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AMPLE DIVISORS ON THE BLOW UP OF $\mathbf{P}^n$ AT POINTS

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ABSTRACT. Fix integers n, k, d with $n \geq 2, d \geq 2$ and $k > 0$; if $n = 2$ assume $d \geq 3$. Let $P_1, \dots, P_k$ be general points of the complex projective space $\mathbf{P}^n$ and let $\pi : X \to \mathbf{P}^n$ be the blow up of $\mathbf{P}^n$ at $P_1, \dots, P_k$ with exceptional divisors $E_i := \pi^{-1}(P_i)$, $1 \leq i \leq k$. Set $H := \pi^*(\mathbf{O}_{\mathbf{P}^n}(1))$. Here we prove that the divisor $L := dH - \sum_{1 \leq i \leq k} E_i$ is ample if and only if $L^n > 0$, i.e. if and only if $d^n > k$.

The aim of this paper is the proof of the following result conjectured in [1].

Theorem 0.1. *Fix integers n, k, d with $n \geq 2, d \geq 2$ and $k > 0$; if $n = 2$ assume $d \geq 3$. Let $P_1, \dots, P_k$ be general points of the complex projective space $\mathbf{P}^n$ and let $\pi : X \to \mathbf{P}^n$ be the blow up of $\mathbf{P}^n$ at $P_1, \dots, P_k$ with exceptional divisors $E_i := \pi^{-1}(P_i)$, $1 \leq i \leq k$. Set $H := \pi^*(\mathbf{O}_{\mathbf{P}^n}(1))$. Then the divisor $L := dH - \sum_{1 \leq i \leq k} E_i$ is ample if and only if $L^n > 0$, i.e. if and only if $d^n > k$.*

Here "general points" means "outside countably many proper subvarieties of the symmetric product $S^k(\mathbf{P}^n)$ of $\mathbf{P}^n$". For $n = 2$ and 3 Theorem 0.1 was proved in [1] (at least for $d \geq 5$). In [1], §3, the proof of Theorem 0.1 was reduced to the proof of a general lemma ([1], Lemma 2.1) which was proved there only for $n = 2$ and 3 (see in particular page 45, lines 4 and 5 from the bottom). We will prove that lemma for every n and hence obtain for free a proof of Theorem 0.1. Indeed we will prove the following more general result which is very classical (for $n = 2$ being due to Kronecker and Castelnuovo).

Lemma 0.2. *Let X be a smooth projective variety of dimension $n \geq 2$. Take $H \in \mathrm{Pic}(X)$, H very ample. Assume that (X, H) is not a scroll over a smooth curve, i.e. assume that X is not a $\mathbf{P}^{n-1}$-bundle $\pi : X \to C$ over a smooth curve with H degree 1 line bundle on each fiber. If $n = 2$ and $X = \mathbf{P}^2$ assume $\deg(H) \neq 2$. Let V be a linear subspace of $H^0(X, H)$ which induces an embedding of X. Let W be a general linear subspace of V with $\dim(W) = n$. Then for every hyperplane M of W the base locus of M is a reduced and irreducible curve.*

Proof. First assume $n = 2$. Consider the embedding $X \subset \mathbf{P}(V)$ associated to V. By Lefschetz' theorem for a general pencil $\{H(\lambda)\}_{\lambda \in \mathbf{P}^1}$ of hyperplanes of X each $H(\lambda) \cap X$ is either a smooth curve or it has a single ordinary double point. We claim that, except for the list given, this implies that every $H(\lambda) \cap X$ is reduced and irreducible. Since $\mathrm{Sing}(H(\lambda) \cap X)$ is finite, $H(\lambda) \cap X$ is reduced. In order to obtain a contradiction we assume that $H(\lambda) \cap X$ is reducible, say $H(\lambda) \cap X = A \cup B$ with

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A irreducible and $B \neq \emptyset$. Since H is very ample, $H(\lambda) \cap X$ is 1-connected (see e.g. [3], Remark after the statement of Th. I). Furthermore, outside the list given $H(\lambda) \cap X$ is 2-connected ([3], Th. I). This implies that either $\mathrm{card}(\mathrm{Sing}(H(\lambda) \cap X)) \geq \mathrm{card}(A \cap B) \geq 2$ or there is $P \in A \cap B$ with $A \cup B$ not an ordinary nodal curve at P, a contradiction. Now assume $n \geq 3$. Taking the 0-locus $F_1, \dots, F_{n-2}$ of $n-2$ general elements of V we obtain a smooth surface $X' := F_1 \cap \cdots \cap F_{n-2}$. Apply the first part to the case $(X', H|X', V|X')$. We may apply the case $n = 2$ because we did not require that $V = H^0(X, H)$, but only that V embeds X. Assume that $X' \cong \mathbf{P}^2$ and $\deg(H|X') = 2$ or that $(X', H|X')$ is a scroll over a smooth curve. By the adjunction formula we have $K_{X'} \cong K_X \otimes H^{\otimes n-2}|X'$. Hence in both cases $K_X \otimes H^{\otimes n-1}$ is not spanned by global sections. By [2], Th. 1, and the assumption $n \geq 3$ the pair (X, H) is a scroll over a smooth curve.

References

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