GENERATING SETS
FOR COMPACT SEMISIMPLE LIE GROUPS

MICHAEL FIELD

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Abstract. Let $\Gamma$ be a compact connected semisimple Lie group. We prove that the subset of $\Gamma^2$ consisting of pairs $(g, h)$ which topologically generate $\Gamma$ is Zariski open.

1. Introduction

Suppose that $\Gamma$ is a compact connected Lie group of dimension $n$. Let $\langle \gamma_1, \ldots, \gamma_N \rangle$ denote the closure (in $\Gamma$) of the group generated by $\gamma_1, \ldots, \gamma_N \in \Gamma$. If $\langle \gamma_1, \ldots, \gamma_N \rangle = \Gamma$, we say $\gamma_1, \ldots, \gamma_N$ are topological generators of $\Gamma$.

If $\Gamma$ is abelian, then $\Gamma$ is isomorphic to a torus $T^n$. It follows from Kronecker's theorem that $\Gamma$ may be topologically generated by one element of $\Gamma$ (see [1, 3]). Indeed, Kronecker's theorem gives necessary and sufficient conditions for an element of $T^n$ to generate $T^n$ topologically. It follows from these conditions that the topological generators of $T^n$ form a dense (full-measure) subset of $\Gamma$ with no interior points.

Since a single element of $\Gamma$ always generates an abelian subgroup of $\Gamma$, it follows that if $\Gamma$ is not abelian, then we need at least two group elements to generate $\Gamma$ topologically. Somewhat surprisingly, basic results for the non-abelian case appear not to be very well known. In fact, Auerbach showed in 1934 [2] that a compact connected (linear) Lie group $\Gamma$ could be generated by two elements and that the set of generating pairs had full measure in $\Gamma^2$. Subsequently, in 1935, Schreier & Ulam [11] showed that for compact connected metrizable topological groups $\Gamma$ the set of pairs generating $\Gamma$ was dense in $\Gamma^2$. We refer to the article by Hoffman and Morris [7] for more recent results on the minimal number of generators needed for locally compact groups.

We recall that $\Gamma$ is semisimple if the center $Z(\Gamma)$ of $\Gamma$ is finite. In 1949, Kuranishi showed that connected semisimple Lie groups could be generated by two elements [8]. Later, in 1951, Kuranishi showed that if $\Gamma$ was a perfect Lie group, then the set of pairs generating $\Gamma$ was open and dense in $\Gamma^2$ [9]. It follows from a result of Goto [6] that every compact connected semisimple Lie group is perfect and so the set of pairs generating a compact connected semisimple Lie group is open and dense.
The proof of Kuranishi’s theorem uses the Hausdorff topology on compact subsets of $\Gamma$. In particular, the proof gives little insight into the structure of the pairs which do not generate $\Gamma$.

In this note we give an elementary proof of the following result.

**Theorem 1.1.** Let $\Gamma$ be a compact connected semisimple Lie group. The set of pairs topologically generating $\Gamma$ is a non-empty Zariski open subset of $\Gamma^2$.

Elsewhere, we use openness results of this type to prove results on the stable ergodicity of Hölder continuous skew-extensions by compact Lie groups [5].

### 2. Preliminaries

In this section we recall some basic definitions and results on compact connected Lie groups and Lie algebras. Details of proofs may be found in standard references (for example [1, 3]). Henceforth, we always assume that the Lie group $\Gamma$ is compact and connected. Since $\Gamma$ admits a faithful representation on some $\mathbb{R}^m$ by matrices of determinant one, it is no loss of generality to assume that $\Gamma \subset \text{SO}(m)$, for some $m \in \mathbb{N}$.

We use the term *Zariski open* for subsets whose complement is a closed algebraic variety. Since every compact Lie group may be given the (unique) structure of a real algebraic variety by embedding in some $\text{SO}(m)$, we may refer, without ambiguity, to Zariski open subsets of a compact Lie group (for uniqueness of algebraic structure, see [10, pg 247]).

If $H$ is a closed subgroup of $\Gamma$, we let $H_0$ denote the identity component of $H$.

Let $\mathfrak{g}$ denote the Lie algebra of $\Gamma$. The group $\Gamma$ is semisimple if and only if $\mathfrak{g}$ is semisimple (as a Lie algebra). The structure of a semisimple group is given in terms of simple groups by the following well-known result.

**Theorem 2.1.** If $\Gamma$ is semisimple, there exist compact simple groups $\Gamma_1, \ldots, \Gamma_p$ and a finite covering homomorphism

$$
\phi: \Gamma \to \Gamma_1 \times \cdots \times \Gamma_p.
$$

Up to order and isomorphism, the $\Gamma_i$ are uniquely determined by $\Gamma$.

