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A GENERALIZATION OF A THEOREM OF EDWARDS

JYH-YANG WU

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Abstract. In this note we extend a theorem of Edwards on the characterization of topological manifolds for polyhedra to a more general class of stratified spaces. We show that a cone-like space X of dimension $n \geq 3$ is a topological manifold if and only if the base space B_p of every point p in X is a simply connected cone-like sphere.

§1. Introduction

The recognition problem of topological manifolds is one of the fundamental problems in geometric topology ([D]). Among other results, R.D. Edwards proved the following characterization of polyhedra which are topological manifolds.

Theorem (Edwards). *A finite polyhedra P is a closed topological n-manifold if and only if*

(1) *the link of every vertex of P is simply connected if $n \geq 3$, and*
(2) *the link of every point $p \in P$ has the homology of an $(n-1)$-sphere.*

The purpose of this note is to extend this result to a more general class of stratified spaces.

Definition 1.1. A point is a 0-dimensional cone-like space. The circle S^1 and the closed interval are 1-dimensional cone-like spaces. A space X is called an n-dimensional (locally) cone-like space ($n \geq 2$) if every point p in X has a neighborhood U such that (U, p) is homeomorphic to an open cone $(C(B_p), *)$ over a connected compact $(n-1)$-dimensional (locally) cone-like space B_p where $*$ denotes the vertex of the cone $C(B_p)$. The space B_p is called a base of p. For simplicity we shall also denote the vertex $*$ of the cone $C(B_p)$ by p.

Example 1.2. Cone-like spaces include a lot of topological spaces of interest. For instance, topological manifolds, orbifolds and Alexandrov spaces ([P]) are cone-like spaces. The product space $X \times Y$ is also a cone-like space if both X and Y are. The suspension ΣX of a compact connected cone-like space X is also cone-like, for example, the suspension ΣT^n of an n-torus T^n.

Note that every cone-like space X of dimension n has a natural stratification $X_0 \subset X_1 \subset \cdots \subset X_n = X$ such that $X_i - X_{i-1}$ is an i-dimensional topological manifold. For the discussion of stratified spaces, we refer to the wonderful book

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[We] by S. Weinberger. See also the paper of Mitchell [M] for discussion of absolute suspensions and cones.

Definition 1.3. A compact cone-like space X of dimension n is called a cone-like sphere if it is a homology manifold and has the homology groups $H_*(X) \simeq H_*(S^n)$.

Our main result can now be stated as

Theorem 1.4. *A cone-like space X of dimension $n \geq 3$ is a topological manifold without boundary if and only if the base space B_p of every point p in X is a simply connected cone-like sphere.*

We give an example to indicate that this result is optimal.

Example 1.5. Let X be a non-simply connected homology sphere. For instance, Poincare's 3-dimensional homology sphere. The suspension ΣX is a cone-like sphere. It is easy to see that ΣX is not a topological manifold since the two vertices of the suspension are not manifold points. On the other hand, the bases of these two vertices are the space X which is not simply connected.

It is also an interesting open problem that if the product space $X \times \mathbb{R}^2$ is a topological manifold, is $X \times \mathbb{R}$ a manifold? As a corollary of Theorem 1.4, we shall give an affirmative answer to this problem for cone-like spaces. To be more precise, we shall obtain

Corollary 1.6. *Let X be a cone-like space. Then the following three statements are equivalent.*

(1) *X is a homology manifold,*
(2) *$X \times \mathbb{R}^k$ is a topological manifold for some $k \geq 1$, and*
(3) *$X \times \mathbb{R}$ is a topological manifold.*

§2. The proof of the main theorem

Since we shall need several results in geometric topology to prove our main theorem, we list here some important ones that will appear in our discussion.

Theorem 1 (Quinn [Q1]- [Q4])**.** *Let X and Y be two spaces with dimensions ≥ 2. Then the product space $X \times Y$ is a topological manifold if and only if X and Y are ANR homology manifolds of local index 1 in the sense of Quinn.*

A general characterization of topological manifolds takes the following form.

Theorem 2 (Edwards-Quinn [D])**.** *A finite dimensional space X is a topological n-manifold, $n \geq 5$, if and only if*

(1) *X is an ANR homology manifold of local index 1 in the sense of Quinn, and*
(2) *X satisfies the disjoint disks property (DDP).*

Let A be a subset of X and $x \in A \cap \overline{(X - A)}$. We say that $X - A$ is locally k-connected, $k \geq 0$, at x provided that each neighborhood U of x contains another neighborhood V such that each map of the sphere S^m into $V - A$, for $m \leq k$, can be extended to a map of the disk D^{m+1} into $U - A$. The subset A is said to be locally k-co-connected (k-LCC) provided that $X - A$ is locally k-connected at each $x \in A \cap \overline{(X - A)}$. For related notions, please see Daverman [D].

Let $S(X)$ denote the set of non-manifold points in X. Cannon, Bryant and Lacher gave a characterization of topological manifolds when the dimension $S(X)$ has large co-dimensions.

