AN APPLICATION OF THE REGULARIZED SIEGEL-WEIL FORMULA ON UNITARY GROUPS TO A THETA LIFTING PROBLEM

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Abstract. Let $U(2)$ and $U(2,1)$ be the pair of unitary groups over a global field $F$ and $\pi$ an irreducible cuspidal representation of $U(2)$ which satisfies a certain $L$-function condition. By using a regularized Siegel-Weil formula, we can show that the global theta lifting of $\pi$ in $U(2,1)$ is non-trivial if every local factor $\pi_\nu$ of $\pi$ has a local theta lifting (Howe lifting) in $U(2,1)(F_\nu)$.

1. Introduction

In this paper, we will apply a regularized Siegel-Weil formula [Tan2] to a theta lifting problem for the dual pair of unitary groups $U(2)$ and $U(2,1)$ defined over a global field $F$. The motivation of this work come from [KR] and [KRS] which proved the regularized Siegel-Weil formula for the symplectic-orthogonal dual pair $Sp(n), O(m)$ and applied it to poles and values of $L$-functions of $Sp(n)$.

Given an irreducible cuspidal representation $\pi$ of $U(2)$, we would like to ask if its global theta lifting $\Theta(\pi)$ in $U(2,1)$, to be defined below, is non-trivial. Let $\otimes \pi_\nu$, be the decomposition of $\pi$ as a restricted tensor product where $\nu$ runs through all the places of $F$. We say that $\pi_\nu$ has a local theta lift (or Howe lift) to $U(2,1)(F_\nu)$ if $\pi_\nu$ occurs in the (local) Howe correspondence for the dual reductive pair. (See for example [MVW].)

We will show that, under a certain condition on $\pi$, if $\pi_\nu$ has a local theta lifting everywhere, then $\pi$ has a global theta lifting $\Theta(\pi)$. More precisely,

**Theorem 1.1.** Let $\pi = \otimes \pi_\nu$ be a cuspidal representation of $U(2)$ such that $\pi_\nu$ belongs to the discrete series for all ramified places $\nu$ and the standard Langlands $L$-function $L(s, \pi, \gamma^3)$ has no pole at $s = 1$. Then the global theta lifting $\Theta(\pi)$ of $\pi$ to $U(2,1)$ is non-trivial if and only if the local theta lifting of $\pi_\nu$ to $U(2,1)_\nu$ is non-trivial for all $\nu$.

All the notations and terminologies will be introduced in section 2 and the proof of the theorem will be given in section 4.

We remark here that, for a general dual pair $U(m)$ and $U(n)$ such that $m > 2n$ (the convergent range), it has been proved in [Li2] that this is the case using the classical Siegel-Weil formula on $U(m)$ and $U(n,n)$. Note that this pair lies in the convergent range for the classical formula to work. Since our dual pair falls outside

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this range, we have to resort to the regularized version of the Siegel-Weil formula which we will recall in section 3.

This paper is based on part of the author’s dissertation [Tan1] under the guidance of Jonathan Rogawski. This result was first announced in [GRS1] based on the approach in [Tan1]. The result was later improved in [GRS2] using a sophisticated arguments involving endoscopic $L$-packets. It should be mentioned that, in that paper, the hypothesis that $\pi_\upsilon$ belongs to the discrete series at all ramified places has been removed. However, this condition is needed in our approach in order to conclude the non-vanishing of certain local integrals defined in section 4. We believe that the local theta lifting of $\pi_\upsilon$ should be enough to imply these local integrals are nonzero provided they converge. Nevertheless, our approach is more elementary and our emphasis is on the application of the Siegel-Weil formula.

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2. Set up

Fix a quadratic extension $E$ of $F$. Let $G = U(2)$ act on the two-dimensional Hermitian space $U$ over $E$ with Hermitian form $\langle \cdot, \cdot \rangle$ and $H = U(2,1)$ act on the three dimensional skew-Hermitian space $V$ over $E$ with skew-Hermitian form $\langle \cdot, \cdot \rangle'$. We form the tensor product $W = U \otimes_E V$ together with the symplectic form $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle \otimes_E \langle \cdot, \cdot \rangle'$. Let $X \oplus Y$ be a complete polarization of $W$.

