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A COMPOSITION FORMULA IN THE RANK TWO FREE GROUP

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ABSTRACT. Using the fundamental group of a punctured torus, a free group F of rank two, and the fact that the natural eipmorphism from $AutF$ onto $Aut(F/F')$ has as kernel the group of inner automorphisms of F, we describe representatives of the conjugacy classes of generating pairs of F and give explicit relations between them.

Let $F = F(S,T)$ be the free group on S and T. By a theorem of Nielsen [N] (see [LS, p. 25]) the natural epimorphism from $AutF$ onto $Aut(F/F')$ ($= GL(2,\mathbb{Z})$) has as kernel the group of inner automorphisms of F. From this it follows easily that, if α is the abelianization homomorphism from F onto F/F' ($= \mathbb{Z}^2$) and $\mathbf{a} \in \mathbb{Z}^2$ is primitive[1], then the inverse image of $\mathbf{a}$ under α is a conjugacy class of primitive elements. Also, if $(\mathbf{a}_1, \mathbf{a}_2)$ is a basis of $\mathbb{Z}^2$, then, up to conjugacy, there is a unique basis (f_1, f_2) of F such that $(f_i)\alpha = \mathbf{a}_i$ $(i = 1, 2)$. (The basis (f_1, f_2) is conjugate to (g_1, g_2) if there exists $w \in F$ such that $w^{-1} f_i w = g_i$ $(i = 1, 2)$).

In the important paper [OZ], Osborne and Zieschang define explicitly primitive words $W_{m,n} \in F(S,T)$, where m and n are relatively prime integers, such that $(W_{m,n})\alpha = (m, n)$. They also state that if $mn - pq = 1$, then $(W_{m,n}, W_{p,q})$ is a basis of F; this, while correct for nonnegative values of m, n, p, q, is not valid in general (for example $W_{-2,-3}$ and $W_{1,1}$ do not generate F). A composition formula is also stated in [OZ, Thm. 3.5] but this, even with the correction of indices in [LTZ, 2.1.3], is incorrect in general.

In the present article we consider elements $V^{\varepsilon}_{\mathbf{a}}$ of F for $\mathbf{a} = (m, n) \in \mathbb{Z}^2$ and $\varepsilon \in \mathcal{D} \subset \mathbb{R}^2$ where $\mathcal{D}$ is the complement of the union of all the lines that intersect $\mathbb{Z}^2$ in more than one point. If $\gcd(m, n) = 1$, then $V^{\varepsilon}_{(m,n)}$ is conjugate to $W_{m,n}$. We show in Theorem 1.i) that $(V^{\varepsilon}_{\mathbf{a}}, V^{\varepsilon}_{\mathbf{b}})$ is a basis of F, if $\mathbb{Z}^2 = \langle \mathbf{a}, \mathbf{b} \rangle$, and obtain in Theorem 1.ii) a composition formula. Everything is obtained by applying the fundamental group functor π to the punctured torus.

Denote by $\mathbb{T}$ the torus $\mathbb{R}^2/\ \mathbb{Z}^2$, by $\mathbb{T}_0$ the punctured torus $(\mathbb{R}^2 - \mathbb{Z}^2)\ /\ \mathbb{Z}^2$ and by $\rho : \mathbb{R}^2 - \mathbb{Z}^2 \to \mathbb{T}_0$ the natural projection. If $\mathbf{a} \in \mathbb{Z}^2$ and $\varepsilon \in \mathcal{D}$, then denote $(\varepsilon)\rho$ by $\overline{\varepsilon}$ and define $\gamma^{\varepsilon}_{\mathbf{a}} \in \pi(\mathbb{T}_0, \overline{\varepsilon})$ as the homotopy class of the loop $(\varepsilon + t\mathbf{a})\rho$, $t \in [0, 1]$. Denote $\gamma^{\varepsilon}_{(1,0)}$ (resp. $\gamma^{\varepsilon}_{(0,1)}$) by S_{ε} (resp. T_{ε}). There is an isomorphism

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[1]An element f of a rank two group G is primitive if there exists $g \in G$ such that f and g generate G.

$\varphi_\varepsilon : F(S,T) \to \pi(\mathbb{T}_0, \overline{\varepsilon})$ sending S to S_ε and T to T_ε. Define $V^\varepsilon_{\mathbf{a}} \in F(S,T)$ by $(V^\varepsilon_{\mathbf{a}})\varphi_\varepsilon = \gamma^\varepsilon_{\mathbf{a}}$. Notice that $(V^\varepsilon_{\mathbf{a}})\alpha = \mathbf{a}$ since the inclusion induced homomorphism $\pi(\mathbb{T}_0, \overline{\varepsilon}) \to \pi(\mathbb{T}, \overline{\varepsilon})$ is the abelianization.

