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THE PONTRYAGIN 4-FORM

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ABSTRACT. The unit 4-planes on which the first Pontryagin form of the Grassmann manifolds achieves its maximum are determined. This is a shorter and unified proof of results first obtained in 1995 by H. Gluck et al. and in 1998 by W. Gu.

INTRODUCTION

Calibrations are closed differential forms with comass bounded by one (see [HL]). This bound means that they are never larger than submanifold volume forms. Calibrations can be used to identify submanifolds that are volume minimizers in their homology classes. On any symmetric space an invariant differential p-form is determined by its restriction to a single tangent space and hence rescales to be a calibration. However, it is a difficult problem to determine the rescaling constant, even in special cases. After rescaling, the next step is to determine the unit p-vectors that are calibrated by the given p-form, i.e. the tangent p-planes that are the maximum points for the p-form.

H. Gluck, D. Mackenzie, and F. Morgan [GMM] determined the rescaling constant and the maximum 4-planes for the first Pontryagin form Φ on the real Grassmannians, $G_m(\mathbf{R}^{n+m})$, of oriented m-planes in $\mathbf{R}^{n+m}$, in the cases $m = 3, 4$. Later Gu [G] completed the general case $m \geq 3$.

In this note we provide a simpler and unified way of determining the four-planes calibrated by Φ on the Grassmannians $G_m(\mathbf{R}^{n+m})$, $m \geq 4$, $n \geq 3$ (Theorem 2). The main idea is to reduce this quartic problem to a quadratic problem concerning the eigenvalues of a selfadjoint quaternionic matrix Q (see Propositions 5 and 6). The methods we use are algebraic with all estimates being immediate consequences of standard linear algebra facts. One interesting aspect of our approach is that the desired estimate, in some sense, measures the "maximal" difference between the eigenvalues of Q and the eigenvalues of its transpose.

The Pontryagin 4-form $\Phi \in \Lambda^4 M_{m,n}(\mathbf{R})$ is defined by the same algebraic formula as the Cayley 4-form $\Phi \in \Lambda^4 \mathbf{O}$ on the octonians [HL]. These are both special cases of a "triality 4-form" constructed using either the double or triple cross product on a "triality algebra" [DH].

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The vector space $M_{n,m}(\mathbf{R})$ of an $n \times m$ real matrix is equipped with the real inner product

$$\langle A, B\rangle = \operatorname{Re} \operatorname{TRACE}(AB^t).$$

Define the cross product by

$$A \times B = AB^t - BA^t,$$

and note $A \times B \in M_n(\mathbf{R})$ is a square matrix (and skew symmetric).

Definition 1. The **Pontryagin 4-Form** $\Phi \in \Lambda^4 M_{n,m}(\mathbf{R})$ is defined by

$$\Phi(A, B, C, D) = \frac{1}{2}\Big(\langle A \times B, C \times D\rangle + \langle A \times C, D \times B\rangle + \langle A \times D, B \times C\rangle\Big).$$

Note. Using the triple cross product defined by

$$A \times B \times C = (A \times B)C + (C \times A)B + (B \times C)A$$

one can easily check that

$$\Phi(A, B, C, D) = \langle A \times B \times C, D\rangle .$$

Theorem 2.

GMM *and* GU	$(m \geq n \geq 4)$	$\Phi(A, B, C, D) \leq \frac{3}{2}\lvert A \wedge B \wedge C \wedge D\rvert,$
GMM	$(m \geq 4, n = 3)$	$\Phi(A, B, C, D) \leq \frac{4}{3}\lvert A \wedge B \wedge C \wedge D\rvert,$
GMM	$(m = n = 3)$	$\Phi(A, B, C, D) \leq \sqrt{\frac{3}{2}}\lvert A \wedge B \wedge C \wedge D\rvert.$

These inequalities are sharp, with equality achieved on a single $\mathrm{O}_n \times \mathrm{O}_m$ *orbit through a point* $A \wedge B \wedge C \wedge D$ *described in the proof.*

Proof (of the first two cases).