Let $G'$ be $U(2,2)$ so that $G' \times H \subset Sp(W')$, where $W' = W \oplus W$ with Hermitian form $\langle \langle \cdot, \cdot \rangle \rangle := \langle \cdot, \cdot \rangle \oplus -\langle \cdot, \cdot \rangle$. Then

$$(X \oplus X) \bigoplus (Y \oplus Y)$$

is a complete polarization of $W'$. We can choose a basis for $W'$ so that, with respect to this decomposition, the Hermitian form $\langle \langle \cdot, \cdot \rangle \rangle$ is given by the matrix

$$
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}.
$$

We also have another complete polarization of $W'$. Let

$$U^d := \{(u, u) | u \in U\}, \quad U_d := \{(u, -u) | u \in U\}$$

and

$$W^d := U^d \otimes V, \quad W_d := U_d \otimes V.$$

Then $W' = W_d \oplus W^d$.

Let $P'$ be the parabolic subgroup of $G'$ that stabilizes $U^d$ (and hence $W^d$).

We can embed the product $G \times G$ into $G'$ by an embedding $\iota$ such that

$$\iota(g_1, g_2)(u_1, u_2) = (g_1 u_1, g_2 u_2)$$

for all $g_i \in G$ and $u_i \in U$. We identify $G \times G$ with its image under $\iota$.

Given a non-trivial additive character $\psi$ of $A/F$, there is a metaplectic representation, depending on $\psi$, on the metaplectic cover of $Sp(W')(A)$. Here $A$ denotes the adele ring of $F$. By [GR], Proposition 3.1.1, we have an explicit splitting of the metaplectic cover over $G'(A) \times H(A)$. This splitting is not unique and depends on the choice of $\psi$ and a Hecke character $\gamma$ whose restriction to $F$ is $\omega_{E/F}$, the
\[ \theta \text{ is the representation whose space is spanned by the functions } \]
\[ \phi \in \mathcal{S}(\mathbb{A}) \text{ where } H \text{ this function is well defined and slowly increasing on } \]
\[ (2.2) \]
\[ \text{Here we have identified } w \in W_d \text{ with } (x, y) \in (X \oplus Y) = W \text{ by mapping } (w, -w) \text{ to } w. \text{ We then have} \]
\[ \varphi^*(0) = \int_X \varphi_1(v) \varphi_2(v) dv = \langle \varphi_1, \varphi_2 \rangle \]
\[ (2.1) \]
\[ \text{where } \langle , \rangle \text{ denotes the inner product in } L^2(X). \text{ Also, the two theta series} \]
\[ \sum_{w \in W_d(F)} \varphi^*(w) \text{ and } \sum_{x, y \in X(F)} \varphi(x, y) \]
\[ \text{agree by the Poisson summation formula.} \]
\[ \text{Now we define the theta lifting } \Theta(\pi) \text{ for a cuspidal representation } \pi \text{ of } G. \text{ First} \]
\[ \text{of all, we have our usual theta function} \]
\[ \theta^\pm(g, h, \varphi) = \sum_{x \in X(F)} \omega_\pm(g, h) \varphi(x) \]
\[ \text{where } \varphi \in \mathcal{S}(X(\mathbb{A})). \text{ For a cusp form } f \text{ of } G \text{ in } \pi, \text{ we define} \]
\[ (2.2) \]
\[ \theta^\pm_f(h) := \int_{G(F) \backslash G(\mathbb{A})} f(g) \theta^\pm(g, h, \varphi) dg. \]
\[ \text{This function is well defined and slowly increasing on } H(F) \backslash H(\mathbb{A}). \text{ Then } \Theta(\pi) \]
\[ \text{is the representation whose space is spanned by the functions } \theta^\pm_f \text{ where } \varphi \text{ runs} \]
\[ \text{through } S(X(\mathbb{A})) \text{ and } f \text{ runs through the cuspidal forms in } \pi. \]
\[ \text{Let us also recall the standard Langlands } L\text{-function } L(s, \pi, \gamma^3) \text{ attached to } \pi \text{ and} \]
\[ \text{twisted by the character } \gamma^3. \text{ (For a general exposition of this type of } L\text{-functions,} \]
\[ \text{please refer to [Bor].} \text{ It is defined by an Euler product of local } L\text{-factors over those} \]
\[ \text{places } v \text{ such that } G_v = G(F_v) \text{ is unramified over } F_v \text{ and } \pi_v \text{ is unramified.} \text{ These} \]
\[ \text{local factors are explicitly given as follows:} \]
\[ \text{When } v \text{ is inert and } \pi_v \subseteq \text{Ind}_{B_v}^G(\tau), \text{ the normalized induced representation of} \]
\[ G_v \text{ from the character } \tau \text{ of } B_v \cong GL(1, F_v), \]
\[ (2.3) \]
\[ L(s, \pi_v, \gamma^3_v) = L_{E_v}(s, \tau \gamma^3_v)L_{E_v}(s, \tau^{-1} \gamma^3_v) \]
\[ \text{(we use a bar to denote the non-trivial Galois automorphism of } E/F). \]
\[ \text{When } v \text{ splits in } E \text{ and } \pi_v = \text{Ind}_{B_v}^G(\chi_1 \chi_2) \text{ where } \chi_i \text{ are characters of } GL(1, F_v), \]
\[ (2.4) \]
\[ L(s, \pi_v, \gamma^3_v) = L_{F_v}(s, \chi_1 \gamma^3_v)L_{F_v}(s, \chi_2 \gamma^3_v)L_{F_v}(s, \chi_1 \gamma^3_v)L_{F_v}(s, \chi_2 \gamma^3_v). \]
Here \( L_{F_v}(s, \chi) = \frac{1}{1-q_v^s} \) and is the local Tate \( L \)-factors associated to the Hecke character \( \chi \) of \( F \) where \( q_v \) is the order of the residue field at \( F_v \).