In comparison with [OZ, 4.2] $V^\varepsilon_{\mathbf{a}}$ is represented by the word obtained by traveling along the segment, in $\mathbb{R}^2 - \mathbb{Z}^2$, from ε to $\varepsilon + \mathbf{a}$ and writing S (resp. S^{-1}) whenever we cross a component of $\mathbb{Z} \times \mathbb{R}$ from left to right (resp. from right to left) and writing T (resp. T^{-1}) whenever we cross a component of $\mathbb{R} \times \mathbb{Z}$ from below (resp. from above).

A matrix $M = \binom{\mathbf{a}}{\mathbf{b}} \in GL(2,\mathbb{Z})$ defines a linear automorphism of $(\mathbb{R}^2, \mathbb{Z}^2)$ that, for any $\varepsilon \in \mathcal{D}$, induces a homeomorphism $\mu^\varepsilon_M : (\mathbb{T}_0, \overline{\varepsilon}) \to (\mathbb{T}_0, \overline{\varepsilon M})$. This induces $\nu^\varepsilon_M : \pi(\mathbb{T}_0, \overline{\varepsilon}) \tilde{\to} \pi(\mathbb{T}_0, \overline{\varepsilon M})$ and an automorphism Ψ^ε_M of F defined by $\Psi^\varepsilon_M = \varphi_\varepsilon \nu^\varepsilon_M \varphi^{-1}_{\varepsilon M}$. We have $\gamma^\varepsilon_{\mathbf{c}} \nu^\varepsilon_M = \gamma^{\varepsilon M}_{\mathbf{c}M}$ since $\rho\mu^\varepsilon_M = M\rho$ and therefore $V^\varepsilon_{\mathbf{c}} \Psi^\varepsilon_M = V^{\varepsilon M}_{\mathbf{c}M}$ for all $\mathbf{c} \in \mathbb{Z}^2$ (this equality was suggested by the referee). In particular $(S)\Psi^\varepsilon_M = V^{\varepsilon M}_{\mathbf{a}}$ and $(T)\Psi^\varepsilon_M = V^{\varepsilon M}_{\mathbf{b}}$. Notice that, for any word $W(S,T)$ we have $(W(S,T))\Psi^\varepsilon_M = W((S)\Psi^\varepsilon_M, (T)\Psi^\varepsilon_M) = W(V^{\varepsilon M}_{\mathbf{a}}, V^{\varepsilon M}_{\mathbf{b}})$. One has the following composition theorem.

Theorem 1. *We have: i) If $\langle \mathbf{a}, \mathbf{b} \rangle = \mathbb{Z}^2$ and $\varepsilon \in \mathcal{D}$, then $\langle V^\varepsilon_{\mathbf{a}}, V^\varepsilon_{\mathbf{b}} \rangle = F$.*

ii) If M, $N \in GL(2,\mathbb{Z})$ and $\varepsilon \in \mathcal{D}$, then $\Psi^\varepsilon_{NM} = \Psi^\varepsilon_N \Psi^{\varepsilon N}_M$.

iii) If $\mathbf{c} \in \mathbb{Z}^2$, then $V^\varepsilon_{\mathbf{c}M} = V^{\varepsilon M^{-1}}_{\mathbf{c}}(V^\varepsilon_{\mathbf{a}}, V^\varepsilon_{\mathbf{b}})$ where $M = \binom{\mathbf{a}}{\mathbf{b}} \in GL(2,\mathbb{Z})$.

Proof. Since $(S)\Psi^{\varepsilon M^{-1}}_M = V^\varepsilon_{\mathbf{a}}$ and $(T)\Psi^{\varepsilon M^{-1}}_M = V^\varepsilon_{\mathbf{b}}$, i) follows.