Step 1. Parallelogram Law. Using $4\langle u, v\rangle = |u + v|^2 - |u - v|^2$ we get

$$\begin{aligned} 8\Phi(A, B, C, D) =& |A \times B + C \times D|^2 + |A \times C + D \times B|^2 + |A \times D + B \times C|^2 \\ & - |A \times B - C \times D|^2 - |A \times C - D \times B|^2 - |A \times D - B \times C|^2. \end{aligned}$$

Step 2. The Quaternions. Set $X \equiv A + iB + jC + kD$, where i, j, k are the unit imaginary quaternions, and define $Q \equiv XX^* \in M_n(\mathbf{H})$. Note that:

$$\operatorname{Re} Q = AA^t + BB^t + CC^t + DD^t,$$

and

$$\operatorname{Im} Q = -(A \times B + C \times D)i - (A \times C + D \times B)j - (A \times D + B \times C)k.$$

The identity in Step 1 can be rewritten as follows.

Proposition 3.

$$8\Phi(A, B, C, D) = |\operatorname{Im} Q|^2 - E,$$

with error E *defined by*

$$E = |A \times B - C \times D|^2 + |A \times C - D \times B|^2 + |A \times D - B \times C|^2.$$

Corollary 4.

$$\Phi(A, B, C, D) \leq \tfrac{1}{8}|\operatorname{Im} Q|^2.$$

Step 3. Reduction to rank one.

Proposition 5. *The maximum of* $|\operatorname{Im} Q|^2$ *over the compact convex set*

$$\{Q \in M_n(\mathbf{H}) : Q = Q^*,\ \mathrm{TRACE}\, Q = 4,\ Q \geq 0\} \tag{1}$$

is achieved only on rank one matrices.

Proof. Suppose P is a maximum point and set $p = \operatorname{rank} P$ (the number of non-zero eigenvalues of P). Since P is $\mathbf{H}$-self adjoint we may choose g quaterionic unitary ($g^{-1} = g^*$) so that

$$gPg^* = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad a \in M_p(\mathbf{H}),$$

has its non-zero entries confined to the upper left $p \times p$ block. Consider the function $F \equiv |\operatorname{Im} g^*Qg|^2$ restricted to $Q = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$, with $b \in M_p(\mathbf{H})$, and satisfying $Q = Q^*$ and $\mathrm{TRACE}\, Q = 4$. The domain of F is an affine subset of $M_p(\mathbf{H})$. Restrict F further to $Q \geq 0$. Note gPg^* must be a maximum point for F. Also, note that gPg^* is an interior point of the domain of F, since $\operatorname{rank} gPg^* = p$. Therefore, $\operatorname{grad} F$ vanishes at gPg^*. But any non-negative polynomial F of degree 2 has only critical points at minimum points. Therefore, F is constant. Finally, if $p > 1$, it is easy to see that F is not constant. $\square$

Step 4. Compute an upper bound for $|\operatorname{Im} Q|^2$.

Proposition 6.

$$(n \geq 4) \quad |\operatorname{Im} Q|^2 \leq 3/4(\mathrm{TR}\, Q)^2,$$
$$(n = 3) \quad |\operatorname{Im} Q|^2 \leq 2/3(\mathrm{TR}\, Q)^2,$$

for all $Q \in M_n(\mathbf{H})$ *with* $Q^* = Q$ *and* $Q \geq 0$.

Proof. By Proposition 5 we may assume that Q has rank 1. Since $Q^* = Q$ it must be of the form

$$Q = xx^*$$

with x a column vector in $\mathbf{H}^n$. Note that $\mathrm{TRACE}\, Q = |x|^2$ and $|Q|^2 = |x|^4$. We may assume $\mathrm{TRACE}\, Q = T$ is constant. Therefore $|Q|^2 = T^2$ is fixed. Consequently $|\operatorname{Im} Q|^2$ obtains a maximum when $|\operatorname{Re} Q|^2$ obtains a minimum. Note that $R \equiv \operatorname{Re} Q \in M_n(\mathbf{R})$ satisfies:

$$R = R^t, \quad \mathrm{TRACE}\, R = T, \quad \operatorname{rank} R \leq 4. \tag{2}$$

The problem of minimizing $|R|^2$ over all $R \in M_n(\mathbf{R})$ satisfying (2) can be restated in terms of the four possibly non-zero eigenvalues λ_1, λ_2, λ_3, λ_4 of R. Namely,

$$|R|^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 \ \text{ is minimized over } \ \mathrm{TRACE}\, R = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = T.$$