Via the base change lift from \( G \) to \( GL(2, E) \) (see [Rog], section 4.2), we can regard \( L(s, \pi, \gamma^3) \) as a standard Langlands \( L \)-function of \( GL(2, E) \). More precisely, if \( \Pi_E \) is the automorphic representation of \( GL(2, E) \) corresponding to \( \pi \) by base change lift, then

\[
\Pi_{E, \psi} = \text{Ind}_{B_2, \psi_2}^{GL(2, E_{\psi_2})}( \begin{pmatrix} \alpha & \beta \\ & \end{pmatrix} \mapsto \tau(\alpha/\beta))
\]

when \( \psi \) is inert and

\[
\Pi_{E, \psi_1} = \text{Ind}_{B_2, \psi_1}^{GL(2, E_{\psi_1})}( \begin{pmatrix} \alpha & \beta \\ & \end{pmatrix} \mapsto \chi_1(\alpha)\chi_2(\beta)),
\]

\[
\Pi_{E, \psi_2} = \text{Ind}_{B_2, \psi_2}^{GL(2, E_{\psi_2})}( \begin{pmatrix} \alpha & \beta \\ & \end{pmatrix} \mapsto \chi_1^{-1}(\alpha)\chi_2^{-1}(\beta))
\]

when \( \psi \) splits as \( \psi_1 \psi_2 \) in \( E \). Then

\[
L(s, \pi, \gamma^3) = L(s, \Pi_E \otimes \gamma^3)
\]

where the right-hand side above is the standard Langlands \( L \)-function attached to \( \Pi_E \otimes \gamma^3 \) (without twist). Therefore, the study of such \( L \)-function of \( U(2) \) reduces to that of \( GL(2) \) (over \( E \)) whose analytic properties are more well known. In particular, we know that the possible pole of the standard \( L \)-function of \( GL(2) \) can only occur at \( s = 1 \). Therefore the \( L \)-function condition of \( G \) in Theorem 1.1 can be replaced by that of \( GL(2) \).

3. Regularized Siegel-Weil Formula

In this section, we recall the regularized Siegel-Weil formula for the dual pair \( G' \) and \( H \) which is the key to the proof of our theta lifting problem. Details of this formula can be found in [Tan2]. As is mentioned in the introduction, the regularized Siegel-Weil formula was first formulated (the so-called first term identity) by S. Kudla and S. Rallis in [KR] for the dual pair \( Sp(n) \times O(m) \). In the other paper [KRS], a further result (second term identity) was obtained for the pair \( Sp(2) \times O(4) \). These two papers have provided a prototype for the regularized Siegel-Weil formula in the unitary case.

