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ITERATION OF A CLASS OF HYPERBOLIC MEROMORPHIC FUNCTIONS

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Dedicated to Professor Noel Baker on the occasion of his retirement

ABSTRACT. We look at the class B_n which contains those transcendental meromorphic functions f for which the finite singularities of f^{-n} are in a bounded set and prove that, if f belongs to B_n, then there are no components of the set of normality in which $f^{mn}(z) \to \infty$ as $m \to \infty$. We then consider the class $\widehat{B}$ which contains those functions f in B_1 for which the forward orbits of the singularities of f^{-1} stay away from the Julia set and show (a) that there is a bounded set containing the finite singularities of all the functions f^{-n} and (b) that, for points in the Julia set of f, the derivatives $(f^n)'$ have exponential-type growth. This justifies the assertion that $\widehat{B}$ is a class of *hyperbolic* functions.

1. INTRODUCTION

Let f be a meromorphic function which is not rational of degree less than two, and denote by f^n, $n \in \mathbf{N}$, the n-th iterate of f. The set of normality, $N(f)$, is defined to be the set of points, $z \in \mathbf{C}$, such that $(f^n)_{n\in\mathbf{N}}$ is well-defined, meromorphic and forms a normal family in some neighbourhood of z. The complement of $N(f)$ is called the Julia set, $J(f)$, of f. An introduction to the properties of these sets can be found in, for example, [3].

We will use the following notation concerning singularities:

$$S(f) = \{z \in \mathbf{C} : z \text{ is a singularity of } f^{-1}\},$$
$$P(f) = \{z \in \mathbf{C} : z \text{ is a singularity of } f^{-n}, \text{ for some } n \in \mathbf{N}\}.$$

It was shown by Herring [7, Theorem 7.1.2] that

$$\{z \in \mathbf{C} : z \text{ is a singularity of } f^{-n}\} \subseteq S_n(f) = \bigcup_{j=0}^{n-1} f^j(S(f)\backslash A_j(f)),$$

where

$$A_j(f) = \{z \in \mathbf{C} : f^j \text{ is not analytic at } z\},$$

and that

$$P(f) = \bigcup_{n=0}^{\infty} S_n(f).$$

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Eremenko and Lyubich [6] investigated the properties of entire functions in the class

$$B = \{f : f \text{ is a transcendental meromorphic function with } S(f) \text{ bounded}\}.$$

In Section 2 we look at the properties of functions in the class

$$B_n = \{f : f \text{ is a transcendental meromorphic function with } S_n(f) \text{ bounded}\}.$$

(Note that B_1 is equal to B.) We prove the following result.

Theorem A. *If $f \in B_n$, then there is no component of $N(f)$ in which $f^{mn}(z) \to \infty$ as $m \to \infty$.*

Remarks. Our proof is based on ideas of Eremenko and Lyubich [6, Theorem 1] who proved this result in the case when f is entire and $n = 1$. The proof of Theorem A given by Bergweiler [3, Theorem 16] uses [3, Lemma 8] which asserts that, if $f \in B$, $p \geq 1$ and 0 is not a pre-image of ∞, then there exist a positive constant R and a curve Γ connecting 0 to ∞ such that $|f^p(z)| \leq R$ for $z \in \Gamma$. Unfortunately, this lemma is not correct, as shown by the counterexample $f(z) = \frac{\tan z}{z} + \frac{\pi}{2}$. Although $f \in B$, f^2 is unbounded on each path to ∞. The rest of the proof of [3, Theorem 16] is correct and the reference to [3, Lemma 8] can be successfully replaced by a reference to Lemma 2.1 of this paper.

It follows from Theorem A that, if $f \in B_n$, then there can be no periodic cycle $\{N_0, \ldots, N_{n-1}\}$ of components of $N(f)$ with $f^{mn}(z) \to \infty$ as $m \to \infty$ in one of the components—such a cycle is known as a cycle of Baker domains or essentially parabolic domains. Thus we have the following Corollary to Theorem A.

