SOME COROLLARIES OF FROBENIUS’ NORMAL $p$-COMPLEMENT THEOREM

YAKOV BERKOVICH

(Communicated by Ronald M. Solomon)

Abstract. For a prime divisor $q$ of the order of a finite group $G$, we present the set of $q$-subgroups generating $O^{q'}(G)$. In particular, we present the set of primary subgroups of $G$ generating the last member of the lower central series of $G$. The proof is based on the Frobenius Normal $p$-Complement Theorem and basic properties of minimal nonnilpotent groups. Let $G$ be a group and $\Theta$ a group-theoretic property inherited by subgroups and epimorphic images such that all minimal non-$\Theta$-subgroups (= $\Theta_1$-subgroups) of $G$ are not nilpotent. Then (see the lemma), if $K$ is generated by all $\Theta_1$-subgroups of $G$ it follows that $G/K$ is a $\Theta$-group.

Let $p, q$ be distinct primes, $\pi(G)$ the set of all prime divisors of the order of a finite group $G$ (we consider only finite groups), $\pi$ a set of primes, $S_p(G)$ the set of Sylow $p$-subgroups of $G$, $\Phi(G)$, $F(G)$, $G'$ and $Z(G)$ the Frattini subgroup, the Fitting subgroup, the commutator subgroup and the center of $G$, respectively. In what follows, $\Theta$ is a nonempty group-theoretic property inherited by subgroups and epimorphic images and such that there exists a non-$\Theta$-group.

A group is said to be minimal nonnilpotent if it is not nilpotent but all its proper subgroups are nilpotent. Let $G$ be minimal nonnilpotent. It is known (see [Hup], Satz 3.5.2 — this theorem is due to O. Yu. Schmidt [S] and Yu. A. Gol'fand [Gol]; L. Redei [R] gave the complete classification of such groups) that

(i) $\pi(G) = \{p, q\}$ and $G = PQ$, where $P \in S_p(G)$, $G' = Q \in S_q(G)$;
(ii) $P$ is cyclic and $|P : P \cap Z(G)| = p$;
(iii) $Q/Q \cap Z(G)$ is a minimal normal subgroup of $G/Q \cap Z(G)$, $Q$ is special.

(Recall that a $q$-group $Q$ is special if it is elementary abelian or $Q' = Z(Q) = \Phi(Q)$; in particular, the exponent of $Q$ is at most $q^2$.) It is known that if $q > 2$, then $\exp(Q) = q$.

It is easy to check that a group possessing properties (i–iii) is minimal nonnilpotent. We call a group with this structure an $S(p, q)$-group.

A group $G$ is said to be $p$-nilpotent if it has a normal $p$-complement. By Frobenius’ Normal $p$-Complement Theorem and Ito’s remark (see [H], Theorem 14.4.7, [Hup], Satz 4.5.4, or [I]) $G$ is $q$-nilpotent if and only if it has no $S(p, q)$-subgroups. A group $G$ is said to be $p$-closed if its Sylow $p$-subgroup is normal.

Received by the editors May 14, 1997.

1991 Mathematics Subject Classification. Primary 20D20.

Key words and phrases. Special $p$-group, minimal nonnilpotent (nonabelian, noncyclic, nonsolvable) group, $p$-nilpotent group, $p$-closed group, $S(p, q)$-group, $B(p, q)$-group.

The author was supported in part by the Ministry of Absorption of Israel.

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Let $O^p(G)$ be the subgroup generated by all $p'$-elements of $G$, and $O^{p,p'}(G)$ the subgroup generated by all $p$-elements of $O^p(G)$. Obviously, $G/O^{p,p'}(G)$ is $p$-nilpotent but, for every normal subgroup $N$ of $G$ properly contained in $O^{p,p'}(G)$, $G/N$ is not $p$-nilpotent. Let $H(G) = \bigcap_{p \in \pi(G)} O^p(G)$; $H(G)$ is the last member of the lower central series of $G$. It is clear that $O^{p,p'}(G) \leq H(G)$.