There are two objects involved in the formula, namely the Siegel-Eisenstein series and the regularized theta integral. Let \( I(s) = \text{Ind}_{P_0}^{G'}(\gamma^3 || \cdot ||^s) \) be the normalized induced representation of \( G'(\mathbb{A}) \) inducing from the character \( \gamma^3 || \cdot ||^s \) of the Levi factor of \( P'(\mathbb{A}) \). We denote an element in (the space of) \( I(s) \) by \( \Phi(s) \) or simply \( \Phi \).

Now we define the Siegel-Eisenstein series

\[
E(g, s, \Phi) = \sum_{\varepsilon \in P'(F) \backslash G'(F)} \Phi(\varepsilon g, s)
\]

for \( \Phi(s) \) an element in \( I(s) \). This series converges absolutely for \( \text{Re}(s) > 1 \) and has a meromorphic continuation to the whole complex plane. It has a simple pole at \( s = \frac{1}{2} \). Let

\[
\frac{A_{-1}(g, \Phi)}{s - \frac{1}{2}} + A_0(g, \Phi) + \cdots
\]

be the Laurent expansion of \( E(g, s, \Phi) \) at \( s = \frac{1}{2} \). Then \( A_{-1} \) defines an intertwining map from \( I(\frac{1}{2}) \) to \( \mathcal{A}(G') \), the space of automorphic forms on \( G' \).
We now turn over to the regularized theta integral. We first define the theta function

$$\theta(g, h, \varphi) = \sum_{w \in W_0(F)} \omega(g, h) \varphi(w)$$

where $\varphi \in S(W_0(\mathbb{A})).$ We also need to introduce a Hecke operator $z$ of $H_\upsilon$ for some unramified inert place $\upsilon.$ This is a locally constant, compactly supported bi $K$-invariant function on $H_\upsilon.$ Given a representation $\sigma$ of $H_\upsilon,$ a Hecke operator $z$ acts on the space of $\sigma$ via

$$\sigma(z) = \int_{H_\upsilon} z(h) \sigma(h) dh.$$ 

If $\sigma$ is unramified, then $z$ acts on the $K$-fixed vectors of $\sigma$ by scalars. More precisely, suppose $\chi$ is some unramified character of the Borel of $H_\upsilon$ such that

$$\chi \left( \begin{array}{ccc} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \bar{a}^{-1} \end{array} \right) = \rho(a^2 b) \|a\|^s$$

for some $s \in \mathbb{C}$ and some unitary character $\rho.$ If $\upsilon^0$ is a $K$-fixed vector in $\sigma,$ then $\sigma(z)\upsilon^0 = P(q^{\pm s})$ where $P(q^{\pm s})$ is a symmetric polynomial in $q^{\pm s}$ over $\mathbb{C}$ and $q$ is the residual characteristic of $F_\upsilon.$ (In fact, $z \mapsto P(q^{\pm s})$ defines a one-to-one correspondence between the set of all Hecke operators of $H_\upsilon$ and $\mathbb{C}[q^s]^{W_{H_\upsilon}}$ where $W_{H_\upsilon}$ is the Weyl group of $H_\upsilon.$ This is called the Satake isomorphism.) For our choice of $z,$ the corresponding $P(q^{\pm s})$ is

$$P(q^{\pm s}) = q^s + q^{-s} - q - q^{-1}.$$ 

By Corollary 2.3.2 in [Tan2], we have

Lemma 3.1. $\theta(g, h, \omega(z)\varphi)$ is rapidly decreasing on $H(F) \setminus H(\mathbb{A})$ for all $g$ and $\varphi.$

