SINGULAR MEASURES WITH ABSOLUTELY CONTINUOUS CONVOLUTION SQUARES ON LOCALLY COMPACT GROUPS

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Abstract. Saeki’s result states that on any locally compact nondiscrete group there exist continuous singular measures, with respect to the left Haar measure, $\mu$ with $\mu \ast \mu$ in $L^p$ for all $p$, $1 \leq p < \infty$. This paper gives a new and short proof of this using Rademacher-Riesz products.

1. Introduction

In 1938, Wiener and Wintner [9], [3], p.209, proved that there exists a singular probability measure $\mu$ on $[-\pi, \pi]$ such that $\mu \ast \mu$ is absolutely continuous. In 1966, Hewitt and Zuckerman [4] showed that every nondiscrete locally compact abelian group supports such a measure (absolutely continuous with respect to Haar measure) and the Lebesgue-Radon-Nikodým derivative of $\mu \ast \mu$ is in $L^p$ for all $p$, $1 \leq p < \infty$. Hewitt has asked whether such measures exist on every (locally) compact group [7], p.217. Ragozin [7] proved that every connected compact simple Lie group has a measure with absolutely continuous square. In 1977 Saeki [8], p. 403, proved Hewitt’s question. We also note two papers on this subject. One by Karanikas and Koumandos [6] which makes use of the Rademacher-Riesz products, and another by Dooley and Gupta [2] which makes use of the theory of compact Lie groups. In this paper we give a new and short proof of Saeki’s result. Our ideas have been adapted from [6]; see also [1] and [5]. Section 2 contains a brief summary, for the reader’s convenience, of some results and techniques from [6] that we shall need to use. Our proof of the main theorem appears in section 3.

2. Terminology

Let $G$ be a locally compact, nondiscrete metrizable group and $E$ a compact subset of $G$ with left Haar measure equal to one. We define a system of Rademacher functions $(r_n)_{n=1}^\infty$ as follows: We divide $E$ into two disjoint subsets, $E_{11}, E_{12}$ of equal measure and define $r_1(x) = 1$ for $x \in E_{11}$ and $r_1(x) = -1$ for $x \in E_{12}$. Similarly, we divide each of these two subsets $E_{11}, E_{12}$ to define $r_2$, etc. We define the Walsh functions $(w_n)_{n=0}^\infty$ by the relations:

$$w_n = r_{j_1}r_{j_2} \ldots r_{j_p}, \quad n = 2^{j_1-1} + 2^{j_2-1} + \ldots + 2^{j_p-1}, \quad j_1 < j_2 < \ldots < j_p.$$
The existence is given by the following Lemma:

**Lemma 1.** On any nondiscrete metrizable group we can find a set $E$ and a partition into sets of equal measure, $E_{n,k}$, $1 \leq k \leq 2^n$, such that:

(i) $\max_{1 \leq k \leq 2^n} \text{diam} E_{n,k} \to 0$, as $n \to 0$, (diam $E_{n,k}$ denotes the diameter of $E_{n,k}$ with respect to the metric of $G$).

(ii) The Walsh system is complete in $L^2(E)$.

(iii) If $f, g \in L^1(G)$, then for any $p \in [1, \infty)$, $\lim_{n \to \infty} \int f w_n = 0$, $\lim_{n \to \infty} ||f*(gw_n)||_p = 0$ and $\lim_{n \to \infty} ||(gw_n) * f||_p = 0$.

**Proof.** For (i) and (ii) see [6], Lemma 2.1, and for (iii) see the proof of Lemma 3.2 in [6].

We have also (see [1], [5], [6]) that

**Lemma 2.** Let $f_n(x) = \prod_{j=1}^{n} (\chi_E(x) + a_j r_j(x))$, where $\chi_E$ is the characteristic function of $E$ and $|a_j| \leq 1$. Then

(i) The sequence $f_n$ converges weak* to some $\mu$ in $M(G)$, the convolution algebra of bounded complex-valued Borel measures on $G$.

(ii) The measure $\mu$ is singular if and only if $\sum_{n=1}^{\infty} a_n^2 = \infty$.

(iii) The measure $\mu$ is continuous if and only if $\sum_{n=1}^{\infty} (1 - |a_n|) = \infty$.