Suppose that $\Phi(G) \leq N \lneq G$. If $N/\Phi(G)$ has a normal $\pi$-Hall subgroup $K_1/\Phi(G)$, then $N$ has a normal $\pi$-Hall subgroup as well. Indeed, $K_1$ has a $\pi$-Hall subgroup $K$ and all such subgroups are conjugate in $K_1$, by the Schur-Zassenhaus Theorem. Therefore, $G = K_1 N_G(K) = K \Phi(G) N_G(K) = N_G(K)$, and our claim follows since $K$ is a $\pi$-Hall subgroup of $N$. It follows from what has just been proved and the Schur-Zassenhaus Theorem that $\pi(G/\Phi(G)) = \pi(G)$.

Let $G$ be an arbitrary group.

**Definition 1.** A group $G$ is said to be a $B(p, q)$-group if $G/\Phi(G)$ is an $S(p, q)$-group for some primes $p, q$.

**Definition 2.** A group $G$ is said to be a $\Theta$-group (= minimal non-$\Theta$-group) if it is not a $\Theta$-group but all its proper subgroups are $\Theta$-groups.

It is clear that $G$ is a $\Theta$-group if and only if it has no $\Theta$-subgroups.

It follows from the remark preceding Definition 1 that a $B(p, q)$-group $G$ has the form $PQ$, where $P \in \text{Syl}_p(G)$, $G' = Q \in \text{Syl}_q(G)$, $|P : P \cap Z(G)| = p$, $P$ is cyclic and $Q/\Phi(Q)$ is a minimal normal subgroup of $G/\Phi(Q)$. It is clear that a non-nilpotent epimorphic image of a $B(p, q)$-group is also a $B(p, q)$-group. Let $Q > \{1\}$ be a $q$-group. There exists a $B(p, q)$-group with Sylow subgroup $Q$ if and only if $Q$ possesses an automorphism of order $p$ that acts irreducibly on $Q/\Phi(Q)$. For any distinct $p$ and $q$, there exists a $B(p, q)$-group. For $p > 2, n \in \mathbb{N}$, the dihedral group of order $2p^n$ is a $B(2, p)$-group.

In this and the following paragraph we shall define some characteristic subgroups of an arbitrary group. Let $\mathfrak{B}_q(G)$ be the subgroup generated by normal Sylow $q$-subgroups (= derived subgroups) of all $B(p, q)$-subgroups in $G$ ($p \in \pi(G) \setminus \{q\}$). By the Frobenius Normal $p$-Complement Theorem, $\mathfrak{B}_q(G) = \{1\}$ if and only if $G$ is $q$-nilpotent. Set $\mathfrak{B}(G) = \prod_{q \in \pi(G)} \mathfrak{B}_q(G)$.

Let $\Theta_1(G)$ denote the subgroup generated by all $\Theta_1$-subgroups of $G$; $\Theta_1(G) = \{1\}$ if and only if $G$ is a $\Theta$-group. $\Theta_1(G)$ is characteristic in $G$. We shall show (see the lemma below) that $G/\Theta_1(G)$ is a $\Theta$-group for some $\Theta$’s.

Let $G = A\Theta_1(G)$ be such that the subgroup $A$ is as small as possible. Then $\Theta_1(A) \leq A \cap \Theta_1(G) \leq \Phi(A)$ is nilpotent. Hence, if $G/\Theta_1(G)$ is not a $\Theta$-group, there exists in $G$ a non-$\Theta$-subgroup $A$ such that $\Theta_1(A) \leq \Phi(A)$; in particular, all $\Theta_1$-subgroups of $A$ are nilpotent. Let, in addition, $\Theta = \text{nilpotency}$. It follows from the remark above that then $A$ is nilpotent so $G/\Theta_1(G)$ is nilpotent. We generalize this observation in the following

**Lemma.** Suppose that all $\Theta_1$-subgroups of a group $G$ are not nilpotent. Then $G$ has a $\Theta$-subgroup $A$ such that $G = A\Theta_1(G)$. In particular, $G/\Theta_1(G)$ is a $\Theta$-group.