In order to compare our theta integral with the Siegel-Eisenstein series, we need to incorporate an auxiliary Eisenstein series $E(h, s)$ (see [Tan1, section 3.3]) of $H$ into the definition. Then the regularized theta integral is defined by

$$E(h, s) \text{ has a simple pole at } s = 1 \text{ with constant residue. So we may write}$$

$$E(h, s) = \frac{\kappa}{s-1} + \kappa_0(h) + \cdots. \tag{3.2}$$

Also $P(q^{\pm s})$ has a zero at $s = 1.$ So

$$\frac{1}{P(q^{\pm s})} = a \frac{1}{s-1} + b + \cdots. \tag{3.3}$$

Hence $E(g, s, \varphi)$ has a double pole at $s = 1.$ Let us write its Laurent expansion as

$$\frac{B_{-2}(g, \varphi)}{(s-1)^2} + \frac{B_{-1}(g, \varphi)}{(s-1)} + B_0(g, \varphi) + \cdots.$$ 

In view of (3.2) and (3.3), we have the expressions

$$B_{-2}(g, \varphi) = ak \int_{H(F) \setminus H(\mathbb{A})} \theta(g, h, \omega(z)\varphi) \gamma^{-2}(\det h) dh.$$
and

\[ B_{-1}(g, \varphi) = \frac{b}{a} B_{-2}(g, \varphi) + a \int_{H(F) \backslash H(\mathbb{A})} \theta(g, h, \omega(z) \varphi) \kappa_0(h) \gamma^{-2}(\det h) dh. \]

Note that the integrals in the above expressions are convergent in view of Lemma 3.1. Again, \( B_{-1} \) and \( B_{-2} \) define \( G'(\mathbb{A}) \)-intertwining maps from \( S(W_d(\mathbb{A})) \) to \( \mathcal{A}(G') \).

In order to link the two objects we have just defined, we define a map

\[ S(W_d(\mathbb{A})) \to \mathcal{I}(\frac{1}{2}), \]

\[ \varphi \mapsto \Phi(\frac{1}{2}) \]

such that

\[ \Phi(\frac{1}{2})(g) = \omega(g) \varphi(0). \]

This gives an intertwining map between the two representations. We denote the image of this map by \( \Pi(V) \).

If we decompose \( V \) as \( V_0 \oplus V_{1,1} \) where \( V_{1,1} \) is a hyperbolic plane in \( V \), then \( V_0 \) is the one dimensional anisotropic subspace associated to the skew-Hermitian form \((,)'|_{V_0} \). We can define similarly \( \Pi(V_0) \) as the space of all \( \Phi(-\frac{1}{2}) \) where

\[ \Phi(-\frac{1}{2})(g) = \omega'(g) \varphi(0) \]

and \( \omega' \) is the oscillator representation of \( G' \times U(1) \) as \( \varphi \) runs over \( S(U_d \otimes V_0(\mathbb{A})) \).

We may now summarize some of the main results in [Tan2]:

(i) \( \text{Im} \mathcal{A}_{-1} \cong \bigoplus \Pi(V_{00}) \) where \( V_{00} \) runs over all one dimensional skew Hermitian spaces;

(ii) \( \text{Im} \mathcal{A}_{-1} \big|_{\Pi(V)} \cong \Pi(V_0) \);

(iii) \( \text{Im} \mathcal{B}_{-2} \cong \Pi(V_0) \);

(iv) \( \Pi(V_0) \) can be embedded in \( \mathcal{A}(G') \) in exactly one way. In particular, we have

(v) (First term identity) There is a non-zero constant \( c \) such that, for all \( \varphi \in S(W_d) \),

\[ A_{-1}(g, \Phi) = cB_{-2}(g, \varphi) \]

where \( \Phi \) is associated to \( \varphi \).

We also have

(vi) (Second term identity) There is a non-zero constant \( c \) such that, for all \( \varphi \in S(W_d) \),

\[ A_0(g, \Phi) = cB_{-1}(g, \varphi) + \Psi(g) \]

for some \( \Psi \) in \( \text{Im} \mathcal{A}_{-1} \).

4. THE NON-VANISHING RESULT

Let us consider the integral

\[ \mathcal{Z}(s, f_1, f_2, \Phi) := \int_{G(F) \times G(F) \backslash G(\mathbb{A}) \times G(\mathbb{A})} f_1(g_1) f_2(g_2) E((g_1, g_2), s, \Phi) dg_1 dg_2 \]

where \( f_1, f_2 \in \pi, \Phi = \Phi(s) \in \mathcal{I}(s) \) and \( E(g', s, \Phi) \) is our Siegel-Eisenstein series.