**Lemma 2.2.** Let $\phi: \Gamma \to G$ be a covering homomorphism of compact connected Lie groups. Let $g, h \in \Gamma$. Then $\langle g, h \rangle = \Gamma$ if and only if $\langle \phi(g), \phi(h) \rangle = G$.

**Proof.** Since $\Gamma$ is compact, $\phi$ is a finite covering homomorphism. It follows that if $\langle \phi(g), \phi(h) \rangle = G$, then $\phi(\langle g, h \rangle) = G$. Hence, $\dim(\langle g, h \rangle) = \dim(\Gamma)$ and so, since $\Gamma$ connected, $\langle g, h \rangle = \Gamma$. The converse is trivial. \qed

### 3. Infinite subgroups generated by two elements

Regard $\Gamma$ as embedded in $\text{SO}(m)$. Let $(\ , \ )$ be the inner product defined on real $m \times m$ matrices by

$$
(\mathbf{A}, \mathbf{B}) = \text{trace}(\mathbf{A}^t \mathbf{B}^t).
$$

Let $\| \cdot \|$ denote the associated norm. If $\mathbf{A}, \mathbf{B} \in \text{SO}(n)$, let $[\mathbf{A}, \mathbf{B}]$ denote the commutator $\mathbf{A} \mathbf{B} \mathbf{A}^{-1} \mathbf{B}^{-1}$. We recall, without proof, the following well-known result on commutators of orthogonal transformations.
Lemma 3.1 ([4, Lemmas 36.15, 36.16]). Let $A, B \in SO(m)$ and set $C = [A, B]$.

1. $\|I - C\| \leq \sqrt{2}\|I - A\|\|I - B\|$.
2. If $AC = CA$ and $\|I - B\| < 2$, then $AB = BA$.

Let $D$ denote the open disk, center $I$, radius $1/\sqrt{2}$ in $\Gamma$. Set $D^* = D \setminus \{I\}$.

Lemma 3.2. If $g, h \in D^*$ and $gh \neq hg$, then $\langle g, h \rangle$ is infinite.

Proof. Define the sequence $(g_n)$ inductively by $g_0 = g$, $g_n = [g_{n-1}, h]$, $n \geq 1$. Since $g, h \in D^*$, it follows from Lemma 3.1 (1), that $g_n \to I$. If $\langle g, h \rangle$ is finite, it follows that there exists $n \geq 1$ such that $g_n = I$. That is, $g_{n-1}h = hg_{n-1}$. It follows from Lemma 3.1(2) that $h$ must commute with $g_{n-2}$. Proceeding inductively, it follows that $gh = hg$, contradicting our assumption that $g, h$ do not commute. Hence $\langle g, h \rangle$ is infinite.

Remark 3.3. The proof of Lemma 3.2 is based on part of the proof of Jordan’s theorem given in [4, §36].

Lemma 3.4. Let $m \geq 1$. The set $Z_m$ of pairs $(g, h)$ such that $g^p h^q \neq h^q g^p$, $1 \leq p, q \leq m$, is a non-empty Zariski open subset of $\Gamma^2$.

Proof. It suffices to show that for all $p, q \geq 1$, the equation $g^p h^q = h^q g^p$ defines a proper Zariski closed subset of $\Gamma^2$. For this, choose $g, h \in \Gamma$ such that $g, h$ generate distinct maximal tori.

The next result follows from the compactness of $\Gamma$.

Lemma 3.5. There exists $N \in \mathbb{N}$ such that for all $g \in \Gamma$, there exists $n, 1 \leq n \leq N$, such that $g^n \in D$.

Lemma 3.6. If $(g, h) \in Z_N$, then $\langle g, h \rangle$ is infinite.

Proof. If $(g, h) \in Z_N$, then there exist $p, q \in \mathbb{N}$, $1 \leq p, q \leq N$, such that $g^p, h^q \in D$ and $g^p h^q \neq h^q g^p$. Since $g^p h^q \neq h^q g^p$, it follows that $g^p, h^q \neq I$ and so $g^p, h^q \in D^*$. The result follows from Lemma 3.2.

4. Generating semisimple groups

Lemma 4.1. Let $\Gamma$ be a finite product of compact connected simple Lie groups. There is a non-empty Zariski open set $S \subset \Gamma^2$ such that if $(g, h) \in S$, then either $\langle g, h \rangle$ is finite, or $\langle g, h \rangle = \Gamma$.