This standard problem is easily solved. If $n \geq 4$, then all four non-zero eigenvalues must agree and equal $\frac{1}{4}T$. Thus $|R|^2$ has minimum value $\frac{1}{4}T^2$ and hence $|\operatorname{Im} Q|^2$ has maximum value $\frac{3}{4}T^2$ as desired. If $n = 3$, then all three non-zero eigenvalues must agree and equal $\frac{1}{3}T$. Thus $|R|^2$ has minimum value $\frac{1}{3}T^2$ and hence $|\operatorname{Im} Q|^2 \leq \frac{2}{3}T^2$ as desired. $\square$

Step 5. The upper bound for $|\operatorname{Im} Q|^2$ is sharp.

Case $n \geq 4$. Define $P \in M_n(\mathbf{H})$ to have its non-zero entries in the upper-left 4×4 block and set this block equal to

$$a = \begin{pmatrix} 1 & -i & -j & -k \\ i & 1 & -k & j \\ j & k & 1 & -i \\ k & -j & i & 1 \end{pmatrix}.$$

Note $P = yy^*$ where y is the column vector with entries 1, i, j, k, $0, \ldots$ so that P has rank one. Since $\operatorname{TRACE} P = 4$ and $R = \operatorname{Re} P$ has its non-zero eigenvalues equal to one, P is a maximum point for $|\operatorname{Im} Q|^2$ with maximum value 12.

For later reference

Note. $P^2 = 4P$ since $yy^*yy^* = |y|^2 yy^*$ and $|y|^2 = 4$.

Case $n = 3$. Similarly define

$$P = \frac{4}{3}\begin{pmatrix} 1 & -i & -j \\ i & 1 & -k \\ j & k & 1 \end{pmatrix} = \frac{4}{3} yy^*,$$

where y is the column vector with entries $1, i, j$. Now P satisfies $P^* = P$, $\operatorname{rank} P = 1$, $\operatorname{TRACE} P = 4$, $P \geq 0$, and $\operatorname{Re} P = \frac{4}{3} I$. Therefore $|\operatorname{Im} P|^2 = 32/3$ achieves the upper bound.

Step 6. Completing the proof. Combining Corollary 4 and Proposition 6 immediately yields the inequalities in the GGMM-Theorem.

Next we show that the inequalities are sharp.

Case $m \geq n \geq 4$. Define $Y \in M_{n,m}(\mathbf{H})$ to have upper left 4×4 block equal to $\frac{1}{2}a$ so that $YY^* = P$.

Case $m \geq 4$, $n = 3$. Define Y to have upper left 3×4 block equal to

$$\frac{1}{\sqrt{3}} yz^* = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & -i & -j & -k \\ i & 1 & -k & j \\ j & k & 1 & -i \end{pmatrix},$$

where y is the column vector with entries 1, i, j and where z is the column vector with entries 1, i, j, k, $0, \ldots$. Note that $YY^* = \frac{1}{3} yz^*zy^* = \frac{4}{3} yy^* = P$.

Now $Y = A + iB + jC + kD$ defines A, B, C, $D \in M_{n,m}(\mathbf{R})$. First, note that in both cases A, B, C, D are orthonormal, so that $|A \wedge B \wedge C \wedge D|^2 = 1$. Now for some good fortune! Check directly (or use Remark 2 at the end) that

$$A \times B = C \times D, \; A \times C = D \times B, \text{ and } A \times D = B \times C,$$

that is, the error E in Proposition 3 vanishes. Consequently,

$$\Phi(A, B, C, D) = \frac{1}{8} |\operatorname{Im} YY^*|^2 = \frac{1}{8} |\operatorname{Im} P|^2,$$

where $|\operatorname{Im} P|^2$ is the maximum value 12 or 32/3 as in Proposition 6.