As $\mu^\varepsilon_{NM} = \mu^\varepsilon_N \mu^{\varepsilon N}_M$ we obtain $\nu^\varepsilon_{NM} = \nu^\varepsilon_N \nu^{\varepsilon N}_M$; therefore $\Psi^\varepsilon_{NM} = \varphi_\varepsilon \nu^\varepsilon_{NM} \varphi^{-1}_{\varepsilon NM} = \varphi_\varepsilon \nu^\varepsilon_N \nu^{\varepsilon N}_M \varphi^{-1}_{\varepsilon NM} = \varphi_\varepsilon \nu^\varepsilon_N \varphi^{-1}_{\varepsilon N} \varphi_{\varepsilon N} \nu^{\varepsilon N}_M \varphi^{-1}_{\varepsilon NM} = \Psi^\varepsilon_N \Psi^{\varepsilon N}_M$ which proves ii).

The identities $(W(S,T))\Psi^\varepsilon_M = W(V^{\varepsilon M}_{\mathbf{a}}, V^{\varepsilon M}_{\mathbf{b}})$ and $V^\varepsilon_{\mathbf{c}} \Psi^\varepsilon_M = V^{\varepsilon M}_{\mathbf{c}M}$ give $V^{\varepsilon M}_{\mathbf{c}M} = V^\varepsilon_{\mathbf{c}}(V^{\varepsilon M}_{\mathbf{a}}, V^{\varepsilon M}_{\mathbf{b}})$ that is equivalent to iii). □

Remarks. 1. By varying ε' along the segment from ε to $\varepsilon + \mathbf{a}$ we obtain as $V^{\varepsilon'}_{(m,n)}$ all the cyclic permutations of $V^\varepsilon_{(m,n)}$; also if ε', $\varepsilon \in \mathcal{D}$, then $V^{\varepsilon'}_{(m,n)}$ is a cyclic permutation of $V^\varepsilon_{(m,n)}$ since they are conjugate and cyclically reduced. Hence, if $\gcd(m,n) = 1$, then $\{V^\varepsilon_{(m,n)} : \varepsilon \in \mathcal{D}\}$ is the set of the primitive elements of F whose image under α is (m,n) and have minimal length (equal to $| m | + | n |$); this set, which contains $W_{m,n}$, has cardinality $| m | + | n |$ since $V^\varepsilon_{(m,n)}$ is not a proper power (cf. [OZ, 4.2]).

2. In Theorem 1.ii) one needs different superscripts: There is no collection $\{\Psi_M \in AutF : M \in GL(2,\mathbb{Z})\}$ such that the image of any Ψ_M under $\lambda : AutF \to OutF$ is M and $\Psi_{NM} = \Psi_N \Psi_M$. That is, λ does not split since there are elements of order 6 in $OutF$ but not in $AutF$ (see [LS, I.4.6]).

However there is a collection $\{\Psi_M \in AutF : M \in GL^+(2,\mathbb{Z})\}$, where $GL^+(2,\mathbf{Z})$ $= \{ \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in GL(2,Z) : m \geq 0, n \geq 0, p \geq 0, q \geq 0\}$, such that $\lambda(\Psi_M) = M$ and $\Psi_{NM} = \Psi_N \Psi_M$. To see this observe that $SL^+(2,\mathbb{Z})$, the set of matrices in $SL(2,\mathbb{Z})$ with nonnegative entries, is a free monoid on A and BAB where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ [CMZ, Lemma 3.5] and from this one obtains the monoid presentation $(A, B : B^2)$ of $GL^+(2,\mathbf{Z})$, that is, every element of $GL^+(2,\mathbf{Z})$ can be written uniquely as a word in A and B where the exponents of A are positive and the exponents of B are 1. One can define $(S)\Psi_A = ST$, $(T)\Psi_A = T$, $(S)\Psi_B = T$, $(T)\Psi_B = S$ and if $M = B^{\delta_1}(\prod_{i=1}^n A^{e_i} B) B^{\delta_2}$, $n > 0$, $e_i > 0$ $(i = 1, \ldots, n)$, $\delta_j = 0, 1$ $(j = 1, 2)$, we define $\Psi_M = \Psi_B^{\delta_1}(\prod_{i=1}^n \Psi_A^{e_i} \Psi_B) \Psi_B^{\delta_2}$. The Ψ's have the desired properties.

