A COMPOSITION FORMULA
IN THE RANK TWO FREE GROUP

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Abstract. Using the fundamental group of a punctured torus, a free group $F$ of rank two, and the fact that the natural epimorphism from $\text{Aut}(F)$ onto $\text{Aut}(F/F')$ has as kernel the group of inner automorphisms of $F$, we describe representatives of the conjugacy classes of generating pairs of $F$ and give explicit relations between them.

Let $F = F(S,T)$ be the free group on $S$ and $T$. By a theorem of Nielsen [N] (see [LS, p. 25]) the natural epimorphism from $\text{Aut}(F)$ onto $\text{Aut}(F/F')$ ($= \text{GL}(2,\mathbb{Z})$) has as kernel the group of inner automorphisms of $F$. From this it follows easily that, if $\alpha$ is the abelianization homomorphism from $F$ onto $F/F'$ ($= \mathbb{Z}^2$) and $a \in \mathbb{Z}^2$ is primitive$^1$, then the inverse image of $a$ under $\alpha$ is a conjugacy class of primitive elements. Also, if $(a_1, a_2)$ is a basis of $\mathbb{Z}^2$, then, up to conjugacy, there is a unique basis $(f_1, f_2)$ of $F$ such that $(f_i)\alpha = a_i$ ($i = 1, 2$). (The basis $(f_1, f_2)$ is conjugate to $(g_1, g_2)$ if there exists $w \in F$ such that $w^{-1}f_iw = g_i$ ($i = 1, 2$)).

In the important paper [OZ], Osborne and Zieschang define explicitly primitive words $W_{m,n} \in F(S,T)$, where $m$ and $n$ are relatively prime integers, such that $(W_{m,n})\alpha = (m,n)$. They also state that if $mn - pq = 1$, then $(W_{m,n}W_{p,q})$ is a basis of $F$; this, while correct for nonnegative values of $m, n, p, q$, is not valid in general (for example $W_{-2,-3}$ and $W_{1,1}$ do not generate $F$). A composition formula is also stated in [OZ, Thm. 3.5] but this, even with the correction of indices in [LTZ, 2.1.3], is incorrect in general.

In the present article we consider elements $V_{a}^\varepsilon$ of $F$ for $a = (m,n) \in \mathbb{Z}^2$ and $\varepsilon \in \mathcal{D} \subset \mathbb{R}^2$ where $\mathcal{D}$ is the complement of the union of all the lines that intersect $\mathbb{Z}^2$ in more than one point. If $\gcd(m,n) = 1$, then $V_{(m,n)}^\varepsilon$ is conjugate to $W_{m,n}$. We show in Theorem 1.i) that $(V_{a}^\varepsilon, V_{b}^\alpha)$ is a basis of $F$, if $\mathbb{Z}^2 = \langle a, b \rangle$, and obtain in Theorem 1.ii) a composition formula. Everything is obtained by applying the fundamental group functor $\pi$ to the punctured torus.

Denote by $T$ the torus $\mathbb{R}^2/\mathbb{Z}^2$, by $T_0$ the punctured torus $(\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$ and by $\rho : \mathbb{R}^2 - \mathbb{Z}^2 \to T_0$ the natural projection. If $a \in \mathbb{Z}^2$ and $\varepsilon \in \mathcal{D}$, then denote $(\varepsilon)\rho$ by $\pi$ and define $\gamma_a^\varepsilon \in \pi(T_0, \pi)$ as the homotopy class of the loop $(\varepsilon + t a)\rho$, $t \in [0,1]$. Denote $\gamma_{(1,0)}^\varepsilon$ (resp. $\gamma_{(0,1)}^\varepsilon$) by $S_{\varepsilon}$ (resp. $T_{\varepsilon}$). There is an isomorphism

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$^1$An element $f$ of a rank two group $G$ is primitive if there exists $g \in G$ such that $f$ and $g$ generate $G$.  

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\( \varphi_e : F(S,T) \to \pi(T_0, \pi) \) sending \( S \) to \( S_e \) and \( T \) to \( T_e \). Define \( V_e^a \in F(S,T) \) by \( (V_e^a)^a = \gamma_e^a \). Notice that \((V_e^a) \alpha = a \) since the inclusion induced homomorphism \( \pi(T_0, \pi) \to \pi(T, \pi) \) is the abelianization.