3. THE MAIN THEOREM

**Theorem 1.** Let $G$ be a nondiscrete locally compact metrizable group. Then there exists a continuous singular measure $\mu \in M(G)$ with absolutely continuous convolution square and the Lebesgue-Radon-Nikodým derivative of $\mu * \mu$ is in $L^p(G)$ for all $p$, $1 \leq p < \infty$.

**Proof.** Let $(a_n)_{n=1}^{\infty}$ be a sequence such that $\sum_{n=1}^{\infty} a_n^2 = \infty$, $\sum_{n=1}^{\infty} a_{n+1}^4 \left( \sum_{k=1}^{n} a_k^2 \right)^q < \infty$, for any $q \in \mathbb{N}$ and $|a_n| \leq 1$. We shall show that there exists a subsequence $(r_{m_k})_{k=1}^{\infty}$ of Rademacher functions such that if $f_n(x) = \prod_{k=1}^{n} (\chi_E(x) + a_k r_{m_k}(x))$, then the weak* limit $\mu$ of $f_n$ is a continuous singular measure $\mu \in M(G)$, with absolutely continuous convolution square and the Lebesgue-Radon-Nikodým derivative of $\mu * \mu$ is in $L^p(G)$ for all $p$, $1 \leq p < \infty$. From Lemma 2 we have that $\mu$ is a continuous singular measure. It is easy to show that $f_n * f_n \to \mu * \mu$, $n \to \infty$, weak* in $M(G)$ and so from completeness of $L^p$ it is enough to show that $f_n * f_n$ is a Cauchy sequence in $L^p$. It is also enough to prove this only for $p$ an even integer. For simplicity of notation we write

$t_n = f_n * f_n, \quad g_n = (f_n r_{m_{n+1}}) * f_n, \quad h_n = f_n * (f_n r_{m_{n+1}}),

s_n = (f_n r_{m_{n+1}}) * (f_n r_{m_{n+1}}), \quad n \in \mathbb{N}.$

We shall estimate the norm $||t_n - t_m||_p, \quad n > m$. Schwarz’s inequality yields

$$||t_n - t_m||_p^p = || \sum_{i=m}^{n-1} (t_{i+1} - t_i)||_p^p = || \sum_{i=m}^{n-1} [a_{i+1}(g_i + h_i) + a_{i+1}^2 s_i]||_p^p$$
\[
\sum_{\nu=0}^{p} \binom{p}{\nu} \int \left[ \sum_{i=m}^{n-1} a_{i+1} (g_i + h_i) \right]^{p-\nu} \left[ \sum_{i=m}^{n-1} a_i^2 s_i \right]^{\nu} \\
\leq \sum_{\nu=0}^{p} \binom{p}{\nu} \left( \int \left[ \sum_{i=m}^{n-1} a_{i+1} (g_i + h_i) \right]^{2(p-\nu)} \right)^{\frac{1}{2}} \left( \int \left[ \sum_{i=m}^{n-1} a_i^2 s_i \right]^{2\nu} \right)^{\frac{1}{2}}.
\]

For \( \nu \geq 1 \) we have
\[
\int \left[ \sum_{i=m}^{n-1} a_i^2 s_i \right]^{2\nu} = \sum_{i=m}^{n-1} a_i^{4\nu} \int s_i^{2\nu} \\
+ \sum_{k=m+1}^{n-1} \sum_{l=1}^{2\nu-1} \binom{2\nu}{l} \int \left[ a_{k+1}^2 s_k \right]^{l} \left( \sum_{i=m}^{k-1} a_i^2 s_i \right)^{2\nu-l}.
\]

Using Lemma 1(iii) and Fubini’s theorem we find \((r_{m,k})_{k \geq 1}\) such that
\[
\max \left\{ \left| \int \left[ \sum_{i=m}^{k-1} a_i^2 s_i \right]^{2\nu-1} \right| \right\}, \ 1 \leq \nu \leq k, \ 1 \leq m \leq k-1 \leq \frac{1}{k^2}
\]
and, at the same time, using Lemma 1(iii),
\[
\max \left\{ \left| g_i \right|, \left| h_i \right|, \ k = 1, \ldots, 2i \right\} < \frac{1}{2i^3}, \ i = 1, 2, \ldots.
\]