Corollary. *If $f \in B_n$, then f has no Baker domains of period n.*

Many authors have considered functions in the class

$$S = \{f : f \text{ is a transcendental meromorphic function with } S(f) \text{ finite}\}.$$

It is easy to see that, if $f \in S$, then $f \in \bigcap_{n=1}^{\infty} B_n$ and so a special case of the above Corollary is that functions in S have no Baker domains.

In Sections 3 and 4 we consider the iteration of functions in the class

$$\widehat{B} = \{f : f \in B \quad \text{and} \quad \bar{P}(f) \cap J(f) = \varnothing\},$$

where $\bar{P}$ denotes closure with respect to the plane. In Section 3 we use Theorem A to prove the following result.

Theorem B. *If $f \in \widehat{B}$, then $P(f)$ is bounded.*

In Section 4, we use Theorem B together with the results of Section 2 to prove the following result for meromorphic functions which has applications to estimating the Hausdorff dimension of $J(f)$ when $f \in \widehat{B}$; see [8].

Theorem C. *If $f \in \widehat{B}$, then there exist $K > 1$ and $c > 0$ such that*

$$|(f^n)'(z)| > cK^n \frac{|f^n(z)| + 1}{|z| + 1},$$

for each $z \in J(f) \backslash A_n(f), n \in \mathbf{N}$.

If f is rational, then the following conditions are equivalent—see [2, Section 9.7] and [4, Section 5.2] and note that, for rational functions, $\bar{P}$ denotes closure in the sphere:

- $\bar{P}(f) \cap J(f) = \varnothing$;

- $\bar{P}(f)$ is a compact subset of $N(f)$;
- f is expanding, in the sense that there exist $K > 1$ and $c > 0$ such that $|(f^n)'(z)| > cK^n$ for each $z \in J(f), n \in \mathbf{N}$.

A rational function with these properties is said to be *hyperbolic*. For transcendental meromorphic functions, these conditions are no longer equivalent and so it is not clear what the definition of a hyperbolic transcendental meromorphic function should be. In view of Theorems B and C, however, it does seem natural to say that the functions in $\widehat{B}$ are hyperbolic.

2. Properties of functions in the class B_n

We use the following notation:

$$B(z,r) = \{w : |w - z| < r\},$$
$$D_R = \{z \in \mathbf{C} : |z| > R\} \cup \{\infty\}.$$

The following lemma is probably 'well known'; we include a proof for the sake of completeness.

Lemma 2.1. *If $f \in B_n$ and $S_n(f) \subseteq B(0,R)$, then each component of $f^{-n}(D_R)$ is simply connected in $\mathbf{C}$.*

Proof. Let V be a component of $f^{-n}(D_R)$, let g denote a branch of f^{-n} which maps a point of D_R into V and let h denote all analytic continuations of $g(e^t)$ to $H = \{t : \operatorname{Re} t > \log R\}$. Then, by the monodromy theorem, h is analytic in H and maps H onto V. There are now two cases to consider.

Case A. The function h is univalent in H and hence $h(H) = V$ is simply connected.

Case B. The function h is $2m\pi i$-periodic in H, for some minimal positive integer m.

Indeed, if h is not univalent in H, then there is some minimal positive integer m for which $h(t_m) = h(t_m + 2m\pi i)$ for some $t_m \in H$ and, if t is close to t_m, then it follows from the open mapping theorem that there exists t' close to $t_m + 2m\pi i$ with $h(t) = h(t')$ and hence $t' = t + 2m\pi i$. Thus h has period $2m\pi i$.

In Case B,

$$h(t) = \varphi(e^{t/m}), \quad \text{for } t \in H,$$

where $\varphi(s) = a_1 s + a_0 + a_{-1}s^{-1} + \cdots$ is univalent in $\{s : |s| > R^{1/m}\}$, and so

$$f^n(z) = (\varphi^{-1}(z))^m, \quad \text{for } z \in \varphi(\{s : |s| > R^{1/m}\}).$$

Now, if $a_1 \neq 0$, then $\varphi(\{s : |s| > R^{1/m}\})$ includes a neighbourhood of ∞, so

$$f^n(z) \approx a_1^{-m} z^m \quad \text{as } z \to \infty. \tag{2.1}$$

But (2.1) is impossible because ∞ is an essential singularity of f^n and not a pole. Thus $a_1 = 0$ and φ maps $\{s : |s| > R^{1/m}\} \cup \{\infty\}$ onto a simply connected region in $\mathbf{C}$ containing a_0, and this region is V.

