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STABILITY OF THE FIXED POINT PROPERTY OF HILBERT SPACES

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Abstract. We prove that any Banach space X whose Banach-Mazur distance to a Hilbert space is less than $\sqrt{\frac{5+\sqrt{13}}{2}}$ has the fixed point property for nonexpansive mappings.

Let C be a nonempty closed convex subset of a Banach space X. A mapping $T : C \to C$ is said to be *nonexpansive* if $\|T\,x - T\,y\| \leq \|x - y\|$ for any $x, y \in C$. A nonempty weakly compact convex set C is said to have the *fixed point property* if every nonexpansive $T : C \to C$ has a fixed point. X is said to have the *fixed point property* if every nonempty weakly compact convex subset C of X has the fixed point property.

Let C be a nonempty weakly compact convex subset of X and $T : C \to C$ be nonexpansive. A closed convex nonempty subset K of C is said to be *minimal* for T if $T(K) \subseteq K$ and for any nonempty closed convex subset K' of K,

$$T(K') \subseteq K' \text{ implies } K' = K.$$

Since C is weakly compact, C has a minimal subset. Hence we can assume that C is minimal for T. Recall that a sequence $\{x_n\}$ in C is called an *approximate fixed point sequence* (afps in short) for T if

$$\lim_{n\to\infty} |x_n - T\,x_n| = 0.$$

It is known that if T is a nonexpansive mapping on a bounded convex set, then T has an afps. Karlovitz [Ka] proved the following theorem.

Theorem 1. *Let $(K, |\cdot|)$ be a minimal weakly compact convex set for a nonexpansive mapping T. For any apfs $\{x_n\}$ of T and any $y \in K$,*

$$\lim_{n\to\infty} |y - x_n| = \operatorname{diam}(K).$$

Using Theorem 1, one can easily prove that the ℓ_2 with the norm

$$\|x\| = \max\{\|x\|_2, 2\|x\|_\infty\}$$

has the fixed point property. In [A], Alspach showed that L_1 does not have the fixed point property. In [M], Maurey introduced the ultraproduct technique and

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he proved c_o and every reflexive subspace of L_1 have the fixed point property (also see [ELOS]). Recently, Domíngues Benavides, Jiménez-Melado and Llorens-Fuster [DB], [JMLF] showed that X has the fixed point property if the Banach-Mazur distance from X to a Hilbert space is less than $\sqrt{2+\sqrt{2}}$. From their proofs, it is natural to ask that whether X has the fixed point property if the Banach-Mazur distance from X to ℓ_2 is 2. (It is still open as to whether every isomorph of ℓ_2 has the fixed point property.) In this article, first, we give a simple proof of Jiménez-Melado and Llorens-Fuster's result. Then we use it to show that X has the fixed point property if its distance to a Hilbert space is less than $\sqrt{\frac{5+\sqrt{13}}{2}}$ (≈ 2.07). For the background and information on the fixed point property, see [AkK], [ELOS] and [GK].

Suppose that there is a Banach space X such that X is isomorphic to ℓ_p for some $1 < p < \infty$, and X does not have the fixed point property. Let $|\cdot|$ denote the norm of $X = \ell_p$, $1 < p < \infty$, and K a nonempty weakly compact convex set of X so that there is a nonexpansive mapping T on $(K, |\cdot|)$ which has no fixed point. The *Banach-Mazur distance* $d(X, \ell_p)$ from X to ℓ_p is the number

$$d(X,\ell_p) = \inf\Big\{\|S\|\cdot\|S^{-1}\| :\ S \text{ is an isomorphism from } X \text{ onto } \ell_p\Big\}.$$

Suppose that $d(X, \ell_p) < B$. Without loss of generality, we assume that for any $x \in X$,

$$(*) \qquad \|x\|_p \le |x| \le B\|x\|_p.$$

Let $\ell_\infty(X)$ denote the set

$$\Big\{[y_n] : y_n \in X \text{ and } \{|y_n|\} \in \ell_\infty\Big\}$$

with the norm $\big\|[y_n]\big\|_{\ell_\infty(X)} = \sup_n |y_n|$ and let $c_o(X)$ be the closed subspace

$$\Big\{[y_n] : y_n \in X \text{ and } \{|y_n|\} \in c_o\Big\}$$

of $\ell_\infty(X)$. Set

$$\widetilde{X} = \ell_\infty(X)/c_o(X).$$

Thus for any $[y_n] \in \widetilde{X}$,

$$(**) \qquad \big\|[y_n]\big\|_{\widetilde{X}} = \limsup_{n\to\infty} |y_n|.$$

Finally, let

$$\widetilde{K} = \{[y_n] \in \widetilde{X} : y_n \in K\}.$$

Clearly, $\widetilde{K}$ is a closed convex subset of $\widetilde{X}$ and it contains the set $\{[x] : x \in K\}$ which is isometrically isomorphic to K. Let $\widetilde{T} : \widetilde{K} \to \widetilde{K}$ be the mapping defined by

$$\widetilde{T}[y_n] = [T\,y_n].$$

Since T is nonexpansive on K, $\widetilde{T}$ is a well-defined nonexpansive mapping on $\widetilde{K}$. Moreover, if $\{x_n\}$ is an afps, then $[x_n]$ is a fixed point of $\widetilde{T}$.

