SPECTRUM PRESERVING LINEAR MAPPINGS FOR SCATTERED JORDAN-BANACH ALGEBRAS

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Abstract. Given two semisimple complex Jordan-Banach algebras with identity $A$ and $B$, we say that $T$ is a spectrum preserving linear mapping from $A$ to $B$ if $T$ is surjective and we have $\text{Sp}(Tx) = \text{Sp}(x)$, for all $x \in A$. We prove that if $B$ is a scattered Jordan-Banach algebra, then $T$ is a Jordan isomorphism.

The aim of this note is to provide a detailed proof of a question raised in [2], saying that surjective linear and spectrum preserving mappings on separable scattered Jordan-Banach algebras are Jordan isomorphisms. This is in fact an extension of Theorem 3.7 of [4] from the associative case to the more general situation of Jordan-Banach algebras. In the proof of our result we will use the extension of Harte’s theorem obtained by the author in [6, 7] and the structure theorem for scattered Jordan-Banach algebras of Aupetit-Baribeau [3]. In [2] Aupetit generalized some results previously obtained in [4], from the associative setting to the Jordan case.

The next theorem contains Theorem 2.1 and Corollary 2.4 of [2].

Theorem 1.1. Let $T$ be a spectrum preserving linear mapping from $A$ onto $B$. Then $Tx^2 - (Tx)^2 \in \text{Ann}(\text{Soc} B)$, for every $x \in A$. Moreover $T$ is a Jordan isomorphism from $\text{kh}(\text{Soc} A)$ onto $\text{kh}(\text{Soc} B)$.

We recall that a complex Jordan algebra $A$ is non-associative and the product satisfies the identities $ab = ba$ and $(ab)a^2 = a(ba^2)$, for all $a, b \in A$. A unital Jordan-Banach algebra is a Jordan algebra with a complete norm satisfying $\|xy\| \leq \|x\| \|y\|$, for $x, y \in A$, and $\|1\| = 1$. An element $a \in A$ is said to be invertible if there exists $b \in A$ such that $ab = 1$ and $a^2b = a$. For an element $x$ of a Jordan-Banach algebra $A$, the spectrum of $x$ is by definition the set of $\lambda \in \mathbb{C}$ for which $\lambda - x$ is not invertible. It is a non-empty subset of $\mathbb{C}$. Moreover $x \mapsto \text{Sp}(x)$ is upper semicontinuous on $A$.

As in the associative case, the set $\Omega(A)$ of invertible elements is open, but unfortunately it is not anymore a multiplicative group (see [6, 7]). We also denote by $\Omega_1(A)$ the connected component of $\Omega(A)$, which contains the identity. At this stage it is appropriate to notice that if $A$ is a Jordan-Banach algebra and we define $\exp(A) = \{e^x : x \in A\}$, then $\exp(A) \subset \Omega_1(A)$. It is clear that $e^{-x}$ is the inverse of $e^x$ for $x \in A$. It is also possible to give a concise notion of exponential spectrum of an element $x$, that we denote $\varepsilon(x)$, as the compact set defined by $\lambda \notin \varepsilon(x)$ if and

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only if \( \lambda - x \in \Omega_1(A) \). As in the associative case we have \( \text{Sp}(x) \subset \varepsilon(x) \subset \sigma(x) \), where \( \sigma(x) \) denotes the full spectrum of \( x \). To see the first inclusion, suppose \( \lambda \notin \varepsilon(x) \); then \( \lambda - x \in \Omega_1(A) \), which means \( \lambda - x \) is invertible, so \( \lambda \notin \text{Sp}(x) \). For the second one, if \( \lambda \notin \sigma(x) \), then by the Holomorphic Functional Calculus \( \lambda - x = e^y \) for \( y \) in the closed associative subalgebra \( C(1, x) \) generated by 1 and \( x \). Then \( x(t) = e^{ty} \) defines a continuous path of invertible elements joining \( \lambda - x \) and 1. Hence \( \lambda - x \in \Omega_1(A) \) and \( \lambda \notin \varepsilon(x) \).

In [6] we proved the following result for which another application is given in this note.

**Theorem 1.2** (Extension of Harte’s Theorem [6, 7]). Let \( T \) be a continuous morphism from a Jordan-Banach algebra \( A \) onto a Jordan-Banach algebra \( B \). Then \( T(\Omega_1(A)) = \Omega_1(B) \). Moreover,

\[
\varepsilon(Tx) = \bigcap_{y \in \ker T} \varepsilon(x + y)
\]

and

\[
\text{Sp}(Tx) \subset \bigcap_{y \in \ker T} \text{Sp}(x + y) \subset \sigma(Tx).
\]

We recall that for a semisimple Jordan-Banach algebra \( A \), the socle \( \text{Soc} A \) is the sum of all the quadratic minimal ideals of \( A \). It is known that \( \text{Soc} A \) is an ideal which is the sum of simple ideals generated by minimal projections. The theory of the socle has been extensively studied in recent years; for more details and references see [3, 7]. The notion of annihilator of a set of a Jordan algebra has been introduced by E. Zelmanov [8]; in particular the annihilator of an ideal is also an ideal. It is shown in [5] that \( \text{Ann}(\text{Soc} A) = \{ a \in A : a \cdot \text{Soc} A = \{0\} \} \).