Using the fundamental identity, which was first introduced in [GPS], we have

\[ \mathcal{Z}(s, f_1, f_2, \Phi) = \int_{G(\mathbb{A})} \Phi((g, 1)) \langle \pi(g) f_1, f_2 \rangle dg \]

where \( \langle f_1, f_2 \rangle = \int_{G(F) \backslash G(\mathbb{A})} \gamma^3(\det g) f_1(g) f_2(g) dg \).
Now suppose $\Phi, f_1, f_2$ are decomposable as local factors. Then the global zeta integral on the right-hand side of (4.1) admits an Euler product:

\[
\prod_v \int_{G_v} \Phi_v((g, 1)) \langle \pi_v(g) f_{1,v}, f_{2,v} \rangle dg.
\]

Our next lemma says that, for almost all $v$, the local zeta integrals in (4.2) are essentially the local factors of the (twisted) Langlands $L$-function associated with $\pi_v$ when $\Phi_v, f_{1,v}, f_{2,v}$ are chosen properly.

Let $S$ be a finite set of places in $F$ including all archimedean places such that everything is unramified outside $S$ (i.e. $G_v, \pi_v, \gamma_v, \psi_v$ etc. are unramified).

Then by a tedious but similar computation as in [Li2] (see [Tan1, chapter 6] for details), we obtain:

**Lemma 4.3.** Let $\nu \not\in S$. For $\Phi^0_v$ the normalized $K'_v$-fixed vector in $I_v(s)$ and $f_{0,v}$ a $K_v$-fixed vector in $\pi_v$ such that $(f_{0,v}, f_{0,v}) = 1$,

\[
\int_{G_v} \Phi^0_v((g, 1)) \langle \pi_v(g) f_{0,v}, f_{0,v} \rangle dg = \frac{1}{\xi_v(s)} L_v(s + \frac{1}{2}, \pi_v, \gamma^3_v)
\]

where $\xi_v(s) = L_{F_v}(2s + 1) L_{F_v}(2s + 2, \omega_{E/F})$ and $L_v(s, \pi_v, \gamma^3_v)$ is given by (2.3) and (2.4).

Let us choose two cusp forms $f_1, f_2$ in $\pi$. For almost all $v$,

\[
f_{1,v} = f_{2,v} = f_{0,v}
\]

where $f_{0,v}$ is as in Lemma 4.3. We may assume (4.4) is satisfied for every $v$ outside $S$. We also choose $\Phi(\frac{1}{2})$ from $\Pi(V)$, i.e. $\Phi(\frac{1}{2})(g') = \omega(g') \varphi^*(0)$ where $\varphi^* \in S(W_d)$ is the partial Fourier transform of $\varphi = \varphi_1 \otimes \varphi_2 \in S(X \oplus X)$. In view of (2.1),

\[
\Phi(\frac{1}{2})(g, 1) = \langle \omega(g) \varphi_1, \varphi_2 \rangle.
\]

At almost all places, $\Phi_v$ is the normalized $K'_v$-fixed vector in $I_v(s)$. So we might as well assume that, outside the finite set $S$, $\Phi_v = \Phi^0_v$. In particular, for $v \not\in S$, $\varphi_{1,v}$ and $\varphi_{2,v}$ are the characteristic functions of the lattice $X(O_{E_v})$.

We shall examine the analytic property of $Z(s, f_1, f_2, \Phi)$ at the point $s = \frac{1}{2}$. From Lemma 4.3,

\[
Z(\frac{1}{2}, f_1, f_2, \Phi) = L^S(1, \pi, \gamma^3) \prod_{v \in S} \int_{G_v} \langle \omega_v(g) \varphi_{1,v}, \varphi_{2,v} \rangle \langle \pi_v(g) f_{1,v}, f_{2,v} \rangle dg.
\]

where $L^S(s, \pi, \gamma^3) = \prod_{v \not\in S} L_v(s, \pi_v, \gamma^3_v)$ is the partial direct product of the local $L$-factors. By the discussion in section 2, the possible pole of this object can only occur at $s = 1$. Whenever $\pi_v$ is a discrete series, the local integrals

\[
\int_{G_v} \langle \omega_v(g) \varphi_{1,v}, \varphi_{2,v} \rangle \langle \pi_v(g) f_{1,v}, f_{2,v} \rangle dg
\]

converge absolutely. In fact, this is true more generally for $G = U(n)$ and $H = U(m)$ with $n \leq m$ and $\pi$ in the discrete series of $G$ (see [Li1]). Hence