In comparison with \([OZ, 4.2]\) \( V_e^a \) is represented by the word obtained by traveling along the segment, in \( \mathbb{R}^2 - \mathbb{Z}^2 \), from \( e \) to \( e + a \) and writing \( S \) (resp. \( S^{-1} \)) whenever we cross a component of \( \mathbb{Z} \times \mathbb{R} \) from left to right (resp. from right to left) and writing \( T \) (resp. \( T^{-1} \)) whenever we cross a component of \( \mathbb{R} \times \mathbb{Z} \) from below (resp. from above).

A matrix \( M = \binom{a}{b} \in GL(2, \mathbb{Z}) \) defines a linear automorphism of \((\mathbb{R}^2, \mathbb{Z}^2)\) that, for any \( e \in D \), induces a homeomorphism \( \mu_M : (T_0, \pi) \to (T, \pi) \). This induces \( \nu_M : \pi(T_0, \pi) \to \pi(T, \pi) \) and an automorphism \( \Psi^M_M \) of \( F \) defined by \( \Psi^M_M = \varphi_e \nu_M \varphi_e^{-1} \).

We have \( \gamma_e \nu_M = \gamma_e M \) since \( \mu_M = M \rho \) and therefore \( V_e^c \Psi^M_M = V_e^c M \) for all \( c \in \mathbb{Z}^2 \) (this equality was suggested by the referee). In particular \((S)\Psi^M_M = V_e^c M \) and \((T)\Psi^M_M = V_e^c M \). Notice that, for any word \( W(S,T) \) we have \((W(S,T))\Psi^M_M = W((S)\Psi^M_M,(T)\Psi^M_M) = W(V_e^c M, V_e^c M) \). One has the following composition theorem.

**Theorem 1.** We have: i) If \( (a, b) = \mathbb{Z}^2 \) and \( e \in D \), then \( (V_e^a, V_e^b) = F \).

ii) If \( M, N \in GL(2, \mathbb{Z}) \) and \( e \in D \), then \( \Psi^M_N M = \Psi^N_N \Psi^M_M \).

iii) If \( c \in \mathbb{Z}^2 \), then \( V_e^c M = V_e^c M^{-1} (V_e^a, V_e^b) \) where \( M = \binom{a}{b} \in GL(2, \mathbb{Z}) \).

**Proof.** Since \((S)\Psi^M_M^{-1} = V_e^a \) and \((T)\Psi^M_M^{-1} = V_e^b \) i) follows.

As \( \mu^M_N = \mu^N_M \mu_M^N \) we obtain \( \nu^e_M = \nu^e_N \nu^e_M \); therefore \( \Psi^M_N = \varphi_e \nu^e_N \nu^e_M \varphi_e^{-1} = \varphi_e \nu^e_M \nu^e_N \nu^e_M \varphi_e \), which proves ii).

The identities \((W(S,T))\Psi^M_M = W(V_e^c M, V_e^c M) \) and \( V_e^c M = V_e^c M^{-1} (V_e^a, V_e^b) \) are equivalent to iii). \( \square \)

**Remarks.** 1. By varying \( e' \) along the segment from \( e \) to \( e + a \) we obtain all the cyclic permutations of \( V_{(m,n)}^e \); also if \( e', e \in D \), then \( V_{(m,n)}^{e'} \) is a cyclic permutation of \( V_{(m,n)}^e \) since they are conjugate and cyclically reduced. Hence, if \( \gcd(m,n) = 1 \), then \( \{V_{(m,n)}^e : e \in D\} \) is the set of the primitive elements of \( F \) whose image under \( \alpha \) is \((m,n)\) and have minimal length (equal to \( |m| + |n| \)); this set, which contains \( W_{m,n} \), has cardinality \( |m| + |n| \) since \( V_{(m,n)}^e \) is not a proper power (cf. \([OZ, 4.2]\)).

2. In Theorem 1.ii) one needs different superscripts: There is no collection \( \{\Psi_M \in Aut F : M \in GL(2, \mathbb{Z})\} \) such that the image of any \( \Psi_M \) under \( \lambda : Aut F \to Out F \) is \( M \) and \( \Psi_N M = \Psi_N \Psi_M \). That is, \( \lambda \) does not split since there are elements of order 6 in \( Out F \) but not in \( Aut F \) (see \([LS, 1.4.6]\)).