Proof. Regard $\Gamma$ as acting on $\mathfrak{g}$ via the adjoint representation. Since $\Gamma$ is a product of simple Lie groups, the adjoint representation is faithful and so we may and shall regard $\Gamma$ as embedded in $GL(\mathfrak{g})$. It follows from Lemma 3.6 that there is a non-empty Zariski open subset $Z$ of $\Gamma^2$ such that if $(g, h) \in Z$, then the projection of $\langle g, h \rangle$ into each simple factor of $\Gamma$ is infinite. Let $L(\mathfrak{g})$ denote the space of $\mathbb{R}$-linear endomorphisms of $\mathfrak{g}$ and $L_\Gamma(\mathfrak{g})$ denote the subspace of endomorphisms which commute with $\Gamma$. Given $g, h \in \Gamma$, let $S(g, h) \subset L(\mathfrak{g})$ denote the solution set of the linear system of equations $[g, T] = 0$ and $[h, T] = 0$. It follows from the density theorem of Auerbach and Kuranishi that there exists a pair $(\gamma, \eta)$ such that $\langle \gamma, \eta \rangle = \Gamma$. Obviously, $S(\gamma, \eta) = L_\Gamma(\mathfrak{g})$ and $S(g, h) \supset S(\gamma, \eta)$, all $g, h \in \Gamma$. Let $S'$ be the set of pairs $(g, h)$ for which we have equality. Since the complement of $S'$ is defined by the vanishing of determinants, depending polynomially on $g, h$, it follows that $S'$ is a non-empty Zariski open set of $\Gamma^2$. Set $S = S' \cap Z$. 
Given \( g, h \in \Gamma \), suppose that \( \langle g, h \rangle_0 \) is a proper non-trivial subgroup of \( \Gamma \). It suffices to prove that \( \langle g, h \rangle \not\in S \). Denote the Lie algebra of \( \langle g, h \rangle_0 \) by \( \mathfrak{n} \). Let \( \mathfrak{v} \) denote the minimal \( \Gamma \)-invariant subspace of \( \mathfrak{g} \) containing \( \mathfrak{n} \). If \( \mathfrak{v} = \mathfrak{n} \), \( \langle g, h \rangle_0 \) must be a proper normal subgroup of \( \Gamma \) and must therefore be a product of simple factors of \( \Gamma \). If \( (g, h) \in \mathbb{Z} \), then the projection of \( \langle g, h \rangle_0 \) on every simple factor of \( \Gamma \) is infinite and so \( (g, h) \not\in \mathbb{Z} \supset \mathcal{S} \). On the other hand, if \( \mathfrak{v} \neq \mathfrak{n} \), then the projection of \( \mathfrak{v} \) on \( \mathfrak{n} \) commutes with \( g \) and \( h \) but not \( \Gamma \).

5. Proof of the main theorem

Lemma 5.1. Let \( \Gamma \) be a connected compact and semisimple Lie group. There is a non-empty Zariski open subset \( \mathbb{Z} \) of \( \Gamma^2 \) consisting of topological generators for \( \Gamma \).

Proof. It follows from Theorem 2.1 and Lemma 2.2 that we may assume that \( \Gamma \) is a finite product of compact connected simple Lie groups. The result follows from Lemmas 3.6, 4.1.

Theorem 5.2. If \( \Gamma \) is a compact connected semisimple Lie group, then the set of pairs of topological generators of \( \Gamma \) is a non-empty Zariski open subset of \( \Gamma^2 \).

Proof. Suppose that \( \langle g, h \rangle = \Gamma \). Then we may find a word \( W(g, h) = (w_1(g, h), w_2(g, h)) \) such that \( (w_1(g, h), w_2(g, h)) \in \mathbb{Z} \). But \( W^{-1}(\mathbb{Z}) \) is a Zariski open neighborhood of \( (g, h) \) consisting of topological generators of \( \Gamma \).

We conclude with a question suggested by Peter Rowley. If \( \Gamma \) is a finite simple group, it is known that for every non-identity element \( g \in \Gamma \), it is possible to choose \( h \in \Gamma \) such that \( \Gamma = \langle g, h \rangle \). Groups with this property are said to have ‘one and a half’ generators. We conjecture that if \( \Gamma \) is a compact simple group and \( g \) is a non-identity element of \( \Gamma \), then the subset of \( h \in \Gamma \) for which \( \langle g, h \rangle = \Gamma \) is non-empty and Zariski open.

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References


Department of Mathematics, University of Houston, Houston, Texas 77204-3476

E-mail address: mf@uh.edu