3. By Theorem 1.iii) if $M = \binom{\mathbf{a}}{\mathbf{b}}$ and $(\eta_1, \eta_2) = \varepsilon M^{-1}$, then we have a general addition formula that implies Theorem 1.3 of [OZ]:

$$V^{\varepsilon}_{\mathbf{a}+\mathbf{b}} = V^{(\eta_1,\eta_2)}_{(1,1)}(V^{\varepsilon}_{\mathbf{a}}, V^{\varepsilon}_{\mathbf{b}}) = \begin{cases} V^{\varepsilon}_{\mathbf{a}} V^{\varepsilon}_{\mathbf{b}} & \text{if } \eta_1 - [\eta_1] > \eta_2 - [\eta_2], \\ V^{\varepsilon}_{\mathbf{b}} V^{\varepsilon}_{\mathbf{a}} & \text{if } \eta_1 - [\eta_1] < \eta_2 - [\eta_2]. \end{cases}$$

4. It may be desirable to modify slightly the definition of the words $W_{m,n}$ given in [OZ] as follows: If $n \geq 0$ define the $W_{m,n}$ as in [OZ], but if $n < 0$ define $W_{m,n}$ as $W^{-1}_{-m,-n}$ not $W_{m,-n}(S, T^{-1})$, as stated in [OZ]. With this modification the analog of Theorem 1.i) holds, that is, if $mq - np = \pm 1$, then $\langle W_{m,n}, W_{p,q} \rangle = F$.

5. If we let ε be a pair of infinitesimals [SL] that does not lie in a line in ${}^*\mathbb{R}^2$ intersecting $\mathbb{Z}^2$ in more than one point and if $\mathbf{a} \in \mathbb{Z}^2$, then $V^{\varepsilon}_{\mathbf{a}}$ can be defined as in the fifth paragraph. Again $\langle V^{\varepsilon}_{\mathbf{a}}, V^{\varepsilon}_{\mathbf{b}} \rangle = F$ if $\langle \mathbf{a}, \mathbf{b} \rangle = \mathbb{Z}^2$ and the assertion 1.iii) is still valid. Then, with our modification of the definition of $W_{m,n}$ given in the previous remark, $W_{m,n} = V^{(-\delta i^2, -i)}_{(m,n)}$ where i is a positive infinitesimal and $\delta = \begin{cases} 1 \text{ if } mn \geq 0, \\ -1 \text{ if } mn < 0. \end{cases}$

If $kl \neq 0$ and $\gcd(k,l) = 1$, the axes and the line through the origin and (k,l) divide the plane ${}^*\mathbb{R}^2$ into six open regions. If the infinitesimal pairs ε and ε' belong to the same region, then $V^{\varepsilon}_{(k,l)} = V^{\varepsilon'}_{(k,l)}$. If $V'_{k,l}$ is the word defined by the open segment from $(0,0)$ to (k,l) (cf. [OZ, Definition 2.1]), $\delta_1 = sgn\, k$ and $\delta_2 = sgn\, l$, then $V^{\varepsilon}_{(k,l)}$ is one of the words $S^{\delta_1} T^{\delta_2} V'_{k,l}$, $T^{\delta_2} S^{\delta_1} V'_{k,l}$, $S^{\delta_1} V'_{k,l} T^{\delta_2}$, $T^{\delta_2} V'_{k,l} S^{\delta_1}$, $V'_{k,l} S^{\delta_1} T^{\delta_2}$ or $V'_{k,l} T^{\delta_2} S^{\delta_1}$ depending on the region in which ε lies.

Let $\overrightarrow{W}_{k,l} = V^{i(l+i,-k+\sqrt{2}i)}_{k,l}$ and $\overleftarrow{W}_{k,l} = V^{-i(l+i,-k+\sqrt{2}i)}_{k,l}$, thus, if $k > 0$, $l > 0$ and $\gcd(k,l) = 1$, then $\overrightarrow{W}_{k,l} = TV'_{k,l}S$ and $\overleftarrow{W}_{k,l} = SV'_{k,l}T$. Now, if k, l, n and q are nonnegative integers, $m > 0$, $p > 0$ with $\gcd(k,l) = 1$ and $mq - np = d = \pm 1$, then

$$W_{km+lp,kn+lq} = \begin{cases} \overrightarrow{W}_{k,l}(W_{m,n}, W_{p,q}) & \text{if } d = 1, \\ \overleftarrow{W}_{k,l}(W_{m,n}, W_{p,q}) & \text{if } d = -1. \end{cases}$$

This follows from Theorem 1.iii) taking $\varepsilon = (-i^2, -i)$ and $M = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$; then $V^{\varepsilon M^{-1}}_{(k,l)} = \begin{cases} \overrightarrow{W}_{k,l} & \text{if } d = 1, \\ \overleftarrow{W}_{k,l} & \text{if } d = -1. \end{cases}$ This gives the modification needed in [OZ, Theorem 3.5] and [LTZ, 2.1.3].

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