**Lemma 4.7.** If $L(s, \pi, \gamma^3)$ does not have a pole at $s = 1$ and $\pi_v$ are discrete for all $v \in S$, then $Z(s, f_1, f_2, \Phi)$ has no pole at $s = \frac{1}{2}$. 

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Now let us recall that $Z(s, f_1, f_2, \Phi)$ is defined by integrating the Eisenstein series $E((g_1, g_2), s, \Phi)$ against $f_1(g_1)\bar{f}_2(g_2)$ over $G(F) \times G(F) \setminus G(\mathbb{A}) \times G(\mathbb{A})$. But $E(g', s, \Phi)$ has a pole at $s = \frac{1}{2}$ with residue $A_{-1}(g', \Phi)$ (see section 3). Hence the discussion above implies that

$$\int_{G(F) \times G(F) \setminus G(\mathbb{A}) \times G(\mathbb{A})} f_1(g_1)\bar{f}_2(g_2)A_{-1}((g_1, g_2), \Phi)dg_1dg_2 = 0. \tag{4.8}$$

We also have $\text{Im} A_{-1}|_{\Pi(V)} = \Pi(V_0)$. Thus we have shown that $f_1\bar{f}_2$ is orthogonal to $\Pi(V_0)$. If we replace $V$ by another three-dimensional skew Hermitian space $V^*$, we get a new group $H^*$. Since $f_1, f_2$ are cusp forms of $G$, by repeating the arguments for the dual pair $G' \times H^*$, we obtain $f_1\bar{f}_2$ being orthogonal to $\Pi(V_0^*)$ where $V_0^*$ is the complementary one-dimensional subspace of $V^*$. In fact, by allowing $V$ to run through all (global) three-dimensional skew-Hermitian spaces, we obtain

**Lemma 4.9.** Under the same condition as in Lemma 4.7, $f_1\bar{f}_2$ is orthogonal to $\bigoplus \Pi(V_0)$ where $V_0$ ranges over all one-dimensional skew-Hermitian spaces.

Now we turn to the non-vanishing of $Z(\frac{1}{2}, f_1, f_2, \Phi)$. We have to check that the factors in the right-hand side of (4.5) are non-zero. By a well known result of [Jac], the $L$-function $L(s, \Pi_E \otimes \gamma^3)$ of $GL(2, E)$ is non-zero at $s = 1$. (In fact, this is true for all $GL(n)$ and all $s$ with $\text{Re}(s) = 1$.) Therefore, $L(s, \pi, \gamma^3)$ is also non-vanishing at $s = 1$.

So it remains to check that the local zeta integrals for $v \in S$ are also non-zero. Whenever $\pi_v$ is discrete and has a theta lifting $\rho_v$ in $H_v$, it follows from [Li1], section 2 that (4.6) is not identically zero for all $\varphi_{i,v}$ and $f_{i,v}$. Hence we have

**Lemma 4.10.** Under the same condition as in Lemma 4.7, if $\pi_v$ has non-trivial theta-lifting in $H_v$ for all $v$, then $Z(s, f_1, f_2, \Phi)$ is holomorphic at $s = \frac{1}{2}$ and non-zero for some choice of $\Phi$ such that $\Phi(\frac{1}{2}) \in \Pi(V)$ and $f_1, f_2 \in \pi$.

Finally, we shall see how the results we have obtained so far together with the regularized Siegel-Weil formula imply that the theta lift of $\pi$ to $H$ is non-trivial.

The point is to express $Z(\frac{1}{2}, f_1, f_2, \Phi)$ in terms of theta integral associated to $G'$ and $H$.