**Proof.** Suppose that the lemma has been proved for all proper subgroups of $G$. We may assume that $\Theta_1(G) > \{1\}$ (otherwise, $G$ is a $\Theta$-group, and the lemma is obvious). Let $A$ be a minimal subgroup of $G$ such that $G = A\Theta_1(G)$. Since $\Theta_1(G)$ is non-nilpotent, it is not contained in $\Phi(G)$. It follows that $A < G$. By the induction hypothesis, $A/\Theta_1(A)$ is a $\Theta$-group. Since $\Theta_1(A) \leq A \cap \Theta_1(G) \leq \Phi(A)$,
we get $\Theta_1(A) = \{1\}$ since all $\Theta_1$-subgroups of $G$ are not nilpotent. This means that $A$ is a $\Theta$-group so $G/\Theta_1(G) \cong A/A \cap \Theta_1(G)$ is also a $\Theta$-group, as desired.

In particular, the lemma is true for $\Theta$ such that all nilpotent groups are $\Theta$-groups, but the assumption in the lemma is weaker. In fact, consider the case when $\Theta = \text{commutativity and } G$ is a nonabelian group all of whose Sylow subgroups are abelian. Then all $\Theta_1$-subgroups of $G$ are nonnilpotent (however, there are nilpotent $\Theta_1$-groups). By the lemma, $G/\Theta_1(G)$ is abelian.

It is easy to show that the lemma is not true for $\Theta$'s such that some $\Theta_1$-groups are nilpotent. In fact, if $\Theta$ is such that only the identity group is a $\Theta$-group, $\Theta_1(G)$ is the subgroup of $G$ generated by the elements of prime orders; in that case, the structure of $G/\Theta_1(G)$ may be very complicated.

Remark 1. Suppose, in addition, that a tower of $\Theta$-groups is a $\Theta$-group and all subgroups of prime orders are $\Theta$-groups (in that case, we will call $\Theta$ strong). We will prove that if $G$ is a $\Theta_1$-group, then $G/\Phi(G)$ is a nonabelian simple group. Indeed, let $N$ be a maximal normal subgroup of $G$ and $H$ a maximal subgroup of $G$. Since $HN$ is a $\Theta$-group, we get $N \leq H$. It follows that $N = \Phi(G)$ and $G/\Phi(G)$ is simple. It remains to show that $G/\Phi(G)$ is nonabelian. Assume that this is not true. Then $|G : \Phi(G)| = p$, a prime, so $\Phi(G)$ is maximal in $G$. It follows that $\Phi(G)$ is a unique maximal subgroup of $G$ so $G$ is a cyclic $p$-group.

Remark 2. For strong properties $\Theta$, one can say more than the lemma says. Indeed, in the lemma, let $\Theta$ be a strong property and let $G/\Theta_1(G)$ be a $\Theta$-group, by the lemma. We claim that, in fact, $G/K$ is not a $\Theta$-group if a normal subgroup $K$ of $G$ is properly contained in $\Theta_1(G)$. Assume that this is not true. Then $G$ has a $\Theta_1$-subgroup $L$ such that $L \not\leq K$. Since $L/L \cap K$ and $K \cap L$ are $\Theta$-groups so is $L$, which is not the case. In particular, if $\Theta = \text{solvability}$, then $\Theta_1(G)$ is the last member of the derived series of $G$.

Let $\mathcal{S}(G)$ be the subgroup generated by all minimal nonnilpotent subgroups of $G$. By the lemma, $H(G) \leq \mathcal{S}(G)$. It follows that if all minimal nonnilpotent subgroups are normal in $G$, then $G/F(G)$ is an extension of a direct product of elementary abelian groups by a nilpotent group. Indeed, if $K$ is generated by normal maximal subgroups of all minimal nonnilpotent subgroups of $G$, then $K \leq F(G)$ and $\mathcal{S}(G)/K$ is generated by normal subgroups of prime orders. But we do not know whether the nilpotent length of $G$ is bounded if all its minimal nonabelian subgroups are normal. In general, the inequality $H(G) \neq \mathcal{S}(G)$ is possible. In fact, let $G = S_n$, $n > 3$. Then it is easy to check that $\mathcal{S}(G) = G$: the general case follows from the case $n = 4$, for which our assertion is trivial. Since $H(G) = A_n$, our claim follows.