We now use Lemma 2.1 to prove the following result.

Lemma 2.2. *Let f be a transcendental meromorphic function. There exists R_f such that, if $R > R_f$, $S_n(f) \subseteq B(0,R)$ and $|z|, |f^n(z)| > R^2$, then*

$$|(f^n)'(z)| > \frac{|f^n(z)| \log |f^n(z)|}{16\pi |z|}.$$

Proof. First let c be a periodic point of f (see, for example, [3, Theorem 2]) and then take R_f so large that $|f^n(c)| < R_f$ for each $n \in \mathbf{N}$.

Now suppose that $R > R_f$, $S_n(f) \subseteq B(0,R)$ and $|z|, |f^n(z)| > R^2$. Let V be the component of $f^{-n}(D_R)$ which contains z and let g denote the branch of f^{-n} that maps $f^n(z)$ to z. Since $c \notin V$, it follows from Lemma 2.1 that we can choose a branch L of log so that $L(z-c)$ is analytic on V.

If $H = \{t : \operatorname{Re} t > \log R\}$, then

$$\Phi(t) = L(g(e^t) - c)$$

can be analytically continued to H, and $\Phi(H)$ does not include any disc of radius greater than π. Thus, by Bloch's Theorem,

$$|\Phi'(t)| \le \frac{\pi}{B(\operatorname{Re} t - \log R)}, \quad \text{for } t \in H,$$

where B denotes Bloch's constant. Hence

$$\left| \frac{g'(e^t)e^t}{g(e^t) - c} \right| \le \frac{\pi}{B(\operatorname{Re} t - \log R)},$$

where $e^t = f^n(z)$, and so

$$\left| \frac{f^n(z)}{(z-c)(f^n)'(z)} \right| = \left| \frac{g'(f^n(z))f^n(z)}{z-c} \right| \le \frac{\pi}{B(\log|f^n(z)| - \log R)}. \tag{2.2}$$

The lemma follows by using $|z-c| \le |z| + |c| < 2|z|$, $\log|f^n(z)| > 2\log R$ and $B > \frac{1}{4}$.

Recall that Theorem A states that, if $f \in B_n$, then there is no component of $N(f)$ in which $f^{mn}(z) \to \infty$ as $m \to \infty$. We are now in a position to give a proof of this result.

Proof of Theorem A. If $f \in B_n$, then there exists $R > \max(e^{16\pi}, R_f)$ with $S_n(f) \subseteq B(0,R)$. If $N(f)$ has a component U in which $f^{mn}(z) \to \infty$ as $m \to \infty$, then there exist $p \in \mathbf{N}$, $w \in N(f)$ and $r > 0$ such that $\bar{B}(w,r) \subset f^{pn}(U)$ and $|f^{mn}(z)| > R^2$, for each $z \in B(w,r)$, $m = 0, 1, 2, \ldots$.

Now let V_m be the component of $f^{-n}(D_R)$ in which $U_m = f^{nm}(B(w,r))$ lies. Then, taking c to be the same periodic point as in the proof of Lemma 2.2, it follows from Lemma 2.1 that there exists a branch L_m of log for which $L_m(z-c)$ is analytic in V_m. Next, put $T_m = L_m(U_m - c)$ and $F_m(t) = L_m(f^n(e^t + c) - c)$, so that $T_{m+1} = F_{m+1}(T_m)$. It follows from (2.2) that, if $t \in T_m$, then

$$\begin{aligned}
|F_m'(t)| &= \left| \frac{(f^n)'(e^t + c)e^t}{f^n(e^t + c) - c} \right| \\
&= \left| \frac{(f^n)'(z)(z-c)}{f^n(z) - c} \right|, \quad \text{where } z = e^t + c \in U_m, \\
&\ge \left| \frac{(f^n)'(z)(z-c)}{2f^n(z)} \right| \\
&\ge \frac{B}{2\pi}(\log|f^n(z)| - \log R) \\
&\ge \frac{B\log R}{2\pi} \ge 2,
\end{aligned}$$

and so

$$|(F_m \circ \cdots \circ F_1)'(t)| \ge 2^m, \quad \text{for } t \in T_1.$$

Thus, by Bloch's Theorem, T_m contains a disc of radius r_m, where $r_m \to \infty$ as $m \to \infty$. This, however, is impossible since $T_m \subseteq L_m(V_m - c)$ which contains no disc of radius greater than π.

