A SEPARABLE SPACE WITH NO SCHAUDEL DECOMPOSITION

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Abstract. We combine some known results to remark that there exists a separable Banach space which fails to have a Schauder decomposition. It can be chosen as a subspace of Gowers-Maury space without any unconditional basic sequence.

The following problem was raised in [Si] (Problem 15.1, p. 494): Does every separable Banach space have a Schauder decomposition? This question goes back to J. R. Retherford [R].

Recall that a sequence \( \{X_n\}_{n=1}^{\infty} \) of closed subspaces of a Banach space \( X \) is said to be a Schauder decomposition of \( X \) if every \( x \in X \) has a unique representation of the form \( x = \sum_{n=1}^{\infty} x_n \), with \( x_n \in X_n \) for every \( n \).

Let \( GM \) be Gowers-Maury space which does not contain any unconditional basic sequence [GM]. As was observed by W. B. Johnson, \( GM \) has in fact a stronger property, namely it is hereditarily indecomposable (H.I.); i.e., no infinite-dimensional closed subspace can be written as a direct sum \( Y \oplus Z \), where \( Y \) and \( Z \) are infinite-dimensional closed subspaces. It is known that every block subspace of \( GM \) contains uniform copies of \( \ell_1^n \). This follows from the lower \( f \)-estimate and Krivine’s theorem as in [S]. Then, by Szankowski’s refinement of Enflo’s criterion (see [LT2, p. 111, Remark 1]), we immediately obtain the following.

Proposition. There exists a subspace \( X \) of \( GM \) which does not have the compact approximation property (C.A.P.).

Remark 1. For the same purpose we can as well use other H.I. spaces constructed after the breakthrough of W. T. Gowers and B. Maurey. For example, there are subspaces without the C.A.P. of the super-reflexive H.I. spaces in [F] in the case when they contain uniform copies of \( \ell_p^n \) for \( p \neq 2 \). One can also use the asymptotic \( \ell_1 \) hereditarily indecomposable spaces constructed in [AD] and [ADKM]. The existence of uniform copies of \( \ell_1^n \) in these spaces follows directly from the definition and one does not need to apply Krivine’s theorem. Therefore, they also have subspaces without the C.A.P.

Corollary. The space \( X \) is an example of a separable Banach space with no Schauder decomposition.
Proof. Assume the contrary, i.e. $X$ has a Schauder decomposition $\{X_n\}_{n=1}^\infty$.

Case 1. $\{X_n\}_{n=1}^\infty$ is a finite-dimensional decomposition. This is impossible since the existence of an $F.D.D.$ implies $B.A.P.$ which in turn implies $C.A.P.$ (see [LT1]) and this contradicts the above Proposition.

Case 2. There exists $m$ such that $X_m$ is infinite-dimensional. Denote $Y = \{X_n : n \neq m\}$. Then $X = X_m \oplus Y$ which is also impossible because $X_m$ and $Y$ are closed infinite-dimensional subspaces of $X$, $X$ is a closed subspace of $GM$, and $GM$ is $H.I.$

Remark 2. Clearly, the result is true hereditarily in all the above mentioned $H.I.$ spaces, e.g. we have that every subspace of $GM$ has a further subspace which has no Schauder decomposition.

References


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