In the proof of the following important lemma, we use implicitly that a projection which is in \( \text{kh}(\text{Soc} B) \) is actually in \( \text{Soc} B \) and if \( u \) is in \( \text{Soc} B \), then its non-zero spectrum consists of isolated points. These two assertions follow from Corollaries 2.5, 2.6 and 2.7 of [6].

**Lemma 1.3.** The ideal \( I = \text{kh}(\text{Soc} B) \cap \text{Ann}(\text{Soc} B) = \{0\} \).

**Proof.** First we prove that \( u \in I \) implies \( \rho(u) = 0 \) where \( \rho \) denotes the spectral radius. Suppose \( \rho(u) \neq 0 \); then there exists \( \alpha \neq 0, \alpha \in \text{Sp} u \). Now, the non-zero Riesz projection \( p \) associated to \( \alpha \) and \( u \) is in the socle of \( B \) and we have \( p = \frac{u}{\varepsilon_1} \int_{\Gamma} \frac{1}{(\lambda - u)^{-1}} \, d\lambda \), where \( \Gamma \) is a small circle centred at \( \alpha \). Since \( u \in \text{Ann}(\text{Soc} B) \), which is an ideal, we deduce that \( p \in \text{Ann}(\text{Soc} B) \), so \( p \in \text{Soc}(B) \cap \text{Ann}(\text{Soc} B) = \{0\} \), and this is absurd. Consequently, the ideal \( I \) contains only quasi-nilpotent elements, so \( I \subset \text{Rad} B = \{0\} \). \( \Box \)

We now prove the main result of this note for scattered Jordan-Banach algebras, that is, Jordan-Banach algebras for which the spectrum of every element is finite or countable. By Aupetit-Baribeau’s theorem their socle is non-empty and they have a very particular algebraic structure [3].

**Theorem 1.4** (Aupetit-Baribeau). Let \( J \) be a complex Jordan-Banach algebra with identity such that the spectrum of every element is at most countable. Moreover, suppose that \( J \) is separable. Then there exists an ordinal \( \alpha_0 \) of the first or second class and a sequence \((I_\alpha)_{\alpha \leq \alpha_0}\) of closed ideals of \( J \) such that \( I_0 = \text{Rad} J \), \( I_{\alpha_0} = J \) and \( I_{\alpha+1}/I_\alpha \) is a modular annihilator for \( \alpha \leq \alpha_0 \).
We are ready now to state and prove the main result of this note, which is an adaptation to the Jordan case of the proof of Theorem 3.7 of [4].

**Theorem 1.5.** Let $T$ be a spectrum preserving linear mapping from a Jordan-Banach algebra $A$ onto a separable scattered Jordan-Banach algebra $B$. Then $T$ is a Jordan isomorphism (i.e., $Tx^2 = (Tx)^2$ for all $x \in A$).

**Proof.** Because $Sp(Tx) = Sp(x)$ for every $x \in A$ and $B$ is a scattered Jordan-Banach algebra, it is clear that $A$ is also a scattered Jordan-Banach algebra. Now let $I_1 = kh(Soc A)$ and $J_1 = kh(Soc B)$. By Theorem 1.1 we have $J_1 = T(I_1)$. Define semisimple Jordan-Banach algebras $A_1$ and $B_1$ by $A_1 = A/I_1, B_1 = B/J_1$. Let $\pi_1 : A \to A_1$ and $\gamma_1 : B \to B_1$ be the corresponding canonical maps. Define a linear map $T_1 : A_1 \to B_1$ by $T_1(\pi) = \overline{T\pi}$. By Theorem 1.2 and since the spectrum of every element of $A$ and $B$ is at most countable, we have $Sp(x) = \sigma(x)$ and $Sp(Tx) = \sigma(Tx)$. Then

$$Sp(\overline{\pi}) = \bigcap_{x \in I_1} Sp(a + x) = \bigcap_{y \in J_1} Sp(Ta + y) = Sp(T_1 \pi),$$

and hence $T_1$ is spectrum preserving. Furthermore if $a \in I_1$, then $Ta^2 - (Ta)^2 \in J_1 = kh(Soc B)$ and also $Ta^2 - (Ta)^2 \in Ann(Soc B)$. Since $Soc B \subset kh(Soc B)$ implies $Ann(kh(Soc B)) \subset Ann(Soc B)$ we conclude that $Ta^2 - (Ta)^2 \in Ann(Soc B)$ which is zero by Lemma 1.3. Then $Ta^2 = (Ta)^2$ for every $a \in I_1$. Continuing inductively, we define