3. Proof of Theorem B

Recall that Theorem B states that, if $f \in \widehat{B}$, then $P(f)$ is bounded. Let $f \in \widehat{B}$. Since $\bar{S}(f) \subseteq \bar{P}(f)$ and $\bar{P}(f) \cap J(f) = \varnothing$, it follows that $\bar{S}(f) \subseteq N(f)$ and so, since $S(f)$ is bounded, we deduce that $f \in \bigcap_{n=0}^{\infty} B_n$. The fact that $S(f) \subseteq N(f)$ also implies that

$$P(f) = \bigcup_{j=0}^{\infty} f^j(S(f)). \tag{3.1}$$

Since $\bar{S}(f)$ is bounded and contained in $N(f)$, there exist $r > 0$ and a finite number of points $w_1, \ldots, w_M \in S(f)$ such that

$$S(f) \subseteq \bigcup_{i=1}^{M} \bar{B}(w_i, r) \subseteq N(f). \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$P(f) \subseteq \bigcup_{i=1}^{M} \bigcup_{j=0}^{\infty} f^j(\bar{B}(w_i, r)). \tag{3.3}$$

Therefore, for $1 \le i \le M$, we let U_i denote the component of $N(f)$ which contains w_i and consider the possible forward orbits of U_i.

We first show that U_i cannot be a wandering domain. If U_i is a wandering domain, that is, $f^n(U_i) \cap f^m(U_i) = \varnothing$ when $n \neq m$, then there cannot exist a non-constant limit function of $\{f^n|_U\}$; see, for example, [1, Lemma 2.1]. Since $f \in B_1$, it follows from Theorem A that there exist a sequence $\{n_k\}$ and a finite value $a \in \mathbf{C}$ such that $f^{n_k}(z) \to a$ in U_i as $n_k \to \infty$. Since $w_i \in S(f) \cap U_i$, it follows that $a \in \bar{P}(f)$ and, since $f \in \widehat{B}$, this implies that $a \in N(f)$. This, however, is impossible if U_i is a wandering domain.

Thus, U_i eventually lands in a periodic cycle $\{N_0, \ldots, N_{n-1}\}$ of components of $N(f)$. Since $\bar{P}(f) \cap J(f) = \varnothing$, there are no Siegel discs or Hermann rings and so, for $0 \le p \le n-1$, there exists $z_p \in \bar{N}_p$ with $f^{mn}(z) \to z_p$ locally uniformly in N_p. Since $f \in \bigcap_{n=0}^{\infty} B_n$, it follows from Theorem A that $z_p \neq \infty$, for $0 \le p \le n-1$, and so $\bigcup_{j=0}^{\infty} f^j(\bar{B}(w_i, r))$ is bounded. The result now follows from (3.3).

4. Proof of Theorem C

The proof of Theorem C uses results from earlier sections and the following two well known results. The first is Koebe's one-quarter theorem; see for example, [5].

Lemma 4.1. *If f is univalent in $B(z, r)$, then*

$$f(B(z, r)) \supset B(f(z), |f'(z)|r/4).$$

The other result we need is a basic property of Julia sets. Let

$$O^-(w) = \{z : f^n(z) = w \text{ for some } n \in \mathbf{N}\},$$

$$E(f) = \{w : O^-(w) \text{ is finite}\}.$$

If f is meromorphic, then $E(f)$ contains at most two points and we have the following result; see, for example, [3, Section 2].

Lemma 4.2. *If U is compact, $U \cap E(f) = \varnothing$, $z \in J(f)$ and V is an open neighbourhood of z, then there exists $N \in \mathbf{N}$ such that, for all $n \geq N$, we have*

$$f^n(V) \supset U.$$

Theorem C states that, if $f \in \widehat{B}$, then there exist $K > 1$ and $c > 0$ such that

$$|(f^n)'(z)| > cK^n \frac{|f^n(z)|+1}{|z|+1},$$

for each $z \in J(f) \backslash A_n(f), n \in \mathbf{N}$.

