LEVEL SETS OF A TYPICAL $C^n$ FUNCTION

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ABSTRACT. We determine the level set structure of a typical $C^n$ function.

INTRODUCTION

Assume that $F$ is a function space complete with respect to some norm. We say that a typical function in $F$ satisfies property $P$ if the subfamily of $F$ consisting of those functions which satisfy $P$ is residual (in the sense of Baire category) in $F$. If $y$ is in the range of $f$, we call $f^{-1}\{y\}$ a level set of $f$. In [2] Bruckner and Garg gave a full description of level sets of a typical function from $C[0,1]$, the space of real-valued, continuous functions defined on $[0,1]$ and equipped with the sup norm. Namely, they proved:

Theorem 1 (Bruckner and Garg). For a typical function $f \in C[0,1]$ there exists a countable set $S_f \subseteq (\min f, \max f)$ such that the level set $f^{-1}\{y\}$ is:

1. a nowhere dense perfect set if $y \notin S_f \cup \{\min f, \max f\}$,
2. a single point if $y = \min f$ or $\max f$, and
3. the union of a nowhere dense perfect set and an isolated point of $f^{-1}\{y\}$ if $y \in S_f$.

In this paper we give the description of level sets of typical functions in $C^n[0,1]$. Namely, we prove that a typical $f \in C^1[0,1]$ is either strictly monotone or $f$ has uncountably many level sets having exactly one accumulation point and all other level sets of $f$ are finite. For a typical function in $C^n[0,1]$ with $n \geq 2$ the situation is simple. All level sets are finite.

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Let us now introduce some notation and definitions.

The symbols $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ will denote the sets of all positive integers, rationals and reals, respectively. We use $|A|$ to denote the cardinality of set $A$. The restriction of a function $f$ to a set $A$ will be denoted by $f|A$. We use $\lambda(A)$ to denote the Lebesgue measure of $A$ for a Lebesgue measurable subset $A$ of $\mathbb{R}$. 

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Let $X$ be a metric space with a norm $|\cdot|$. We use $B_{r}(x)$ to denote the open ball in $X$ with center $x$ and radius $r$.

We shall consider the space $C^{n}[0,1]$ $(n \geq 0)$ endowed with Sobolev’s norm, i.e. $\|f\|_{n} = \sum_{i=0}^{n} \|f^{(i)}\|$, where $\|\cdot\|$ stands for the sup norm and $f^{(i)}$ stands for $i$th derivative of $f$.

We prove a series of lemmas which lead us to our main results.

**Lemma 1.** Fix $y_{0}\in \mathbb{R}$. For a typical $f \in C[0,1]$, $f^{-1}(\{y_{0}\})$ is either empty or perfect and nowhere dense.

**Proof.** Let $p, q \in [0,1]$ and $p < q$. Let $\mathcal{F}_{p,q} = \{ f \in C[0,1] : |f^{-1}(\{y_{0}\}) \cap [p,q]| = 1 \}$. We first want to show that $\mathcal{F}_{p,q}$ is nowhere dense in $C[0,1]$. We will accomplish this by showing that for each $h \in \mathcal{F}_{p,q}$ and $\epsilon > 0$, there exists a nonempty open set which misses $\mathcal{F}_{p,q}$ and is contained in $B_{\|\cdot\|}(h, \epsilon)$. To this end, let $h \in \mathcal{F}_{p,q}$ and $\epsilon > 0$. Let $(u,v) \subset [p,q]$ be such that $h((u,v)) \subset (y_{0}-\epsilon/4, y_{0}+\epsilon/4)$. Let $a < b < c$ be any points of $(u,v)$. Let us define a new function $\phi$ as follows: $\phi(x) = h(x)$ for $x \notin (u,v)$, $\phi(a) = \phi(c) = y_{0} - \epsilon/4$, $\phi(b) = y_{0} + \epsilon/4$ and make $\phi$ linear on intervals $(u,a)$, $(a,b)$, $(b,c)$ and $(c,v)$. Every function from $B_{\|\cdot\|}(\phi, \epsilon/4)$ now takes the value $y_{0}$ in at least two points on the interval $(u,v) \subset [p,q]$ whence $B_{\|\cdot\|}(\phi, \epsilon/4) \cap \mathcal{F}_{p,q} = \emptyset$. As $\|h - \phi\| \leq \epsilon/2$, $B_{\|\cdot\|}(\phi, \epsilon/4) \subseteq B_{\|\cdot\|}(h,\epsilon)$. Thus, $\mathcal{F}_{p,q}$ is nowhere dense.

Let us notice now that for every function $f$ in the residual set

$$C[0,1] \setminus \bigcup \{ \mathcal{F}_{p,q} : p, q \in \mathbb{Q} \cap [0,1], p < q \}$$

there is no isolated point in the level set $f^{-1}(\{y_{0}\})$. So $f^{-1}(\{y_{0}\})$ must be either empty or perfect.