Theorem 3 (Cannon, Bryant and Lacher [CBL], [D])**.** *If X is an n-dimensional ANR homology manifold whose singular set $S(X)$ is 1-LCC embedded and has dimension k where $2k+3 \leq n$, then X is a topological manifold.*

Now we shall use the method developed in [Wu1], [Wu2] to prove Theorem 1.4. First of all we point out that a cone-like space is always an ANR since it is locally compact, locally path connected and locally contractible. Next we show that if a cone-like space X is a homology manifold, then so is the base of every point in X.

Proposition 2.1. *Let X be a cone-like space of dimension n. If X is a homology manifold, then the base B_p of p is a cone-like sphere of dimension $n-1$ for any point $p \in X$.*

Proof. We need to verify (1) $H_*(B_p) \simeq H_*(S^{n-1})$ and (2) B_p is also a homology manifold for any point $p \in X$. (1) Since X is cone-like, there is a neighborhood U of p such that (U, p) is homeomorphic to $(C(B_p), p)$. The excision theorem in homology theory gives

$$H_*(\mathbb{R}^n, \mathbb{R}^n - o) \simeq H_*(X, X-p) \simeq H_*(U, U-p) \simeq H_*(C(B_p), C(B_p)-p).$$

On the other hand the exact sequences for pairs $(C(B_p), C(B_p)-p)$ and $(\mathbb{R}^n, \mathbb{R}^n - o)$ give

$$H_j(C(B_p)-p) \to H_j(C(B_p)) \to H_j(C(B_p), C(B_p)-p) \to H_{j-1}(C(B_p)-p) \to \dots$$

and

$$\to H_j(\mathbb{R}^n - o) \to H_j(\mathbb{R}^n) \to H_j(\mathbb{R}^n, \mathbb{R}^n - o) \to H_{j-1}(\mathbb{R}^n - o) \to \dots .$$

Since $C(B_p)$ is a cone over B_p, $C(B_p)$ is contractible. Hence one can easily derive from these exact sequences that $H_*(B_p) \simeq H_*(S^{n-1})$.

(2) To see that B_p is also a homology manifold, we know from (1) that $U-p$ is homeomorphic to the cone-like space $B_p \times \mathbb{R}$. For any $v \in B_p$ and $o \in \mathbb{R}$ the base $B_{(v,o)}$ of (v,o) in $B_p \times \mathbb{R}$ is the spherical suspension $\Sigma(B_v)$ of the base B_v of v in B_p. Note that the dimension of B_v is $n-2$. From (1) and the suspension theorem of homology groups [Wh], we have for $j \geq 2$

$$H_j(S^{n-1}) \simeq H_j(B_{(v,o)}) \simeq H_j(\Sigma(B_v)) \simeq H_{j-1}(B_v).$$

In particular, $H_*(B_v) \simeq H_*(S^{n-2})$. Then a similar argument as in (1) implies that $H_*(B_p, B_p - v) \simeq H_*(\mathbb{R}^{n-1}, \mathbb{R}^{n-1} - o)$. Since v is arbitrary, B_p is a homology manifold. □

To prove Theorem 1.4 we first discuss the lower dimension case.

Proposition 2.2. *Let X be a cone-like space of dimension* 3*. If X is a homology manifold, then it is a topological manifold.*

Proof. Proposition 2.1 implies for every point p that the base B_p of p is a two-dimensional homology manifold, and hence a topological manifold ([D]), with the homology group $H_*(B_p) \simeq H_*(S^2)$. Since the sphere S^2 is the only 2-manifold with this property, B_p is homeomorphic to S^2. Thus the cone $C(B_p)$ is homeomorphic to the 3-disk D^3. Since X is cone-like, there is a neighborhood U such that U is homeomorphic to D^3 and thus p is a manifold point. Hence X is a topological manifold. □

Based on Propositions 2.1 and 2.2, one may suspect that Theorem 1.4 can be obtained easily by an induction on the dimension of X. This is true if we assume the Poincare conjecture. However, since the Poincare conjecture is still open, we need to handle the singular set $S(X)$ in a more detailed manner in order to overcome this difficulty. Before we investigate the structure of the singular set $S(X)$, we first give a rough estimate of the dimension of $S(X)$.

Proposition 2.3. *Let X be a cone-like space of dimension $n \geq 4$. If X is a homology manifold, then one has* $\dim S(X) \leq 1$.

