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SELECTIVE ULTRAFILTERS AND $\omega \longrightarrow (\omega)^\omega$

TODD EISWORTH

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ABSTRACT. Mathias (*Happy families*, Ann. Math. Logic. **12** (1977), 59–111) proved that, assuming the existence of a Mahlo cardinal, it is consistent that CH holds and every set of reals in $L(\mathbb{R})$ is $\mathcal{U}$-Ramsey with respect to every selective ultrafilter $\mathcal{U}$. In this paper, we show that the large cardinal assumption cannot be weakened.

1. Introduction

Ramsey's theorem [5] states that for any $n \in \omega$, if the set $[\omega]^n$ of n-element sets of natural numbers is partitioned into finitely many pieces, then there is an infinite set $H \subseteq \omega$ such that $[H]^n$ is contained in a single piece of the partition. The set H is said to be homogeneous for the partition.

We can attempt to generalize Ramsey's theorem to $[\omega]^\omega$, the collection of infinite subsets of the natural numbers. Let $\omega \longrightarrow (\omega)^\omega$ abbreviate the statement "for every $\mathcal{X} \subseteq [\omega]^\omega$ there is an infinite $H \subseteq \omega$ such that either $[H]^\omega \subseteq \mathcal{X}$ or $[H]^\omega \cap \mathcal{X} = \emptyset$." Using the axiom of choice, it is easy to partition $[\omega]^\omega$ into two pieces in such a way that any two infinite subsets of ω differing by a single element lie in different pieces of the partition. Obviously such a partition can admit no infinite homogeneous set. Thus under the axiom of choice, $\omega \longrightarrow (\omega)^\omega$ is false.

Having been stymied in this naive attempt at generalization, we have several obvious ways to proceed. A natural project would be to attempt to find classes of partitions of $[\omega]^\omega$ that do admit infinite homogeneous sets. The following definition gives us some useful standard terminology.

Definition 1.1. A set $\mathcal{X} \subseteq [\omega]^\omega$ is called Ramsey if there is an infinite $H \subseteq \omega$ homogeneous for the partition of $[\omega]^\omega$ determined by $\mathcal{X}$ and $[\omega]^\omega \setminus \mathcal{X}$.

By identifying sets with their characteristic functions, we can topologize $[\omega]^\omega$ as a subspace of 2^ω with the product topology. Galvin and Prikry [1] showed that sets Borel with respect to this topology are Ramsey. Silver [6] improved this result by showing that all analytic sets (= the continuous images of Borel sets) are Ramsey. Thus if we require our partition of $[\omega]^\omega$ to be "well-behaved", homogeneous sets can be found.

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Another problem to consider is the role of the axiom of choice in proving the negation of $\omega \longrightarrow (\omega)^\omega$ — perhaps it is possible to construct models of ZF in which the axiom of choice fails and in which $\omega \longrightarrow (\omega)^\omega$ holds. This is in fact the case, for Mathias [4], continuing work of Solovay [7], has shown that in the model obtained by Lévy collapsing an inaccessible $L(\mathbb{R}) \models \omega \longrightarrow (\omega)^\omega$; it is an open problem whether this large cardinal assumption can be eliminated.

The third and final extension of Ramsey's theorem we shall consider deals with finding homogeneous sets that satisfy certain properties. We shall concern ourselves with trying to get homogeneous sets that are large in the sense of lying in a given ultrafilter.

Definition 1.2. An ultrafilter $\mathcal{U}$ is said to be selective if for every partition of $[\omega]^2$ into two pieces, there is a set $H \in \mathcal{U}$ homogeneous for the partition.

For obvious reasons, such ultrafilters are also called Ramsey ultrafilters. The name selective arises in connection with another characterization of this class of ultrafilters due to Kunen — we use this terminology simply to avoid the overuse of the adjective "Ramsey" in the sequel.

Such ultrafilters do not necessarily exist — in [3] Kunen constructs a model of ZFC in which there are no selective ultrafilters. However, if we assume the Continuum Hypothesis or Martin's Axiom, such ultrafilters are fairly easy to construct, and in fact it can be shown that under these hypotheses many isomorphism classes of selective ultrafilters exist.

We can also ask to what extent selective ultrafilters can encompass Silver's generalization of Ramsey's theorem. This is done in [4] by Mathias, where it is shown that given a selective ultrafilter $\mathcal{U}$ and an analytic set $\mathcal{X} \subseteq [\omega]^\omega$, there is a set $H \in \mathcal{U}$ homogeneous for the partition of $[\omega]^\omega$ into $\mathcal{X}$ and its complement. Given a selective ultrafilter $\mathcal{U}$, we describe this situation by saying $\mathcal{X}$ is $\mathcal{U}$-Ramsey.

