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NOETHERIAN DOWN-UP ALGEBRAS

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Abstract. Down-up algebras $A = A(\alpha, \beta, \gamma)$ were introduced by G. Benkart and T. Roby to better understand the structure of certain posets. In this paper, we prove that $\beta \neq 0$ is equivalent to A being right (or left) Noetherian, and also to A being a domain. Furthermore, when this occurs, we show that A is Auslander-regular and has global dimension 3.

§1. Introduction

Motivated by the study of posets, G. Benkart and T. Roby introduced certain *down-up algebras* in [BR], see also [B]. Specifically, let K be a field, fix parameters $\alpha, \beta, \gamma \in K$ and let $A = A(\alpha, \beta, \gamma)$ be the K-algebra with generators d and u, and relations

$$d^2u = \alpha dud + \beta ud^2 + \gamma d = (\alpha du + \beta ud + \gamma)d, \tag{1.1}$$

$$du^2 = \alpha udu + \beta u^2d + \gamma u = u(\alpha du + \beta ud + \gamma). \tag{1.2}$$

Note that

$$(ud)(du) = u(d^2u) = u(\alpha du + \beta ud + \gamma)d = (du^2)d = (du)(ud),$$

by (1.1) and (1.2), and therefore ud and du commute in A. Furthermore, A is clearly isomorphic to its opposite ring A^{op} via the map $d \mapsto u^{\mathrm{op}}$ and $u \mapsto d^{\mathrm{op}}$.

Our main result here is

Theorem. *If $A = A(\alpha, \beta, \gamma)$, then the following are equivalent.*

(1) $\beta \neq 0$.
(2) *A is right (or left) Noetherian.*
(3) *A is a domain.*
(4) *$K[ud, du]$ is a polynomial ring in the two generators.*

Furthermore, if these conditions hold, then A is Auslander-regular and has global dimension 3.

In particular, we answer some questions posed in a preliminary version of [B]. Note that condition (4) above is significant because $K[ud, du]$ plays the role of the enveloping algebra of a Cartan subalgebra in the highest weight theory of [BR].

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As is apparent, all our positive results for A occur when the parameter β is not zero. In this situation, we offer two distinct approaches to the study of A, namely via filtered rings and via generalized Weyl algebras.

§2. The generalized Weyl algebra approach

2.1. Suppose that $\beta \neq 0$. We first show that A embeds in a skew group ring. To this end, let $R = K[x, y]$ be a polynomial ring in two variables, and define $\sigma \in \mathrm{Aut}_K(R)$ by $\sigma(x) = y$ and $\sigma(y) = \alpha y + \beta x + \gamma$. Note that σ is indeed an automorphism since $\beta \neq 0$, and we can form $S = R[z, z^{-1}; \sigma]$, the skew group ring of the infinite cyclic group $\langle z \rangle$ over R, with $rz = z\sigma(r)$ for all $r \in R$. Now consider the elements $D = z^{-1}$ and $U = xz$ in S. Then $UD = xzz^{-1} = x$ and $DU = z^{-1}xz = \sigma(x) = y$. In addition,

$$\begin{aligned} D^2U &= D{\cdot}DU = z^{-1}y = \sigma(y)z^{-1} \\ &= (\alpha y + \beta x + \gamma)z^{-1} = (\alpha DU + \beta UD + \gamma)D \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} DU^2 &= DU{\cdot}U = yxz = xz\sigma(y) \\ &= U(\alpha y + \beta x + \gamma) = U(\alpha DU + \beta UD + \gamma). \end{aligned} \tag{2.2}$$

Hence, there is an algebra homomorphism $\theta\colon A(\alpha, \beta, \gamma) \to S$ given by $\theta(d) = D$ and $\theta(u) = U$. Since $\theta(ud) = UD = x$ and $\theta(du) = DU = y$ are algebraically independent, the same is true of the commuting elements ud and du. In particular, $K[ud, du]$ is isomorphic to a polynomial ring in the two variables and σ lifts to an automorphism of this algebra satisfying $\sigma(ud) = du$ and $\sigma(du) = \alpha du + \beta ud + \gamma$.