In view of (4.8), we can write $Z(\frac{1}{2}, f_1, f_2, \Phi)$ as

$$\int_{G(F) \times G(F) \setminus G(\mathbb{A}) \times G(\mathbb{A})} f_1(g_1)\bar{f}_2(g_2)A_0((g_1, g_2), \Phi)dg_1dg_2$$

where $A_0(g, \Phi)$ is the second term in the Laurent expansion of $E(g, s, \Phi)$ at $s = \frac{1}{2}$. Now we invoke the second term identity that we have stated in section 3 to pass from the Eisenstein series to the regularized theta integral. We recall that

$$A_0(g', \Phi) = cB_{-1}(g', \varphi^*) + \Psi(g')$$

where $B_{-1}$ is the second term of the regularized theta integral, $c$ is a constant and $\Psi \in \bigoplus \Pi(V_{00})$. By Lemma 4.9, $\Psi$ is orthogonal to $f_1\bar{f}_2$. So we get

$$Z(\frac{1}{2}, f_1, f_2, \Phi) = c\int_{G(F) \times G(F) \setminus G(\mathbb{A}) \times G(\mathbb{A})} f_1(g_1)\bar{f}_2(g_2)B_{-1}((g_1, g_2), \varphi^*)dg_1dg_2.$$
In view of (3.4) and the fact that $B_{-2}(g', \varphi^*)$ belongs to $\Pi(V_0)$ and hence is orthogonal to $f_1, f_2$, $Z(\frac{1}{2}, f_1, f_2, \Phi)$ becomes a double integral

$$\int_{G(F) \times G(F) \backslash G(\mathbb{A}) \times G(\mathbb{A})} f_1(g_1) f_2(g_2) \times \int_{H(F) \backslash H(\mathbb{A})} \alpha_k(h)(\det h)^{\gamma-2} \theta(g', h, \omega(z) \varphi^*) dh \, dg_1 \, dg_2.$$ 

$\theta((g_1, g_2), h, \omega(z) \varphi^*)$ is rapidly decreasing on $H(F) \backslash H(\mathbb{A})$. Furthermore, $f_1, f_2$ are cusp forms on $G$ and hence also rapidly decreasing on $G(F) \backslash G(\mathbb{A})$. We can then apply Fubini's Theorem to interchange the two integrals. So we get

$$Z(\frac{1}{2}, f_1, f_2, \Phi) = \int_{H(F) \backslash H(\mathbb{A})} \alpha_k(h)(\det h)^{\gamma-2} \theta(g_1, g_2, h, \omega(z) \varphi^*) \, dg_1 \, dg_2 \, dh.$$ 

(4.11)

Now we are only one step from proving the non-vanishing of $\Theta(\pi)$. What we really have to show is that the function $\theta^+_{\varphi^1} f_1$ we defined in section 2 is non-zero for some $f$ and $\varphi$. Using our choice of $f_1, f_2, \varphi_1, \varphi_2$, we compute that

$$\theta^+_{\varphi^1} f_1(h) \theta^-_{\varphi^2} f_2(h)$$

$$= \int_{G(F) \backslash G(\mathbb{A})} \sum_{x \in X(F)} \omega_+(g, h) \varphi_1(x) f_1(g) \, dg$$

$$\times \int_{G(F) \backslash G(\mathbb{A})} \sum_{x \in X(F)} \omega_-(g, h) \varphi_2(x) f_2(g) \, dg$$

$$= \int_{G(F) \backslash G(\mathbb{A})} f_1(g_1) f_2(g_2) \sum_{x, y \in X(F)} \omega((g_1, g_2), h) \varphi(x, y) \, dg_1 \, dg_2.$$ 

If we convolve $\theta^+_{\varphi^1} f_1(h) \theta^-_{\varphi^2} f_2(h)$ with $z'$, the Hecke operator in $H_v$ corresponding to $z$ under Howe correspondence (see [MVW]), we get precisely the inner integral in (4.11). Therefore, under the conditions of Lemma 4.10, the non-vanishing of $Z(\frac{1}{2}, f_1, f_2, \Phi)$ implies $\theta^+_{\varphi^1} f_1$ is non-zero and hence $\Theta(\pi)$ is non-trivial.

We hence proved Theorem 1.1.

REFERENCES


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