S[W_{(N, W)}]$ and use “tilde” to denote pre-images in $B$. Since $J^{-1}$ is free over $S$, we have $\text{p.d.}_B(J^{-1}) = 1$, so $J^{-1}$ is a grade one perfect $B$-module. By [KU, Proposition 3.6], $J$ is a grade one perfect $B$-module, so $J$ is a grade two perfect ideal. On the other hand, $\text{depth}_B(B/J) = 1 + \text{depth}_B(B/(Q + P))$. But, $Q + P = (W, x^3y^3, 3)B$, so $B/(Q + P) = S/(3, x^3y^3)S$, which is easily seen to have depth equal to $\text{depth}(S) - 3 = \text{depth}(B) - 4$. This is a contradiction, so it must hold that $R$ is not a free $S$-module.

Now, $Q^{-1}$ is generated as an $S[\omega]$-module by $\{1, \frac{ab}{\omega}\}$. If we set $K := b \cdot Q^{-1}$, then $K = (\omega^2, b)S[\omega]$. The proof of Theorem 3.8 shows that $M := (K \cap P)^{-1}$ is a finite, birational $R$-module satisfying $\text{depth}_R(M) = \text{dim}(R)$. In other words, $M$ is an $R$-module which is free over $S$. To calculate a basis for $M$, one must calculate $K \cap P$ and then use Proposition 2.1. We leave it to the reader to check that $K \cap P = (\omega^2 - h^2 - 9x^2y^3, b(\omega - h), 3b)$. Therefore, $K \cap P = I_2(\phi)$ for

$$
\phi = \begin{pmatrix}
-b & 0 & 0 \\
\omega + h & -3 & \omega - h \\
-3x^2y^3 & \omega - h & t
\end{pmatrix}.
$$

The augmented matrix that determines $(K \cap P)^{-1} = M$ is the $3 \times 3$ matrix

$$
\begin{pmatrix}
-b & 0 & \omega \\
\omega + h & -3 & x^2y^3 \\
-3x^2y^3 & \omega - h & t
\end{pmatrix},
$$

where $t$ is defined by the equation $x^3y^3 = ab + 3t$. By Proposition 2.1, $M$ is generated as an $S[\omega]$-module by the set $\{1, \gamma, \delta\}$, for

$$
\gamma := \frac{-3t - x^2y^3(\omega - h)}{\omega^2 - h^2 - 9x^2y^3} = \frac{\omega}{b}, \quad \delta := \frac{-bt + 3x^2y^3\omega}{b(\omega - h)} = \frac{\omega^2 + \omega h + h^2 + 9x^2y^3}{3b}.
$$

If we show that $\{1, \gamma, \delta\}$ also generate $M$ as an $S$-module, then since they are clearly linearly independent over $S$, they form a basis for $M$ as an $S$-module. To see that $\{1, \gamma, \delta\}$ generate $M$ as an $S$-module, it suffices to show that $\omega, \gamma$ and $\omega \cdot \delta$ can be expressed as $S$-linear combinations of $\{1, \gamma, \delta\}$. This clearly holds for $\omega$. Using $9x^2y^3 = bxy^3 - x^3y^3$, we obtain

$$
\omega \cdot \gamma = \frac{\omega^2}{b} = -x^2y^3 \cdot 1 - h \cdot \gamma + 3 \cdot \delta.
$$

Since $\omega^3 = h^3 + 9g$ and $g = x^3y^3 + bxy^4 + b^2$, we get

$$
\omega \cdot \delta = (3xy^4 + 3b) \cdot 1 + 3x^2y^3 \cdot \gamma + h \cdot \delta,
$$

and the example is complete.
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