It remains to show that θ is a monomorphism. To this end, note that (1.1) implies

$$\begin{aligned} d(ud) &= (du)d = \sigma(ud)d, \\ d(du) &= (\alpha du + \beta ud + \gamma)d = \sigma(du)d, \end{aligned} \tag{2.3}$$

and similarly, (1.2) yields

$$\begin{aligned} (du)u &= u(\alpha du + \beta ud + \gamma) = u\sigma(du), \\ (ud)u &= u(du) = u\sigma(ud). \end{aligned} \tag{2.4}$$

Thus

$$dc = \sigma(c)d, \qquad cu = u\sigma(c) \qquad \text{for all } c \in K[ud, du], \tag{2.5}$$

and therefore $dK[ud, du] = K[ud, du]d$ and $K[ud, du]u = uK[ud, du]$. In particular, if we set

$$B = \sum_{k \geq 0} K[ud, du]d^k + \sum_{k \geq 0} K[ud, du]u^{k+1} \subseteq A,$$

then $dB \subseteq B$ and $uB \subseteq B$. Hence $AB \subseteq B$ and consequently $A = B$. In other words, A is spanned by the set $\mathcal{B} = \{\, (ud)^i(du)^jd^k,\ (ud)^i(du)^ju^{k+1} \mid i, j, k \geq 0 \,\}$. But $\theta(\mathcal{B}) = \{\, x^iy^jz^{-k},\ x^iy^j(xz)^{k+1} \mid i, j, k \geq 0 \,\}$, and these elements are clearly linearly independent in S. It follows that $\mathcal{B}$ is a basis for A and that θ is indeed a one-to-one map.

Corollary. *If $\beta \neq 0$, then $K[ud, du]$ is a polynomial ring in the two generators and $A(\alpha, \beta, \gamma)$ is a domain.*

2.2. If R is any K-algebra, σ any K-automorphism of R and x any central element of R, then the generalized Weyl algebra $R(\sigma, x)$ is defined to be the algebra generated by R and the two variables X^+ and X^- subject to the relations

$$\begin{gathered} X^-X^+ = x, \qquad X^+X^- = \sigma(x), \\ X^+r = \sigma(r)X^+, \qquad X^-\sigma(r) = rX^- \qquad \text{for all } r \in R. \end{gathered} \tag{2.6}$$

Here, we take $R = K[x, y]$ and we let σ be described as in the preceding section. Then $\sigma(x) = y$, $\sigma(y) = \alpha y + \beta x + \gamma$, and it follows from the above that

$$\begin{aligned} X^+(X^+X^-) &= X^+y = \sigma(y)X^+ = (\alpha y + \beta x + \gamma)X^+ \\ &= (\alpha X^+X^- + \beta X^-X^+ + \gamma)X^+ \end{aligned}$$

and

$$\begin{aligned} (X^+X^-)X^- &= yX^- = X^-\sigma(y) = X^-(\alpha y + \beta x + \gamma) \\ &= X^-(\alpha X^+X^- + \beta X^-X^+ + \gamma). \end{aligned}$$

Thus there exists an algebra homomorphism $\varphi\colon A \to R(\sigma, x)$ given by $d \mapsto X^+$ and $u \mapsto X^-$. On the other hand, (2.5) implies that the map $\varphi'\colon R(\sigma, x) \to A$ given by $X^+ \mapsto d$ and $X^- \mapsto u$ is also an algebra homomorphism. Therefore $\varphi' = \varphi^{-1}$ and φ is an isomorphism. In other words, we have shown

Theorem. *If $\beta \neq 0$, then the algebra $A = A(\alpha, \beta, \gamma)$ is isomorphic to a generalized Weyl algebra $R(\sigma, x)$ with $R = K[x, y]$.*

Consequently, [Bv1, Proposition 7] yields

Corollary. *If $\beta \neq 0$, then $A(\alpha, \beta, \gamma)$ is right and left Noetherian.*