It is a well-known fact that a typical function in $C[0,1]$ is nowhere constant. (For example, this follows from the aforementioned theorem of Bruckner and Garg.) This together with what we just showed finishes the proof of the lemma.

The following is a $C^{1}$ counterpart of Lemma 1.

**Lemma 2.** Fix $y_{0} \in \mathbb{R}$. For a typical $f \in C^{1}[0,1]$, $f^{-1}(\{y_{0}\})$ is finite.

**Proof.** Of course, it is enough to show the lemma for $y_{0} = 0$. It is easy to see that a function $f$ that has infinite level set $f^{-1}(\{0\})$ must satisfy at some point the equation $f(x) = f'(x) = 0$. We shall show that the family of functions

$$\mathcal{F} = \{ f \in C^{1}[0,1] : \text{there exists } x \text{ such that } f(x) = f'(x) = 0 \}$$

is of first category. One can readily observe that it is closed, so it is enough to show that its interior is empty.

Let $f \in \mathcal{F}$. From Sard’s theorem (see for instance Theorem 6.3, p. 226, [3]) we know that $\lambda(f(\{x : f'(x) = 0\})) = 0$. Thus there exists as small a positive $\epsilon$ as we want such that $f(x) \neq \epsilon$ at any point $x$ where $f'(x) = 0$. Let us now consider the function $h = f - \epsilon$. If $h(x) = h'(x) = 0$ at some $x \in [0,1]$, then $\epsilon \in f(\{x : f'(x) = 0\})$ which is impossible. Obviously, $h \in B_{\|\cdot\|}(f, 2\epsilon)$. Thus the interior of $\mathcal{F}$ is empty. This finishes the proof.

Let us now recall the following definition. We say that a function $f : [0,1] \rightarrow \mathbb{R}$ is monotone at a point $x$ if there exists an open (relative to $[0,1]$) neighborhood $V$...
Lemma 3. A typical function $f$ in $C^1[0, 1]$ has the property that if there is an $x$ such that $f'(x) = 0$, then $f$ is not monotone at $x$.

Proof. Let

$$F = \{ f \in C^1[0, 1] : \text{there exists } x \text{ such that } f'(x) = 0 \text{ and } f \text{ is monotone at } x \}. $$

We first show that the complement of $F$ is dense in $C^1[0, 1]$ and $\epsilon > 0$. Let us define $g \in B_{\| \cdot \|_1} f, \epsilon/3$. If $f'(0) = 0$ or $f'(1) = 0$, let us put $g = f + \alpha x$ with $\alpha < \epsilon/6$ such that $g'(0) \neq 0$ and $g'(1) \neq 0$. If $f'(0) \neq 0$ and $f'(1) \neq 0$, then we put $g = f$. One can easily see that $g \in B_{\| \cdot \|_1} f, \epsilon/3)$. As the set $M = \{ x \in [0, 1] : f'(x) = 0 \}$ is compact and the set $\{ x \in [0, 1] : |g'(x)| < \epsilon/6 \}$ is open in $[0, 1]$ and contains $M$, there exists a finite sequence of points $0 \leq a_1 < b_1 < \ldots < a_n < b_n \leq 1$ such that $|g'|((a_i, b_i)) < \epsilon/6$, for each $i \leq n$, and $M \subset \bigcup \{(a_i, b_i) : i \leq n\}$. Let $\phi \in C[0, 1]$ be defined as follows: $\phi(x) = g'(x)$ if $x \not\in \bigcup \{(a_i, b_i) : i \leq n\}$ and $\phi$ is linear on every interval $(a_i, b_i), i \leq n$. Finally, let $h \in C^1[0, 1]$ be defined as

$$h(x) = g(0) + \int_0^x \phi(t) dt. $$

It is easy to observe that $h$ has strict extremum at every point $x$ where $h'(x) = 0$ and thus is not monotone at any such point. We also have $\|f - h\|_1 \leq \|f - g\|_1 + \|g - h\|_1 < \epsilon$. Thus, the complement of $F$ is dense in $C^1[0, 1]$.

To complete the proof of the lemma, we now show that $F$ is $F_\sigma$. Let us first observe that

$$F = \bigcup_{n=1}^{\infty} (F_n^+ \cup F_n^-),$$

where

$$F_n^+ = \{ f \in C^1[0, 1] : \text{there exists } p \in [0, 1] \text{ such that } f'(p) = 0 \text{ and } f|[p - 1/n, p] \cap [0, 1] \leq f(p) \text{ and } f|[p, p + 1/n] \cap [0, 1] \geq f(p) \}$$

and

$$F_n^- = \{ f \in C^1[0, 1] : \text{there exists } p \in [0, 1] \text{ such that } f'(p) = 0 \text{ and } f|[p - 1/n, p] \cap [0, 1] \geq f(p) \text{ and } f|[p, p + 1/n] \cap [0, 1] \leq f(p) \}. $$

It is easy to see that $F_n^+$ and $F_n^-$ are closed in $C^1[0, 1]$. Thus $F$ is of first category.