However there is a collection \( \{\Psi_M \in Aut F : M \in GL^+(2, \mathbb{Z})\} \), where \( GL^+(2, \mathbb{Z}) = \{A \in GL(2, \mathbb{Z}) : m \geq 0, n \geq 0, p \geq 0, q \geq 0\} \), such that \( \lambda(\Psi_M) = M \) and \( \Psi_N M = \Psi_N \Psi_M \). To see this observe that \( SL^+(2, \mathbb{Z}) \), the set of matrices in \( SL(2, \mathbb{Z}) \) with nonnegative entries, is a free monoid on \( A \) and \( BAB \) where \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), \( B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) [CMZ, Lemma 3.5] and from this one obtains the monoid presentation \( \{A, B, B^2\} \) of \( GL^+(2, \mathbb{Z}) \), that is, every element of \( GL^+(2, \mathbb{Z}) \) can be written uniquely as a word in \( A \) and \( B \) where the exponents of \( A \) are positive and the exponents of \( B \) are 1. One can define \((S)\Psi_A = ST, (T)\Psi_A = T, (S)\Psi_B = T, (T)\Psi_B = S \) and if \( M = B^{c_1} (\prod_{i=1}^{l} A^{e_i} B^{B_{2i}}), n > 0, c_i > 0 (i = 1, \ldots, n), \delta_j = 0, 1 (j = 1, 2) \), we define \( \Psi_M = \Psi_B^j (\prod_{i=1}^{l} A^{e_i} \Psi_B \Psi_B^j) \). The \( \Psi \)’s have the desired properties.
3. By Theorem 1.iii) if $M = (\begin{smallmatrix} a \\ b \end{smallmatrix})$ and $(\eta_1, \eta_2) = \varepsilon M^{-1}$, then we have a general addition formula that implies Theorem 1.3 of [OZ]:

$$V_{a+b} = V_{(1,1)}^{(\eta_1, \eta_2)} (V_a, V_b) = \begin{cases} V_a V_b^{\varepsilon} & \text{if } \eta_1 - [\eta_1] > \eta_2 - [\eta_2], \\ V_a V_b^{\varepsilon} & \text{if } \eta_1 - [\eta_1] < \eta_2 - [\eta_2]. \end{cases}$$

4. It may be desirable to modify slightly the definition of the words $W_{m,n}$ given in [OZ] as follows: If $n \geq 0$ define the $W_{m,n}$ as in [OZ], but if $n < 0$ define $W_{m,n}$ as

$$W = \begin{cases} \text{not } & \text{if } mn \geq 0, \\ 1 & \text{if } mn < 0. \end{cases}$$

If $kl \neq 0$ and $\gcd(k,l) = 1$, the axes and the line through the origin and $(k,l)$ divide the plane $\mathbb{R}^2$ into six open regions. If the infinitesimal pairs $\varepsilon$ and $\varepsilon'$ belong to the same region, then $V_{(k,l)}' = V_{(k,l)}''$. If $V_{(k,l)}'$ is the word defined by the open segment from $(0,0)$ to $(k,l)$ (cf. [OZ, Definition 2.1]), $\delta_1 = \text{sgn } k$ and $\delta_2 = \text{sgn } l$, then $V_{(k,l)}$ is one of the words $S^{\delta_1} T^{\delta_2} V_{(k,l)}'$, $S^{\delta_1} V_{(k,l)}' T^{\delta_2}$, $T^{\delta_2} V_{(k,l)}' S^{\delta_1}$, $V_{(k,l)} S T^{\delta_2}$ or $V_{(k,l)} T^{\delta_2} S^{\delta_1}$ depending on the region in which $\varepsilon$ lies.

Let $W_{k,l} = V_{(l+i,-k+\sqrt{2}l)}$ and $W_{k,l} = V_{(l+i,-k+\sqrt{2}l)}^{-1}$, thus, if $k > 0$, $l > 0$ and $\gcd(k,l) = 1$, then $W_{k,l} T = TV_{k,l}' S$ and $W_{k,l} T = SV_{k,l}' T$. Now, if $k$, $l$, $n$ and $q$ are nonnegative integers, $m > 0$, $p > 0$ with $\gcd(k,l) = 1$ and $mq - np = d = \pm 1$, then

$$W_{km+lp,kn+lq} = \begin{cases} W_{k,l} (W_{m,n}, W_{p,q}) & \text{if } d = 1, \\ W_{k,l}^{-1} (W_{m,n}, W_{p,q}) & \text{if } d = -1. \end{cases}$$

This follows from Theorem 1.iii) taking $\varepsilon = (-i^2, -i)$ and $M = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$; then

$$V_{(k,l)}^{M^{-1}} = \begin{cases} W_{k,l} & \text{if } d = 1, \\ W_{k,l}^{-1} & \text{if } d = -1. \end{cases}$$

This gives the modification needed in [OZ, Theorem 3.5] and [LTZ, 2.1.3].

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