§3. The filtered ring approach

3.1. To start with, let α, β, γ be arbitrary parameters and let $A = A(\alpha, \beta, c)$. We define a filtration on A for which the associated graded ring is isomorphic to $A(\alpha, \beta, 0)$. To this end, let $V = K + Ku + Kd$ and let $V_n = V^n$. Then $V_0 = K$, $V_1 = V$, and $\{ V_n \mid n = 0, 1, 2, \dots \}$ is obviously a filtration of A. Certainly, $\bar{u} = u + K$ and $\bar{d} = d + K$ generate the associated graded ring $\operatorname{Gr} A$, and it is clear, from (1.1) and (1.2), that $\bar{u}$ and $\bar{d}$ satisfy the generating relations of $A(\alpha, \beta, 0)$. Thus, there exists an epimorphism $\rho\colon A(\alpha, \beta, 0) \to \operatorname{Gr} A$ given by $U \mapsto \bar{u}$ and $D \mapsto \bar{d}$. Here, of course, we use U and D to denote the obvious generators of $A(\alpha, \beta, 0)$. To see that ρ is an isomorphism, we use the PBW Theorem for down-up algebras as given in [B, Theorem 4.1]. Specifically, that result asserts that $\mathcal{C} = \{ u^i(du)^j d^k \mid i, j, k \geq 0 \}$ is a basis for A and that $V_n = V^n$ is spanned by those monomials with $i + 2j + k \leq n$. With this observation, it is clear that V_n/V_{n-1} has basis $\mathcal{C}_n = \{ \bar{u}^i(\bar{d}\bar{u})^j \bar{d}^k \mid i, j, k \geq 0,\ i + 2j + k = n \}$, and hence $\bar{\mathcal{C}} = \bigcup_{n=0}^{\infty} \mathcal{C}_n$ is a basis of $\operatorname{Gr} A$. But $\{ U^i(DU)^j D^k \mid i, j, k \geq 0 \}$ is a basis of $A(\alpha, \beta, 0)$, by [B, Theorem 4.1] again, and ρ maps this basis to $\bar{\mathcal{C}}$. Consequently, ρ is one-to-one, and we have shown

Lemma. *$A = A(\alpha, \beta, \gamma)$ has a filtration whose associated graded ring is isomorphic to $A(\alpha, \beta, 0)$.*

3.2. We now show that if $\beta \neq 0$ and if K is sufficiently big, then the algebra $A(\alpha, \beta, 0)$ is an iterated Ore extension. To start with, fix nonzero elements $\lambda, \mu \in K$, and let B be the algebra with generators a and b, and with relation

$$ba = \mu ab. \tag{3.1}$$

Then B is clearly an Ore extension of its polynomial subalgebra $K[a]$, and hence $\{ a^i b^j \mid i, j \geq 0 \}$ is a basis of B. Now let τ be the automorphism of B defined by $\tau(a) = \lambda a$, $\tau(b) = \mu b$, and let $\delta \colon B \to B$ be the K-linear map determined by

$$\delta(a^m b^n) = p_m(\lambda, \mu) a^{m-1} b^{n+1} \qquad \text{for all } m, n \geq 0,$$

where

$$p_m = p_m(\lambda, \mu) = \sum_{i=0}^{m-1} \lambda^i \mu^{m-1-i}.$$

Since $p_{m+t} = \mu^t p_m + \lambda^m p_t$, it is easy to see that δ is a τ-derivation of B, that is,

$$\delta(rs) = \delta(r)s + \tau(r)\delta(s) \qquad \text{for all } r, s \in B.$$

Hence, we can form the Ore extension $C = B[c; \tau, \delta] = C(\lambda, \mu)$. Basic properties of such extensions can be found in [GW, Chapter 1]. In particular, C is a free left and right B-module with basis $\{ c^i \mid i \geq 0 \}$, and with multiplication determined by

$$cr = \tau(r)c + \delta(r) \qquad \text{for all } r \in B.$$

Indeed, since $\delta(a) = b$ and $\delta(b) = 0$, we have

$$ca = \lambda ac + b, \qquad cb = \mu bc. \tag{3.2}$$

Furthermore, C has a basis over K consisting of all monomials $a^i b^j c^k$ with $i, j, k \geq 0$, and it is clear that $C = C(\lambda, \mu)$ is the K-algebra generated by a, b and c subject to the relations (3.1) and (3.2).

3.3. If $\eta \in K$ and if r and s are elements of any K-algebra, we introduce the notation $[r, s]_\eta = rs - \eta sr$. Now suppose that $\beta \neq 0$. If $\alpha^2 + 4\beta$ is a square in K, we say that $A = A(\alpha, \beta, \gamma)$ is *split*, and we can let λ and μ be the roots of the quadratic equation

$$\zeta^2 - \alpha\zeta - \beta = 0.$$

Thus $\lambda + \mu = \alpha$, $\lambda\mu = -\beta$, and $\lambda, \mu \neq 0$ since $\beta \neq 0$.

Now it is easily seen that the defining relations (1.1) and (1.2) for $A(\alpha, \beta, 0)$ can be expressed in the form

$$[[D, U]_\lambda, U]_\mu = 0 = [D, [D, U]_\lambda]_\mu. \tag{3.3}$$

Indeed,

$$\begin{aligned} [D, [D, U]_\lambda]_\mu &= [D, DU - \lambda UD]_\mu = D(DU - \lambda UD) - \mu(DU - \lambda UD)D \\ &= D^2U - (\lambda + \mu)DUD + \lambda\mu UD^2 = D^2U - \alpha DUD - \beta UD^2, \end{aligned}$$

so $0 = [D, [D, U]_\lambda]_\mu$ is equivalent to (1.1), and similarly, $[[D, U]_\lambda, U]_\mu = 0$ is equivalent to relation (1.2).