Proposition 1. The family

$$S = \{ f \in C^1[0, 1] : \text{there is } 0 \leq u < v \leq 1 \text{ such that } f(u) = f(v)$$

and $(u = 0 \text{ or } f'(u) = 0)$ and $(v = 1 \text{ or } f'(v) = 0)$

is of first category in $C^1[0, 1]$.
Proof. We shall use the fact that the space $C^1[0,1]$ is homeomorphic with the product space $\mathbb{R} \times C[0,1]$ where the homeomorphism $\Psi : \mathbb{R} \times C[0,1] \to C^1[0,1]$ is given by the formula

$$(1) \quad \Psi(a,f)(x) = a + \int_0^x f(t) \, dt.$$ 

We have $\Psi^{-1}(S) = \mathbb{R} \times \mathcal{G}$, where

$$\mathcal{G} = \left\{ \varphi \in C[0,1] : \text{there exist } 0 \leq u < v \leq 1 \text{ such that} \right.$$ 

$$(u = 0 \text{ or } \varphi(u) = 0) \text{ and } (v = 1 \text{ or } \varphi(v) = 0) \text{ and } \int_u^v \varphi(t) \, dt = 0 \bigg\}.$$ 

Thus to show that $S$ is of first category in $C^1[0,1]$ or, equivalently, that $\Psi^{-1}(S)$ is of first category in $\mathbb{R} \times C[0,1]$ it is enough to show that $\mathcal{G}$ is of first category in $C[0,1]$.

Let us first notice that $\mathcal{G} = \bigcup_{n=1}^\infty \mathcal{G}_n$, where

$$\mathcal{G}_n = \left\{ \varphi \in C[0,1] : \text{there exist } 0 \leq u < v \leq 1 \text{ such that } v - u \geq 1/n, \right.$$ 

$$(u = 0 \text{ or } \varphi(u) = 0) \text{ and } (v = 1 \text{ or } \varphi(v) = 0) \text{ and } \int_u^v \varphi(t) \, dt = 0 \bigg\}.$$ 

It is easy to check that $\mathcal{G}_n$ is closed in $C[0,1]$. Now to show that $\mathcal{G}$ is of first category it is enough to show that the complement of $\mathcal{G}$ is dense in $C[0,1]$.

Let $f \in C[0,1]$. Let $\epsilon > 0$. By the Weierstrass approximation theorem there exists a nonzero polynomial $w$ such that $\|f - w\| < \epsilon/2$ (of course, we consider only the restriction of $w$ to $[0,1]$). Let $0 = x_0 < x_1 < \ldots < x_{n-1} \leq x_n = 1$ and the set $\{x_1, \ldots, x_{n-1}\}$ be the set of all roots of $w$ which belong to $[0,1]$. Let

$$\delta = \min \left\{ \left| \int_{x_i}^{x_j} w(t) \, dt \right| : 0 \leq i < j \leq n \right\}.$$ 

Let

$$0 < \alpha < \min \left\{ \frac{\epsilon}{2\|w^2\|}, \frac{1}{\|w\|}, \frac{\delta}{\|w^2\|} \right\}.$$ 

We define a new polynomial $v(t) = w(t) + \alpha w^2(t)$. Let us notice that

- $v$ has the same roots in the interval $[0,1]$ as $w$,
- $\|v(t) - w(t)\| < \epsilon/2$, and
- for $x_i < x_j$ we have that
  $$\left| \int_{x_i}^{x_j} v(t) \, dt \right| \geq \min \left\{ \delta - \alpha \|w^2\|, \alpha \cdot \int_{x_i}^{x_j} w^2(t) \, dt \right\} > 0.$$ 

Thus, we have found a $v \in C[0,1]$ such that $\|v - f\| < \epsilon$ and $v \notin \mathcal{G}$. Therefore, the complement of $\mathcal{G}$ is dense in $C[0,1]$ and the proof of the proposition is complete.

The next three lemmas are immediate corollaries to Proposition 1.

**Lemma 4.** No level set of a typical function in $C^1[0,1]$ contains two points at which $f$ attains a local extremum.
Let us notice that if an extremum is attained at 0 or at 1 the derivative need not be equal to zero, and this was the reason for considering \( u = 0 \) and \( v = 1 \) in Proposition 1.

**Lemma 5.** A typical function \( f \) in \( C^1[0,1] \) has the property that if \( f \) attains a local extremum at \( x \), then \( f^{-1}(f(x)) \) is finite.