Let $f \in \widehat{B}$. We know that $\bar{P}(f) \cap J(f) = \varnothing$ and, from Theorem B, that $\bar{P}(f)$ is bounded. Thus there exist $C > 1$ and an open set G containing $\bar{P}(f)$, such that

$$B\left(z, \frac{|z|+1}{C}\right) \cap G = \varnothing, \tag{4.1}$$

for each $z \in J(f)$.

Since $\bar{P}(f)$ is bounded, it follows from Lemma 2.2 that there exists $R > 0$ such that

$$|(f^n)'(z)| > 16C \frac{|f^n(z)|+1}{|z|+1}, \quad \text{for } n \in \mathbf{N},\ |z| > R,\ |f^n(z)| > R. \tag{4.2}$$

We now claim that there exists $N_1 \in \mathbf{N}$ such that

$$|(f^n)'(z)| > 16C \frac{|f^n(z)|+1}{|z|+1}, \quad \text{for } n \geq N_1,\ z \in (J(f) \backslash A_n(f)) \cap \bar{B}(0, R). \tag{4.3}$$

Otherwise, there exists a sequence of points $z_{n_k} \in (J(f) \backslash A_{n_k}(f)) \cap \bar{B}(0, R)$ such that

$$|(f^{n_k})'(z_{n_k})| \leq 16C \frac{|f^{n_k}(z_{n_k})|+1}{|z_{n_k}|+1},$$

with $z_{n_k} \to \alpha \in J(f) \cap \bar{B}(0, R)$ as $n_k \to \infty$.

It follows from (4.1) and Lemma 4.1 that, if g is the branch of f^{-n_k} that maps $f^{n_k}(z_{n_k})$ to z_{n_k}, then

$$g\left(B\left(f^{n_k}(z_{n_k}), \frac{|f^{n_k}(z_{n_k})|+1}{C}\right)\right) \supseteq B\left(z_{n_k}, \frac{|z_{n_k}|+1}{64C^2}\right).$$

Thus, for large n_k,

$$\begin{aligned} f^{n_k}\left(B\left(\alpha, \frac{1}{100C^2}\right)\right) &\subseteq f^{n_k}\left(B\left(z_{n_k}, \frac{|z_{n_k}|+1}{64C^2}\right)\right) \\ &\subseteq B\left(f^{n_k}(z_{n_k}), \frac{|f^{n_k}(z_{n_k})|+1}{C}\right) \end{aligned}$$

and so, by (4.1),

$$f^n\left(B\left(\alpha, \frac{1}{100C^2}\right)\right) \cap G = \varnothing,$$

for arbitrarily large values of n. Since $\alpha \in J(f)$, this contradicts Lemma 4.2, and hence (4.3) is true.

Our next claim is that there exists $N_2 \in \mathbf{N}$ such that, for each $n \geq N_2$, $z \in J(f) \backslash A_n(f)$, we have

$$|(f^n)'(z)| > \frac{1}{8C} \frac{|f^n(z)| + 1}{|z| + 1}. \tag{4.4}$$

Otherwise, there exists a sequence of points $z_{n_k} \in J(f) \backslash A_{n_k}(f)$ such that

$$|(f^{n_k})'(z_{n_k})| \leq \frac{1}{8C} \frac{|f^{n_k}(z_{n_k})| + 1}{|z_{n_k}| + 1},$$

with $z_{n_k} \to \alpha \in J(f)$ or $z_{n_k} \to \infty$ as $n_k \to \infty$.