Finally, if we set $H = [D, U]_\lambda$, then (3.3) translates to

$$HU = \mu UH, \qquad DH = \mu HD, \qquad DU = \lambda UD + H. \tag{3.4}$$

In other words, $A(\alpha, \beta, 0)$ is generated by the elements U, D and H subject to the relations (3.4) and, in view of the comment at the end of §3.2, there is an algebra isomorphism $\psi\colon C \to A(\alpha,\beta,0)$ given by $\psi(c) = D$, $\psi(a) = U$ and $\psi(b) = H$. By combining all of this, we have therefore proved

Theorem. *Assume that $A = A(\alpha,\beta,\gamma)$ is split and that $\beta \neq 0$. Then A has a filtration whose associated graded ring $\operatorname{Gr} A \cong A(\alpha,\beta,0)$ is isomorphic to an iterated Ore extension of the form $K[a][b;\eta][c;\tau,\delta]$.*

We can also use the above result to prove that $A = A(\alpha,\beta,\gamma)$ is Noetherian when $\beta \neq 0$. Indeed, for this it suffices to extend the field and assume that A is split. Then $\operatorname{Gr} A$ is an iterated Ore extension, so $\operatorname{Gr} A$ is right and left Noetherian, and hence so is A.

§4. Main results

4.1. We start by considering the global dimension of $A = A(\alpha,\beta,\gamma)$. Since A and its associated graded ring $\operatorname{Gr} A \cong A(\alpha,\beta,0)$ are both isomorphic to their opposite rings, left and right global dimensions are equal here. Thus we can use $\operatorname{gl\,dim}$ to denote this common dimension.

Theorem. *If $\beta \neq 0$, then* $\operatorname{gl\,dim} A(\alpha,\beta,\gamma) = 3$.

Proof. We first show that $\operatorname{gl\,dim} A < \infty$ and for this, it suffices to assume that A is split. Indeed, if F is a field extension of K, then $A^F = F \otimes A$ is a free A-module and hence $\operatorname{gl\,dim} A \leq \operatorname{gl\,dim} A^F$ by [McR, Theorem 7.2.8]. Now if A is split, then Theorem 3.2 implies that A has a filtration with $\operatorname{Gr} A$ isomorphic to an iterated Ore extension. Thus $\operatorname{gl\,dim} \operatorname{Gr} A < \infty$ by [McR, Theorem 7.5.3], and consequently $\operatorname{gl\,dim} A < \infty$ by [McR, Corollary 7.6.18].

Now, by Theorem 2.2, A is isomorphic to a generalized Weyl algebra $R(\sigma,x)$ with $R = K[x,y]$. Here the automorphism σ of R is given by $\sigma(x) = y$ and $\sigma(y) = \alpha y + \beta x + \gamma$. Thus the maximal ideals $Q = (x,y)$ and $P = \sigma^{-1}(Q)$ of R both contain x, and it follows from [Bv2, Theorem 3.7] that $\operatorname{gl\,dim} A = 3$. □

4.2. Recall that a Noetherian ring R is said to be *Auslander-regular* if R has finite global dimension and if, for every finitely generated R-module M and positive integer q, we have $j(N) \geq q$ for every submodule N of $\operatorname{Ext}^q_R(M,R)$. Here $j(M) = \min\{\, j \mid \operatorname{Ext}^j_R(M,R) \neq 0 \,\}$. Furthermore, R is said to be *Cohen-Macaulay* if R has finite GK-dimension, and if the equality $\operatorname{GKdim} M + j(M) = \operatorname{GKdim} R$ holds for every finitely generated R-module M.