$$A_n = A_{n-1}/kh(Soc A_{n-1}), \quad B_n = B_{n-1}/kh(Soc B_{n-1})$$

and we denote respectively by $\pi_n, \gamma_n$ the canonical maps from $A_{n-1}$ onto $A_n$ and from $B_{n-1}$ onto $B_n$. Let $I_n = ker(\pi_1 \circ \cdots \circ \pi_n), J_n = ker(\gamma_1 \circ \cdots \circ \gamma_1)$ and note that $J_n = T(I_n)$. Define a linear map $T_n : A_n \to B_n$ by $T_n(\pi) = \overline{T_{n-1}(\pi)}$. Then $T_n$ is spectrum preserving and by using Lemma 1.3 it is easy to see that for every $a \in I_n$, we have $Ta^2 = (Ta)^2$. If $\omega$ is the first ordinal number, we define

$$I_\omega = kh \left( \bigcup_{n \geq 1} I_n \right), \quad J_\omega = kh \left( \bigcup_{n \geq 1} J_n \right)$$

and also notice that $T((\bigcup_{n \geq 1} I_n)) = (\bigcup_{n \geq 1} J_n)$. Define the linear mapping

$$T'_\omega : A/\left( \bigcup_{n \geq 1} I_n \right) \to B/\left( \bigcup_{n \geq 1} J_n \right) \text{ by } T'_\omega \pi = \overline{T_\omega \pi}.$$

By using Theorem 1.2 it follows that $T'_\omega$ is spectrum preserving and we have $J_\omega = T(I_\omega)$. Let $A_\omega = A/I_\omega, B_\omega = B/J_\omega$ and define a linear operator $T_\omega : A_\omega \to B_\omega$ by $T_\omega(\pi) = \overline{T_\omega \pi}$. Notice that $T_\omega$ is spectrum preserving and that $A_\omega$ and $B_\omega$ are semisimple. We claim that $Ta^2 = (Ta)^2$ for every $a \in I_\omega$. Let $a \in I_\omega$ and suppose that $u = Ta^2 - (Ta)^2 \neq 0$. By Lemma 1.3 we conclude that $\gamma_n \circ \cdots \circ \gamma_1(u) \neq 0$ for $n = 1, 2, \ldots,$ and by Theorem 1.1 $(\gamma_n \circ \cdots \circ \gamma_1(u))y = 0$ for every $y \in Soc B_\omega$. Since $u \neq 0$ and $B$ is semisimple, there exists $b \in B$ such that $Sp(ub) \neq \{0\}$ and since $ub \in J_\omega$, once more by Theorem 1.2 we have

$$(**) \quad \bigcap_{y \in \bigcup_{n \geq 1} J_n} Sp(ub + y) = 0.$$
We now prove that there exists an integer \( n \) such that \( \text{Sp}(\gamma_n \circ \cdots \circ \gamma_0(ub)) \neq \text{Sp}(\gamma_{n+1} \circ \cdots \circ \gamma_0(ub)) \), where \( B_0 = B \) and \( \gamma_0 \) is the identity map on \( B \). Suppose the contrary: then \( \text{Sp}(ub) = \text{Sp}(\gamma_n \circ \cdots \circ \gamma_0(ub)) \subset \text{Sp}(ub + y) \) for every \( n \geq 1 \) and \( y \in J_n \), consequently, by continuity of the spectrum, \( \text{Sp}(ub) \subset \text{Sp}(ub + y) \) for every \( y \in \bigcup_{n \geq 1} J_n \). Now by (**) we have \( \text{Sp}(ub) = \{0\} \), which is a contradiction. Hence suppose \( \text{Sp}(B_n(\gamma_n \circ \cdots \circ \gamma_0(ub)) \neq \text{Sp}(B_{n+1}(\gamma_{n+1} \circ \cdots \circ \gamma_0(ub))) \) for some \( n \geq 0 \) and let \( x = (\gamma_n \circ \cdots \circ \gamma_0)(ub) \). Then there exists an isolated point \( \lambda \neq 0 \) of \( \text{Sp}(B_{n+1}(x)) \) such that \( \lambda \notin \text{Sp}(B_{n+1}(\gamma_{n+1}(x))) \). If we denote by \( p \) the spectral projection associated to \( p \) and \( \lambda \) we have \( p \in \text{Soc} B_0 \) and \( 0 \neq xp = (\gamma_n \circ \cdots \circ \gamma_0(u))(\gamma_n \circ \cdots \circ \gamma_0(b))p \), which contradicts the fact that \( (\gamma_n \circ \cdots \circ \gamma_0(u))y = 0 \) for every \( y \in \text{Soc} B_n \). So finally we have proved that for every \( a \in I_0 : Ta^2 - (Ta)^2 = 0 \). Now, continuing by transfinite induction, there exists an ordinal \( \beta \) in the first class of ordinals such that \( A = I_\beta \). By the previous arguments it is easy to see that \( B = I_\beta \) and \( Ta^2 = (Ta)^2 \) for every \( a \in A \).

\[ \square \]

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References