Proof. Given any point $p \in X$ there is a neighborhood U such that $U - p$ is homeomorphic to the product space $B_p \times \mathbb{R}$. Since B_p is a compact cone-like space, one can find that there are finitely many points $\{v_i\}_{i=1}^m$ of B_p and a neighborhood U_i of v_i such that $B_p \subset \bigcup_{i=1}^m U_i$ and each open set $U_i - v_i$ in B_p is homeomorphic to a product space $B_{v_i} \times \mathbb{R}$. Here B_{v_i} is the base of v_i in B_p and has dimension $n-2 \geq 2$. We know that the set of manifold points is dense in X. Hence $B_{v_i} \times \mathbb{R}^2$ contains manifold points for each i. Proposition 2.1 also gives that B_{v_i} is an ANR homology manifold. Since the local index of Quinn is locally defined and locally constant and multiplicative (cf. [Q4]), B_{v_i} has local index 1 everywhere and $B_{v_i} \times \mathbb{R}^2$ is a topological manifold for each i, by Theorem 1. From this, one can easily conclude that $S(X) \cap (U - p)$ is contained in $\bigcup_{i=1}^m v_i \times \mathbb{R}$ if we identify $U - p$ with $B_p \times \mathbb{R}$. Thus $S(X)$ is locally contractible, and has $\dim S(X) \leq 1$. □

We are now in a position to investigate the product space $X \times \mathbb{R}$.

Theorem 2.4. *Let X be a cone-like space. If X is a homology manifold, then $X \times \mathbb{R}$ is a topological manifold.*

Proof. If the space X has $\dim X \leq 3$, then we know from Proposition 2.2 that X itself is a topological manifold and so is $X \times \mathbb{R}$. Thus we can assume $\dim X \geq 4$. Proposition 2.3 shows in particular that the set of manifold points is dense in X. Hence X and $X \times \mathbb{R}$ have the local index 1 in the sense of Quinn and are ANR homology manifolds.

Since we have $\dim X \times \mathbb{R} \geq 5$, to show that $X \times \mathbb{R}$ is a topological manifold we only need, by Theorem 2, to verify that $X \times \mathbb{R}$ has the disjoint disk property. To do so, we shall follow the argument in the proof of Corollary 3C in [D]. Consider two maps $f_1,\ f_2 : D^2 \to X \times \mathbb{R}$. Let $\pi_1 : X \times \mathbb{R} \to X$ and $\pi_2 : X \times \mathbb{R} \to \mathbb{R}$ denote the projection maps. By Proposition 2.3, we know that $\dim S(X) \leq 1$, so $S(X)$ must be nowhere dense and 0-LCC in X. Thus, the maps $\pi_1 f_1$ and $\pi_1 f_2$ can be modified to send increasingly dense 1-skeleta of D^2 into $X - S(X)$ and obtain in the limit approximations $m_1,\ m_2 : D^2 \to X$ such that $m_i^{-1}(S(X))$ $(i = 1, 2)$ is a compact 0-dimensional set K_i. Moreover, the maps $\pi_2 f_i : K_i \to \mathbb{R}$ can be approximated by disjoint embeddings $g_i : K_i \to \mathbb{R}$, where g_i is so close to $\pi_2 f_i|_{K_i}$ that g_i extends to a map $\alpha_i : D^2 \to \mathbb{R}$ close to $\pi_2 f_i : D^2 \to \mathbb{R}$. Define the approximations $h_1,\ h_2 : D^2 \to X \times \mathbb{R}$ to f_1, f_2 by $h_i(x) = (m_i(x), \alpha_i(x))$. Hence one has $h_1(D^2) \cap h_2(D^2) \subset (X - S(X)) \times \mathbb{R}$. Since $(X - S(X)) \times \mathbb{R}$ is a topological manifold of dimension at least 5, one can produce disjoint approximations and thus the space $X \times \mathbb{R}$ satisfies the disjoint disk property. Hence $X \times \mathbb{R}$ is a topological manifold. □

Corollary 1.6 now follows easily from Theorem 2.4 and Theorem 1. We are now in a position to prove Theorem 1.4.

The proof of Theorem 1.4. Let X be a cone-like space of dimension $n \geq 3$. Suppose that the base B_p of every point p in X is a simply connected cone-like sphere. We claim that $\dim S(X) \leq 0$. Indeed, consider a neighborhood U of p such that $U - p$ is homeomorphic to the product space $B_p \times \mathbb{R}$ and B_p is a homology manifold. Since $B_p \times \mathbb{R}$ is, by Theorem 2.4, a topological manifold, we can conclude that the point p is the only possible non-manifold point in U. Hence $S(X)$ is either empty or a discrete subset of X. Thus $\dim S(X) \leq 0$. To prove Theorem 1.4, Theorem 3 tells us that it is sufficient to check that every point p in $S(X)$ is a 1-LCC subset of X. Consider any neighborhood W of p. Since X is cone-like, there is a neighborhood U of p such that $U - p$ is homeomorphic to $C(B_p) - p$. We can find a neighborhood V with $V \subset U \cap W$ such that $V - p$ is still homeomorphic to $C(B_p) - p$. Thus, $V - p$ is simply connected because B_p is simply connected. Therefore a map from S^1 into $V - p$ extends to a map from D^2 into $V - p$ and, thus, into $W - p$. Hence p is a 1-LCC subset of X and Theorem 1.4 follows. $\square$

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Department of Mathematics, National Chung Cheng University, Chia-Yi 621, Taiwan

E-mail address: jywu@math.ccu.edu.tw