Also in [4], Mathias pushes his proof of the consistency of $\omega \longrightarrow (\omega)^\omega$ further and obtains a model of ZFC in which the Continuum Hypothesis holds (so there are plenty of selective ultrafilters) and every set of reals in $L(\mathbb{R})$ is $\mathcal{U}$-Ramsey for every selective ultrafilter $\mathcal{U}$. In order to obtain this stronger result, he assumes the existence of a Mahlo cardinal. In this note, we prove that the assumption of the existence of a Mahlo cardinal is necessary.

2. Forcing with $([\omega]^\omega, \subseteq)$

Let P be the partially ordered set $[\omega]^\omega$ ordered by inclusion. Notice that P is not a separative partial order, but this causes us no problems as we need only keep in mind that, when forcing with P, two conditions X and Y force the same statements to hold in the generic extension if $X \bigtriangleup Y$ is finite. The separative quotient of this partially ordered set is isomorphic to $[\omega]^\omega$ modulo ideal of finite sets.

We should also notice that after we force with P, the generic object G is a selective ultrafilter on ω. To see this, notice that P (or rather its separative quotient) is countably closed. This implies that G is an ultrafilter in the extension, as no new reals are added. For any partition of $[\omega]^2$ into two pieces in the ground model, an easy density argument using Ramsey's theorem tells us that G contains a set homogeneous for this partition. Since such a partition is coded by a real, any such partition in the extension is also in the ground model, and thus G is selective.

Assuming appropriate combinatorial hypotheses, a kind of converse of this statement is true.

Theorem 2.1 (Folklore)**.** *If $\mathcal{U}$ is a selective ultrafilter and every $\mathcal{X} \subseteq [\omega]^\omega$ that lies in $L(\mathbb{R})$ is $\mathcal{U}$-Ramsey, then $\mathcal{U}$ is P-generic over $L(\mathbb{R})$.*

Proof. Assume $\mathcal{U}$ is a selective ultrafilter and let $\mathcal{D} \in L(\mathbb{R})$ be a dense open subset of $[\omega]^\omega$. By our assumptions, there is a set $H \in \mathcal{U}$ homogeneous for the partition defined by $\mathcal{D}$ and $[\omega]^\omega \setminus \mathcal{D}$. Since $\mathcal{D}$ is dense, it cannot be the case that $[H]^\omega \cap \mathcal{D} = \emptyset$, so we must have $[H]^\omega \subseteq \mathcal{D}$. In particular, this implies $H \in \mathcal{D}$. Thus $\mathcal{U} \cap \mathcal{D} \neq \emptyset$ for every dense open subset $\mathcal{D}$ of P that lies in $L(\mathbb{R})$, and so $\mathcal{U}$ is P-generic over $L(\mathbb{R})$. □

In the course of our investigations, we shall also need the following curious result. Once again, we restrict ourselves to forcing over $L(\mathbb{R})$.

Theorem 2.2 (Henle, Mathias, and Woodin)**.** *If $L(\mathbb{R})$ satisfies the partition relation $\omega \longrightarrow (\omega)^\omega$, then forcing with P over $L(\mathbb{R})$ adjoins no function from an ordinal into $L(\mathbb{R})$.*

Proof. See [2]. □

As a corollary to this theorem, we have that if $\omega \longrightarrow (\omega)^\omega$ holds in $L(\mathbb{R})$ and $\mathcal{U}$ is P-generic over $L(\mathbb{R})$, then $\mathcal{P}(\omega)$ cannot be well-ordered in the generic extension $L(\mathbb{R})[\mathcal{U}]$. This follows because our assumption about $L(\mathbb{R})$ implies no such well-ordering exists in $L(\mathbb{R})$, and Theorem 2.2 implies no such well-ordering is introduced in the extension.

3. Main theorem

Theorem 3.1. *Assume that the Continuum Hypothesis holds and every set of reals in $L(\mathbb{R})$ is $\mathcal{U}$-Ramsey with respect to every selective ultrafilter $\mathcal{U}$. Then $\aleph_1$ is Mahlo in L.*

Proof. Assume by way of contradiction that $\aleph_1$ is not a Mahlo cardinal in L. Fix a closed unbounded $C \subseteq \Lambda$(= the countable limit ordinals) so that for $\lambda \in C$,

$$L \models \text{“}\lambda \text{ is not a regular cardinal.”} \tag{3.1}$$

Fix a sequence $\langle a_\lambda : \lambda \in C\rangle$ so that $a_\lambda \subseteq \omega$ and

$$L[a_\lambda] \models \text{“}\lambda \text{ is countable.”} \tag{3.2}$$

Fix (using CH) an enumeration $\langle \pi_\xi : \xi < \omega_1\rangle$ of all partitions $\pi : [\omega]^2 \to 2$. Also fix once and for all a recursive uniform procedure for coding such partitions by subsets of ω; we will not distinguish between π and its code.