Thus, for \( p < m < n \), using Minkowski’s inequality we obtain
\[
\left| t_n - t_m \right|^p \leq \frac{1}{m^{2p}} + \sum_{\nu=1}^{p} \binom{p}{\nu} \frac{1}{m^{2(p-\nu)}} \left[ \sum_{i=m}^{n-1} a_i^{4\nu} \int s_i^{2\nu} + 2\nu \sum_{k=m+1}^{n-1} \frac{1}{k^2} \\
+ \sum_{k=m+1}^{n-1} \sum_{l=2}^{2\nu-1} \binom{2\nu}{l} \int \left[ a_{k+1}^2 s_k \right]^{l} \left( \sum_{i=m}^{k-1} a_i^2 s_i \right)^{2\nu-l} \right]^{\frac{1}{2}}.
\]

Applying Schwarz’s and Minkowski’s inequalities successively, in the last term of the above, we observe that, in view of our hypothesis on the sequence \((a_n)_{n=1}^{\infty}\) and since
\[
\left| s_n \right| \leq |t_n|, \ n \in \mathbb{N},
\]
it suffices to prove that \( \sup \left| t_n \right|^p < c_p \), where \( c_p \) is a constant which depends only
on \( p \). We have
\[
\left| t_{n+1} \right|^p = \left| t_n + a_{n+1} (g_n + h_n) + a_{n+1}^2 s_n \right|^p \\
= \sum_{k_2=0}^{p} \sum_{k_1=0}^{k_2} \binom{p}{k_2} \binom{k_2}{k_1} C_{p,n}(k_1, k_2),
\]
where
\[
C_{p,n}(k_1, k_2) = a_{n+1}^{2p-k_2-k_1} \int t_n^{k_1} (g_n + h_n)^{k_2-k_1} s_n^{p-k_2}.
\]
If \( k_1 < k_2 \), then by Lemma 1(iii) and (3) we find \( r_{m,n+1} \) such that
\[
\max \{ |C_{p,n}(k_1, k_2)|, \ 0 \leq k_1 < k_2 \leq p, \ 2 \leq p \leq n+1 \} \leq \frac{a_{n+1}^2}{3n+1} \min_{2 \leq k \leq n+1} \left| t_n \right|^k.
\]
If \( k_1 = k_2 \), then we have
\[
C_{p,n}(k_1, k_1) = a_{n+1}^{2(p-k_1)} \int t_n^{k_1} s_n^{p-k_1}.
\]

If \( k_1 \leq p - 2 \), then using (3) we have
\[
\sum_{k_1=0}^{p-2} \binom{p}{k_1} |C_{p,n}(k_1, k_1)| \leq \sum_{k_1=0}^{p-2} \binom{p}{k_1} a_{n+1}^{2(p-k_1)} \int t_n^{k_1} s_n^{p-k_1}
\leq a_{n+1}^4 (2^p - (p + 1)) |t_n|^p.
\]

If \( k_1 = p - 1 \), then Fubini’s theorem and Lemma 1(iii) show that we can find \( r_{m+1} \) such that
\[
\max_{2 \leq p \leq n+1} |C_{p,n}(p-1, p-1)| \leq a_{n+1}^4 \min_{2 \leq k \leq n+1} |t_n|^k.
\]

Finally, for \( k_1 = p \) we have
\[
|C_{p,n}(p, p)| \leq |t_n|^p.
\]

Subsequently, we can find \( r_{m+1} \) satisfying (4) and (5) such that
\[
||t_{n+1}||_p \leq (1 + 2^p a_{n+1}^4) ||t_n||_p, \quad 2 \leq p \leq n + 1.
\]

This implies that sup \( ||t_{n+1}||_p < c_p \). It is clear that we can find a subsequence \( (r_m)_{k=1}^\infty \), using induction, satisfying the relations (1), (2), (4) and (5) simultaneously. This completes the proof.

**Corollary 1.** The result of Theorem 1 holds for any nondiscrete locally compact group.

**Proof.** As in [6]; see also [10].

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**References**


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