Let $\pi$ be a set of primes. A group $G$ is said to be $\pi$-decomposable if a $\pi$-Hall subgroup is a direct factor of $G$. Let $K$ be the subgroup generated by all minimal nonnilpotent subgroups of $G$ of orders divisible by a fixed prime $p$. We claim that $G/K$ is $p$-decomposable. Indeed, let $\Theta = p$-decomposability. By [Hup], Satz 4.5.4, and basic properties of $p$-solvable groups, $\Theta_1$-groups are minimal nilpotent of orders divisible by $p$. Therefore, $K = \Theta_1(G)$. By the lemma, $G/K$ is a $\Theta$-group (i.e., it is $p$-decomposable), as claimed. Similarly, if $K$ is generated by all minimal
non-$\pi$-decomposable subgroups of $G$, then, by the lemma, $G/K$ is $\pi$-decomposable. (Note that the minimal non-$\pi$-decomposable groups are not classified.)

Our principal result is the following

**Theorem.** (a) $\mathfrak{B}_q(G) = \mathfrak{O}^{q,q}(G)$, i.e., commutator subgroups of all $B(p,q)$-subgroups of $G$ ($p \in \pi(G) - \{q\}$) generate $\mathfrak{O}^{q,q}(G)$.

(b) $\mathfrak{B}(G) = H(G)$. In other words, the subgroup, generated by commutator subgroups of all $B(p,q)$-subgroups of $G$, where $p, q$ run over the set $\pi(G)$, coincides with the last member of the lower central series of $G$.

**Proof.** (a) Assume that $\mathfrak{B}_q(G) \not\subseteq \mathfrak{O}^{q,q}(G)$. Then $G$ has a $B(p,q)$-subgroup $F = P \cdot Q$, where $p \in \pi(G) - \{q\}$, $P \in \text{Syl}_p(F)$ and $F' = Q \in \text{Syl}_q(F)$ such that $Q \not\subseteq \mathfrak{O}^{q,q}(G)$. Then $\overline{F} = F/F \cap \mathfrak{O}^{q,q}(G)$ is of order divisible by $q$; therefore, $\overline{F}$ is not $q$-nilpotent (otherwise, $F$ has a normal subgroup of index $q$ which is not the case since $|F : F'|$ is a power of $p \neq q$). Since every epimorphic image of $F$ is nilpotent or a $B(p,q)$-group, it follows that $\overline{F}$ is a $B(p,q)$-group. Thus, a non-$q$-nilpotent group $\overline{F}$ is isomorphic to a subgroup of the $q$-nilpotent group $G/\mathfrak{O}^{q,q}(G)$, which is a contradiction. Hence $\mathfrak{B}_q(G) \subseteq \mathfrak{O}^{q,q}(G)$.

Recall that $\mathfrak{O}^{q,q}(G)$ is contained in every normal subgroup $N$ of $G$ such that $G/N$ is $q$-nilpotent. Therefore, to prove the reverse inclusion, it is enough to show that $G/\mathfrak{B}_q(G)$ is $q$-nilpotent. Assume that $G/\mathfrak{B}_q(G)$ is not $q$-nilpotent. Then it has an $S(p,q)$-subgroup $S = S/\mathfrak{B}_q(G)$, by the Frobenius Normal $q$-Complement Theorem. Let $A$ be a smallest subgroup such that $S = A\mathfrak{B}_q(G)$. Then $A \cap \mathfrak{B}_q(G) \leq \Phi(A)$ and $A/A \cap \mathfrak{B}_q(G)$ is an $S(p,q)$-group since it is isomorphic to $S$. It follows that $A/\Phi(A)$ is an $S(p,q)$-group as a non-nilpotent epimorphic image of the $S(p,q)$-group $S$. Therefore, by Definition 1, $A$ is a $B(p,q)$-subgroup of $G$. Since $q$ divides $|S| = |A\mathfrak{B}_q(G)/\mathfrak{B}_q(G)|$, the Sylow $q$-subgroup of $A$ is not contained in $\mathfrak{B}_q(G)$, contrary to the definition of the last subgroup. Thus, $G/\mathfrak{B}_q(G)$ is $q$-nilpotent so $\mathfrak{O}^{q,q}(G) \leq \mathfrak{B}_q(G)$, and (a) follows.