Our goal is to construct a $\subseteq^*$–decreasing sequence $\langle Z_\xi : \xi < \omega_1\rangle$ in $[\omega]^\omega$, generating a selective ultrafilter $\mathcal{U}$, so that in $L(\mathbb{R})[\mathcal{U}]$ the reals can be well–ordered. This gives a contradiction by the discussion after Theorem 2.2.

Our construction of the Z_ξ's is in blocks of length $\omega + \omega$, i.e., proceeding from stage α to stage $\alpha + 1$, involves defining $\langle Z_{\omega\cdot 2\alpha+\nu} : \nu < \omega + \omega\rangle$. We will let $\mathcal{Z}_\alpha$ stand for $\langle Z_\xi : \xi < \omega \cdot 2\alpha\rangle$.

Definition 3.1. If λ is a limit ordinal, $\mathcal{Z} = \langle Z_\alpha : \alpha < \lambda\rangle$ is a $\subseteq^*$–decreasing sequence in $[\omega]^\omega$, and λ is countable in $L[\mathcal{Z}]$, then the *canonical continuation* of $\mathcal{Z}$ is the $\leq_{L[\mathcal{Z}]}$–least infinite Z that is $\subseteq^*$ each member of $\mathcal{Z}$.

Definition 3.2. Given A and B infinite subsets of ω, we say $\langle B_n : n \in \omega\rangle$ is the *filter–coding* of A from B if $B_0 = B$, and

$$B_{k+1} = \begin{cases} \mathrm{Even}(B_k), & \text{if } k \in A, \\ \mathrm{Odd}(B_k), & \text{if } k \notin A. \end{cases} \tag{3.3}$$

(If $D = \{d_n : n \in \omega\}$ listed in increasing order, then $\mathrm{Even}(D) = \{d_{2n} : n \in \omega\}$ and similar for $\mathrm{Odd}(D)$.)

Note that in the above situation A is definable in an absolute fashion from B and the filter generated by $\{B_n : n \in \omega\}$.

Now we turn to the construction. We define for each $\alpha < \omega_1$ the sequence

$$\mathcal{Z}_\alpha = \langle Z_\xi : \xi < \omega \cdot 2\alpha\rangle \tag{3.4}$$

(so $\mathcal{Z}_{\alpha+1}$ is obtained from $\mathcal{Z}_\alpha$ by appending a sequence of order–type $\omega + \omega$). Along the way we will also be defining a strictly increasing and continuous function f mapping ω_1 into C. We do this in such a way as to maintain

$$L[\mathcal{Z}_\alpha] \models \text{“}\omega \cdot 2\alpha \leq f(\alpha) \text{ is countable”}. \tag{3.5}$$

From what is written above, we see that $\mathcal{Z}_0$ must be the empty sequence, so we start by describing how $\mathcal{Z}_{\alpha+1}$ is obtained from $\mathcal{Z}_\alpha$.

Let $\lambda = \omega \cdot 2\alpha$. By (3.5), we let Z_λ be the canonical continuation of $\mathcal{Z}_\alpha$. Next we choose $Z_{\lambda+k}$ for $1 \leq k < \omega$ so that $\langle Z_{\lambda+k} : k < \omega\rangle$ is the filter–coding of (the code for) π_α from Z_λ.

Let $\mathcal{Z}'_\alpha$ denote the sequence $\mathcal{Z}_\alpha {}^\frown \langle Z_{\lambda+k} : k < \omega\rangle$. Clearly $\lambda + \omega$ is countable in $L[\mathcal{Z}'_\alpha]$, so we let $Z_{\lambda+\omega}$ be the canonical continuation of $\mathcal{Z}'_\alpha$.

Since both $Z_{\lambda+\omega}$ and π_α are in $L[\mathcal{Z}'_\alpha]$, we can apply Ramsey's theorem in this model and let $Z_{\lambda+\omega+1}$ be the least (in the canonical well–ordering of the model) infinite subset of $Z_{\lambda+\omega}$ homogeneous for π_α.