Step 7. Characterizing equality. Suppose A', B', C', $D' \in M_{n,m}(\mathbf{R})$ is any other maximum point. We may assume that A', B', C', D' is orthonormal. Define $Y' = A' + iB' + jC' + kD'$, and $P' = Y'(Y')^*$. Since Proposition 3 expresses $\Phi(Y')$ as the difference of two positive quantities and the maximum value for Φ is the same as the maximum value of $\frac{1}{8}|\operatorname{Im} Q|^2$, the square matrix $P' \in M_n(\mathbf{H})$ must be a maximum point for $|\operatorname{Im} Q|^2$. Therefore, P' must have rank one by Proposition 5.

Consequently, Y' must also have rank one, i.e. $Y' = ab^*$ where $a \in \mathbf{H}^n$, $b \in \mathbf{H}^m$ are column vectors. By the proof of Proposition 6, $\operatorname{Re} P'$ must have the following non-zero eigenvalues:

Case $m \geq n \geq 4$. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$.

Case $m \geq 4$, $n = 3$. $\lambda_1 = \lambda_2 = \lambda_3 = 4/3$.

In the first case, conjugating by an element of O_n we may assume $\operatorname{Re} P' = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. That is,

$$\operatorname{Re} ab^*ba^* = |b|^2 \operatorname{Re}(a_i \overline{a}_j) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Normalizing so that $|b| = 1$ we have that a_1, a_2, a_3, $a_4 \in \mathbf{H}$ are real orthonormal and $a_j = 0$, $j = 5, \ldots$.

Therefore, by utilizing another element of O_n we may assume that $a^t = (1, i, j, k, 0, \ldots,)$. Finally, since $|\operatorname{Im} XX^*|^2 = |\operatorname{Im} X^*X|^2$, $\widetilde{P} \equiv (Y)^*Y'$ is a maximum point for $|\operatorname{Im} \widetilde{Q}|^2$. This can be used to show that b is O_m equivalent to $\frac{1}{2}(1, i, j, k, 0, \ldots)^t$.

This proves that Y' is $\mathrm{O}_n \times \mathrm{O}_m$ equivalent to Y.

In the second case, conjugating by an element of SO_n we may assume $\operatorname{Re} ab^*b^*a = |b|^2 \operatorname{Re}(a_i \overline{a}_j) = \frac{4}{3} I$. Normalizing by $|b|^2 = \frac{4}{3}$ we have that a_1, a_2, $a_3 \in \mathbf{H}$ are orthonormal. Utilizing another element of O_3 we may assume a is a column with entries 1, i, j. Finally, b can be seen to be O_m equivalent to $\frac{1}{\sqrt{3}}(1, i, j, k, 0, \ldots, 0)^t$. This proves that $Y' = ab^*$ is $\mathrm{O}_3 \times \mathrm{O}_m$ equivalent to Y. □

Remark 1. If $m = n = 3$, then at points where $|\operatorname{Im} Q|^2$ is maximal the error E is not zero. This case cannot be reduced to an elementary quadratic inequality on the eigenvalues of Q.

Remark 2. It is easy to see that the error $E = |\operatorname{Im} X^t(X^t)^*|^2$ and that

$$\Phi(X) = \tfrac{1}{8}(|XX^*|^2 - |X^t(X^t)^*|^2).$$

Therefore if X is self-adjoint, then $\Phi(X)$ measures the difference between the sum of fourth powers of the eigenvalues of X and the sum of fourth powers of the eigenvalues of X^t. Such a difference would be zero for a complex matrix!

References

[DH] Jiri Dadok and Reese Harvey, *Calibrations associated with triality algebras* (in preparation).

[GMM] Herman Gluck, Dana Mackenzie, and Frank Morgan, *Volume-minimizing cycles in Grassmann manifolds*, Duke Math. J. **79** (2) (1995), 335–404. MR **96d:**53061

[GMZ] Herman Gluck, Frank Morgan, and Wolfgang Ziller, *Calibrated geometries in Grassmann manifolds*, Comm. Math. Helv. **64** (1989), 256–268. MR **90h:**53077

[G] Weiqing Gu, *The stable* 4*-dimensional geometry of the real Grassmann manifolds*, Duke Math. J. **93** (1998), 155–178. MR **99e:**53091

[HL] Reese Harvey and H. Blaine Lawson, Jr., *Calibrated geometries*, Acta Math. **148** (1982), 47–157. MR **85i:**53058

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