Lemma. *Let $\beta \neq 0$ and write $A = A(\alpha,\beta,\gamma)$.*

(i) *A is Auslander-regular.*
(ii) *If A is split, then it is also Cohen-Macaulay.*

Proof. (i) As is shown in [Bv2, pp. 88–89], a generalized Weyl algebra $R(\sigma,x)$ is always a factor ring of an iterated skew polynomial extension of R. The argument is as follows. First form the polynomial ring $R[z]$ and consider the generalized Weyl algebra $R[z](\sigma, x+z)$, where σ is extended to $R[z]$ by taking $\sigma(z) = z$. Thus, since z is in the center of $R[z](\sigma,x+z)$ and since $R[z](\sigma,x+z)/(z) \cong R(\sigma,x)$, it suffices to show that $R[z](\sigma,x+z)$ is an iterated skew polynomial ring extension of R. For this, note that $R[z](\sigma,x+z) \cong R[X^-;\sigma^{-1}][X^+;\sigma,\delta]$, where the automorphism σ is extended to $R[X^-;\sigma^{-1}]$ by $\sigma(X^-) = X^-$, and the σ-derivation δ is defined by $\delta(r) = 0$ for all $r \in R$ and $\delta(X^-) = \sigma(x) - x$.

We now proceed to show that A is Auslander-regular. To start with, Lemma 3.1 and [Bj, Theorem 4.1] allow us to assume that $\gamma = 0$. Consequently, σ is a graded automorphism of $R = K[x, y]$ and hence the Ore extensions above are all constructed via graded automorphisms. In other words, $R[z](\sigma, x + z)$ is an iterated Ore extension of a connected graded K-algebra, with each automorphism graded. Thus, [GZ, Lemma 3.8(2)] implies that $R[z](\sigma, x + z)$ is Auslander-regular (and also Cohen-Macaulay). But z is a central regular element, so we can conclude from [L, §3.4 Remark (3)] that the Auslander condition carries over to the factor ring $R[z](\sigma, x + z)/(z) \cong R(\sigma, x) \cong A$.

(ii) Now suppose that A is split. Then, by Lemma 3.1, A has a filtration with $A_0 = K$ and with $\operatorname{Gr} A \cong A(\alpha, \beta, 0)$. Hence, by [GZ, Lemma 3.8(1)], it suffices to show that $A(\alpha, \beta, 0)$ is Auslander-regular and Cohen-Macaulay. To this end, observe that $A(\alpha, \beta, 0) \cong K[a][b; \eta][c; \tau, \delta]$, by Theorem 3.3, where η and τ are graded algebra automorphisms. Furthermore, in each Ore extension, the set of elements of total degree 0 is precisely equal to K. Thus, [GZ, Lemma 3.8(2)] implies that $A(\alpha, \beta, 0)$ is Auslander-regular and Cohen-Macaulay, as required. □

4.3. Lemma. *If $\beta = 0$, then A is not right or left Noetherian.*

Proof. For convenience, set $x = du$ so that $\{ u^i x^j d^k \mid i, j, k \geq 0 \}$ is a K-basis for A. Since $\beta = 0$, (1.1) yields $dx = (\alpha x + \gamma)d$ and hence $dx^j = (\alpha x + \gamma)^j d$ for all $j \geq 0$. Furthermore, by (1.2), we have $(\alpha ud + \gamma - x)u = 0$.

For each $n \geq 0$, set

$$I_n = \sum_{i=0}^{n} u^i(\alpha ud + \gamma - x)A.$$

Then, since $(\alpha ud + \gamma - x)u = 0$, we have

$$I_n = \sum_{i=0}^{n} \sum_{j,k=0}^{\infty} K u^i(\alpha ud + \gamma - x)x^j d^k.$$

In particular, since $udx^j d^k = u(\alpha x + \gamma)^j d^{k+1}$, it follows that no element of I_n can contain the monomial $u^{n+1}x$ in its support. Thus $u^{n+1}(\alpha ud + \gamma - x) \notin I_n$ and hence I_{n+1} is properly larger than I_n. In other words, we have shown that $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$ is a properly increasing sequence of right ideals of A, and therefore A is not right Noetherian. Since $A \cong A^{\mathrm{op}}$, the result follows. □

4.4. Proof of the main theorem. Suppose first that $\beta \neq 0$. Then, by Corollary 2.1, A is a domain and $K[ud, du]$ is a polynomial ring in the two generators. Furthermore, Corollary 2.2 implies that A is right and left Noetherian, Theorem 4.1 yields the appropriate information on the global dimension of A, and Lemma 4.2 asserts that A is Auslander-regular.

Conversely, suppose $\beta = 0$. Then A is not a domain since, as observed in [B], $d(du - \alpha ud - \gamma) = 0$. Futhermore, multiplying this relation on the left by u shows that du and ud are algebraically dependent. Finally, A is not left or right Noetherian by Lemma 4.3.

We remark in closing that the recent manuscript [Z] completely determines the center of $A(\alpha, \beta, \gamma)$, while [K] proves the equivalence of (1) and (3) by showing that A is a hyperbolic ring if $\beta \neq 0$.

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