(b) Let us prove that $K = \prod_{p \in \pi(G)} \mathfrak{O}_p^{q,q}(G)$ is equal to $H(G)$. Since $G/K$ is $q$-nilpotent for all $q \in \pi(G)$, by (a), it is nilpotent, and so $H(G) \leq K$. The reverse inclusion is evident since $\mathfrak{O}^{q,q}(G) \leq H(G)$ for all $q \in \pi(G)$. Therefore, $K = H(G)$.

By (a),

$$\mathfrak{B}(G) = \prod_{p \in \pi(G)} \mathfrak{B}_p(G) = \prod_{p \in \pi(G)} \mathfrak{O}^{p,p}(G) = K = H(G),$$

completing the proof of (b). \qed

If $K$ is the subgroup generated by normal Sylow subgroups of all minimal non-nilpotent subgroups of $G$, then $G/K$ is not necessarily nilpotent (let $G$ be the dihedral group of order $2p^n$, $p > 2$, $n > 1$). Moreover, we cannot prove that, in the case under consideration, $G/K$ is solvable.

In the following paragraph we will use the Baer-Suzuki Theorem (see [HB], Theorem 9.7.8):

(*) If $x$ is a $p$-element of $G$, then $x \in O_p(G)$ if and only if $\langle x, x^y \rangle$ is a $p$-subgroup for all $y \in G$.

The following result is a consequence of (*) (see [B], p. 27, where another proof is given):

(**) If $G$ has no $S(2,p)$-subgroups for all odd $p \in \pi(G)$, it is 2-closed.
Let us prove that if $\Theta = \text{commutativity (or cyclicity)}$, then $G/\Theta_1(G)$ is 2-closed. Let $G$ be a counterexample of minimal order. Suppose that $\Theta_1(G)$ is not nilpotent. Then $\Theta_1(G)$ has a nonnormal Sylow subgroup $P$. By Frattini’s Lemma, $G = \Theta_1(G)N_G(P)$. Since $\Theta_1(N_G(P)) \leq \Theta_1(G) \cap N_G(P)$ and $N_G(P) < G$, it follows by the induction hypothesis that $N_G(P)/\Theta_1(N_G(P))$ (and so its epimorphic image $N_G(P)/\Theta_1(G)$) is 2-closed. Since $G/\Theta_1(G)$ is isomorphic to the last group, it is also 2-closed. Thus, $\Theta_1(G)$ is nilpotent so all $\Theta_1$-subgroups of $G$ are nilpotent. Since $S(2, p)$-groups, $p > 2$, are $\Theta_1$-groups (see [Hup], Satz 3.5.2), $G$ has no $S(2, p)$-subgroups for all odd $p \in \pi(G)$. It follows from (**) then that $G$ is 2-closed, as desired.

If, as in the previous paragraph, $\Theta = \text{commutativity}$, then all nonabelian Sylow subgroups are contained in $\Theta_1(G)$ (see added in proof).

In view of what has been said, it is interesting to study the groups whose minimal nonabelian subgroups are all nilpotent (such groups are 2-closed, by (**)).

I am indebted to R. Solomon and the referee for useful comments and suggestions.

**Added in proof**

It is easy to deduce from this that $G/K$, where $K = \Theta_1(G)$, is abelian. Indeed, assume that $G/K$ is nonabelian. Let $S/K$ be a minimal nonabelian subgroup of $G/K$; then $S/K$ is nonnilpotent. Let $A$ be a minimal subgroup such that $S = AK$; then $A$ is a $B(p, q)$-subgroup. Since Sylow subgroups of $A$ are abelian, $Z(A)$ is a $p$-subgroup. It follows that $A = \Theta_1(A) \leq K$, a contradiction.

**References**


Department of Mathematics, University of Haifa, Mount Carmel, Haifa 31905, Israel

*E-mail address*: berkov@mathcs2.haifa.ac.il