It follows from (4.1) and Lemma 4.1 that, if g is the branch of f^{-n_k} that maps $f^{n_k}(z_{n_k})$ to z_{n_k}, then

$$g\left(B\left(f^{n_k}(z_{n_k}), \frac{|f^{n_k}(z_{n_k})| + 1}{C}\right)\right) \supseteq B\left(z_{n_k}, 2\left(|z_{n_k}| + 1\right)\right) \supseteq B(0, |z_{n_k}| + 1).$$

Since $z_{n_k} \to \alpha \in J(f)$ or $z_{n_k} \to \infty$ as $n_k \to \infty$, there exist $\beta \in J(f)$, $r > 0$ such that, for large values of n_k,

$$B(\beta, r) \subseteq B(0, |z_{n_k}| + 1),$$

and hence

$$f^{n_k}(B(\beta, r)) \subseteq f^{n_k}(B(0, |z_{n_k}| + 1)) \subseteq B\left(f^{n_k}(z_{n_k}), \frac{|f^{n_k}(z_{n_k})| + 1}{C}\right).$$

Thus, by (4.1),

$$f^n(B(\beta, r)) \cap G = \varnothing,$$

for arbitrarily large values of n. Since $\beta \in J(f)$, this contradicts Lemma 4.2, and hence (4.4) is true.

We now put $N = \max(N_1, N_2)$. If $z \in J(f) \backslash A_{2N+p}(f)$, then it follows from (4.2) and (4.3) that

$$|(f^{2N+p})'(z)| > 16C \frac{|f^{2N+p}(z)| + 1}{|z| + 1}, \tag{4.5}$$

for each $p \in \mathbf{N} \cup \{0\}$, provided that either $|z| \leq R$ or $|z|, |f^{2N+p}(z)| > R$.

If $z \in J(f) \backslash A_{2N+p}(f)$, $|z| > R$ and $|f^{2N+p}(z)| \leq R$, then either $|f^N(z)| \leq R$ in which case, by (4.3) and (4.4),

$$|(f^{2N+p})'(z)| > \frac{1}{8C} \frac{|f^N(z)| + 1}{|z| + 1} 16C \frac{|f^{2N+p}(z)| + 1}{|f^N(z)| + 1} = 2 \frac{|f^{2N+p}(z)| + 1}{|z| + 1}, \tag{4.6}$$

or $|f^N(z)| > R$ in which case, by (4.2) and (4.4),

$$|(f^{2N+p})'(z)| > 16C \frac{|f^N(z)| + 1}{|z| + 1} \frac{1}{8C} \frac{|f^{2N+p}(z)| + 1}{|f^N(z)| + 1} = 2 \frac{|f^{2N+p}(z)| + 1}{|z| + 1}. \tag{4.7}$$

It follows from (4.5), (4.6) and (4.7) that, for each $z \in J(f) \backslash A_{2N+p}(f)$, $p \in \mathbf{N} \cup \{0\}$, we have

$$|(f^{2N+p})'(z)| > 2 \frac{|f^{2N+p}(z)| + 1}{|z| + 1}. \tag{4.8}$$

If $n \geq 2N$, then there exist $m \in \mathbf{N}$, $0 \leq p < 2N$ such that $n = m2N + p$ and so, if $z \in J(f) \backslash A_n(f)$ and $n \geq 4N$, then it follows from (4.8) that

$$|(f^n)'(z)| > 2^m \frac{|f^n(z)| + 1}{|z| + 1} > (2^{\frac{1}{4N}})^n \frac{|f^n(z)| + 1}{|z| + 1}.$$

To complete the proof of Theorem C we need to show that there exist $c_n > 0$, for $n = 1, 2, \ldots, 4N - 1$, such that

$$|(f^n)'(z)| > c_n \frac{|f^n(z)| + 1}{|z| + 1},$$

for $z \in J(f) \backslash A_n(f)$.

If this is not true, then there exist $m \in \mathbf{N}$ and a sequence of points $z_k \in J(f) \backslash A_m(f)$ such that

$$\varepsilon_k = \frac{|(f^m)'(z_k)|(|z_k| + 1)}{|f^m(z_k)| + 1} \to 0 \quad \text{as } k \to \infty.$$

An argument similar to the proof of (4.4) with ε_k instead of $1/(8C)$ and m instead of n_k now leads to the fact that, for large k,

$$f^m \left(B \left(0, \frac{|z_k| + 1}{8C\varepsilon_k} \right) \right) \cap G = \varnothing.$$

Thus $f^m(\mathbf{C}) \cap G = \varnothing$, which is a contradiction, and so the proof of Theorem C is now complete.

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