**Lemma 6.** No level set of a typical function in \( C^1[0,1] \) contains two accumulation points.

**Lemma 7.** A typical function \( f \in C^1[0,1] \) has the property that \( \{ x : f'(x) = 0 \} \) is either empty or perfect.

**Proof.** Let us consider the family \( \mathcal{T} \) of functions \( f \in C^1[0,1] \) for which the set \( \{ x : f'(x) = 0 \} \) is non-empty and is not perfect. We want to show that \( \mathcal{T} \) is of first category. We again use the fact that the space \( C^1[0,1] \) is homeomorphic with the product space \( \mathbb{R} \times C[0,1] \) where the homeomorphism \( \Psi : \mathbb{R} \times C[0,1] \to C^1[0,1] \) is given by the formula (1). Let us notice that \( \Psi^{-1}(T) = \mathbb{R} \times \mathcal{H} \), where

\[
\mathcal{H} = \{ \phi \in C[0,1] : \phi^{-1}([0]) \neq 0 \text{ and } \phi^{-1}(\{0\}) \text{ is not perfect} \}.
\]

By Lemma 1 \( \mathcal{H} \) is of first category in \( C[0,1] \) and thus \( \mathcal{T} \) is of first category in \( C^1[0,1] \).

We are now in a position to prove our main theorem.

**Theorem 2.** A typical function \( f \in C^1[0,1] \) is either strictly monotone or there exist a perfect nowhere dense subset \( P_f \) of \( (\min f, \max f) \) and a countable dense subset \( D_f \) of \( P_f \) such that the level set \( f^{-1}(\{y\}) \) is:

1. a set with exactly one accumulation point if \( y \in P_f \setminus D_f \),
2. a finite set if \( y \in D_f \cup ((\min f, \max f) \setminus P_f) \); and
3. a single point if \( y \in (\min f, \max f) \).

**Proof.** As the intersection of countably many residual families is residual in a complete space, we have that a typical function satisfies the properties of Lemmas 3 - 7. Let \( f \) be such a function.

Assume that \( f \) is not monotone. Then, there exists a point \( x \in (0,1) \) where \( f'(0) = 0 \). As \( f \) satisfies the property of Lemma 7, \( M_f = \{ x : f'(x) = 0 \} \) is perfect. As \( f \) satisfies the property of Lemma 6, the set \( P_f = f(M_f) \) is also perfect. Let

\[
D_f = \{ y \in P_f : \text{ there is } x \in f^{-1}(\{y\}) \text{ such that } f \text{ has a local extremum at } x \}.
\]

It is well-known that \( D_f \) has to be countable [1]. Of course, \( D_f \) is dense in \( P_f \).

Let \( y \in P_f \setminus D_f \). As \( f \) has no extremum at any point of the level set \( f^{-1}(\{y\}) \) and at some point \( x \in f^{-1}(\{y\}) \) we have \( f'(x) = 0 \), then, as \( f \) satisfies the property of Lemma 3, this must be an accumulation point of the level set \( f^{-1}(\{y\}) \). But, as \( f \) satisfies the property of Lemma 6, there is no other accumulation point of the level set \( f^{-1}(\{y\}) \). Thus 1. is satisfied.

As \( f \) satisfies the property of Lemma 5 and the fact that for any \( y \) for which \( f^{-1}(\{y\}) \) is infinite there must be a point \( x \in f^{-1}(\{y\}) \) such that \( f'(x) = 0 \), we have that 2. is satisfied.

That 3. is satisfied follows from the fact that \( f \) satisfies the property of Lemma 4.

Let us finally state the following theorem on the level sets of a typical function in \( C^n[0,1] \) for \( n \geq 2 \).
Theorem 3. A typical function in $C^n[0,1]$, $n \geq 2$, has all level sets finite.

Proof. Let us first prove the theorem for $n = 2$. Notice that the space $C^2[0,1]$ is homeomorphic with the product space $\mathbb{R} \times C^1[0,1]$ where the homeomorphism $\Psi: \mathbb{R} \times C^1[0,1] \to C^2[0,1]$ is again given by the formula (1). Let

$\mathcal{W} = \{ f \in C^2[0,1]: \text{there exists } y \text{ such that the set } f^{-1}\{y\} \text{ is infinite}\}$.

Let us notice that

$\Psi^{-1}(\mathcal{W}) \subseteq \mathbb{R} \times \mathcal{J},$

where

$\mathcal{J} = \{ \phi \in C^1[0,1]: \text{the set } \phi^{-1}\{0\} \text{ is infinite}\}$.

By Lemma 2 $\mathcal{J}$ is of first category in $C^1[0,1]$ whence $\mathcal{W}$ is of first category in $C^2[0,1]$. Thus the conclusion of our theorem is proved for $n = 2$.

For $n > 2$, the theorem can be easily proved in a similar fashion with induction.

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