Now let $f(\alpha+1)$ be the least member of C that is $> \max\{\lambda+\omega+\omega, f(\alpha)\}$. The sequence $\langle Z_{\lambda+\omega+k+1} : k < \omega\rangle$ is the filter–coding of $a_{f(\alpha+1)}$ from $Z_{\lambda+\omega+1}$. This finishes the definition of $\mathcal{Z}_{\alpha+1}$.

Note that since $a_{f(\alpha+1)} \in L[\mathcal{Z}_{\alpha+1}]$, we have

$$L[\mathcal{Z}_{\alpha+1}] \models \text{“}\omega \cdot 2\alpha \leq f(\alpha+1) \text{ is countable, ”} \tag{3.6}$$

and so (3.5) has been maintained.

For α a limit ordinal, $\mathcal{Z}_\alpha$ is just the limit of the $\mathcal{Z}_\xi$'s for $\xi < \alpha$, and $f(\alpha)$ is the supremum of $\{f(\xi) : \xi < \alpha\}$. Note that since (3.5) holds for each $\xi < \alpha$, each $f(\xi)$ is countable in $L[\mathcal{Z}_\alpha]$, so

$$L[\mathcal{Z}_\alpha] \models \text{“}f(\alpha) \text{ is the supremum of a set of countable ordinals.”} \tag{3.7}$$

Since C is a club and $f(\xi) \in C$ for $\xi < \alpha$, we have $f(\alpha) \in C$. In particular, $f(\alpha)$ is not a regular cardinal in L, and so $f(\alpha)$ is not equal to $\omega_1^{L[\mathcal{Z}_\alpha]}$. In conjunction with (3.7), this implies that $f(\alpha)$ is countable in $L[\mathcal{Z}_\alpha]$, and so (3.5) has been maintained.

Once the sequence $\langle Z_\xi : \xi < \omega_1\rangle$ has been constructed, let $\mathcal{U}$ be the filter generated by $\{Z_\xi : \xi < \omega_1\}$. Clearly $\mathcal{U}$ is non–principal.

Claim 1. $\mathcal{U}$ is a selective ultrafilter.

Proof. Selectivity is immediate, as $Z_{\omega\cdot 2\alpha+\omega+1}$ is homogeneous for π_α. The fact that $\mathcal{U}$ is an ultrafilter is almost as easy — given $A \subseteq \omega$, we consider the partition

$f_A : [\omega]^2 \to 2$ defined by $f_A(m < n) = 1$ if and only if $m \in A$. An infinite set homogeneous for f_A is either a subset of A or disjoint from A. □

Claim 2. $\mathcal{P}(\omega)$ can be well–ordered in $L(\mathbb{R})[\mathcal{U}]$.

Proof. It suffices to show that the sequence of partitions $\langle \pi_\alpha : \alpha < \omega_1 \rangle$ is in $L(\mathbb{R})[\mathcal{U}]$, as from this sequence we get a well–ordering of $\mathcal{P}(\omega)$ by saying A precedes B if the first occurrence of f_A precedes the first occurrence of f_B in the sequence of partitions.

Now to see that $\langle \pi_\alpha : \alpha < \omega_1 \rangle$ is in $L(\mathbb{R})[\mathcal{U}]$, we need only verify that $\langle Z_\xi : \xi < \omega_1 \rangle$ is in $L(\mathbb{R})[\mathcal{U}]$ — looking at the sequence $\langle \langle Z_{\omega \cdot 2\alpha + k} : k < \omega \rangle : \alpha < \omega_1 \rangle$ allows us to read off $\langle \pi_\alpha : \alpha < \omega_1 \rangle$ by the absoluteness of our filter–coding.

Now the sequence $\langle Z_\xi : \xi < \omega_1 \rangle$ is in $L(\mathbb{R})[\mathcal{U}]$ because we can "recreate" it using $\mathcal{U}$ — at places where the next element of the sequence is not canonically defined from the previous ones (namely, the stages where we had to decide between "even" or "odd"), we have at our disposal the filter $\mathcal{U}$ to tell us which choice to make. □

We finish by the discussion after Theorem 2.2; our assumptions tell us that $\mathcal{U}$ is P–generic over $L(\mathbb{R})$, and Theorem 2.2 tells us that in the generic extension $L(\mathbb{R})[\mathcal{U}]$ the reals cannot be well–ordered, which contradicts the preceding claim. □

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Institute of Mathematics, The Hebrew Univeristy, Jerusalem, Israel

E-mail address: eisworth@math.huji.ac.il