Let $\{x_n\}$ be any fixed afps of T. By passing to a weakly convergent subsequence of $\{x_n\}$, and then translating K, we may assume that $\{x_n\}$ converges to 0 weakly (so $0 \in K$). For convenience, we assume that diam$(K) = 1$ and we denote the fixed point $[x_n]$ by $\tilde{x}$. For any $0 < t < 1$, let $\widetilde{W}_t$ be the smallest invariant closed convex subset of $\widetilde{K}$ of $\widetilde{T}$ which contains $t\tilde{x}$ (we do not know whether this $\widetilde{W_t}$ is minimal or not). Before proving Jiménez-Melado and Llorens-Fuster Theorem, we need the following two lemmas.

Lemma 2. *For $0 < t < 1$, let $[w_n]$ be an element in $\widetilde{W_t}$.*

(a) $\lim_{n\to\infty} |x_n - w_n| = 1 - t$.
(b) *There is $x \in K$ such that* $\lim_{n\to\infty} |w_n - x| = t$.
(c) *For any $x \in K$,* $\liminf_{n\to\infty} |w_n - x| \geq t$.
(d) $\limsup_{n\to\infty} \limsup_{m\to\infty} |w_n - w_m| \leq t$. *Moreover, if $\{w_{n_k}\}_{k=1}^{\infty}$ converges weakly to $w \in K$, then* $\lim_{n\to\infty} |w_n - w| = t$ *and* $\limsup_{k\to\infty} |x_{n_k} - w_{n_k} + w| \geq 1 - t$.

Proof. Note: we assume that K is minimal for T and $\operatorname{diam}(K) = 1$. By Theorem 1, we have

$$\|\tilde{x} - t\,\tilde{x}\|_{\widetilde{X}} = (1-t)\|\widetilde{x}\|_{\widetilde{X}} = 1 - t,$$
$$\lim_{n\to\infty}\lim_{m\to\infty} |tx_n - tx_m| = t \lim_{n\to\infty}\lim_{m\to\infty} |x_n - x_m| = t,$$
$$|tx_n - 0| = t|x_n| = t.$$

We claim that

$$\limsup_{n\to\infty} |x_n - w_n| = \big\|[x_n] - [w_n]\big\|_{\widetilde{X}} \leq 1 - t, \tag{1}$$
$$\limsup_{n\to\infty}\limsup_{m\to\infty} |w_n - w_m| \leq t, \tag{2}$$
$$\exists\, x \in K \text{ such that } \limsup_{n\to\infty} |w_n - x| \leq t. \tag{3}$$

(1) follows from the fact that the intersection of $\widetilde{K}$ and the closed ball in $\widetilde{X}$ centered at $\widetilde{x}$ of radius $1 - t$ is invariant under $\widetilde{T}$ and it contains $t\widetilde{x}$.

Proof of (2). Note: T is nonexpansive. For any sequence $\{y_n\}$ in C,

$$\limsup_{n\to\infty}\limsup_{m\to\infty} |T\,y_n - T\,y_m| \leq \limsup_{n\to\infty}\limsup_{m\to\infty} |y_n - y_m|.$$

If (2) holds for $[w_n]$, then (2) holds for the convex hull $\operatorname{co}\big([w_n], [T\,w_n]\big)$. $\widetilde{W_t}$ may be constructed by iterating this beginning with $t\widetilde{x}$,

$$W_1 = \operatorname{co}\big(t\widetilde{x}, \widetilde{T}(t\widetilde{x})\big), \cdots, W_{k+1} = \operatorname{co}\big(\widetilde{T}\,W_k, W_k\big),$$

and finally $\widetilde{W_t} = \overline{\bigcup_k W_k}$.

Proof of (3). Note: suppose there exist $x^1, x^2 \in K$ and $[w_n^1], [w_n^2] \in \widetilde{K}$ such that $\big\|[w_n^1] - [x^1]\big\|_{\widetilde{X}} \leq t$ and $\big\|[w_n^2] - [x^2]\big\|_{\widetilde{X}} \leq t$; then

$$\Big\|\widetilde{T}\,[w_n^1] - [T\,x^1]\Big\|_{\widetilde{X}} \leq t$$

and for any $0 < \alpha < 1$,

$$\Big\|\alpha[w_n^1] + (1-\alpha)[w_n^2] - [\alpha x^1 + (1-\alpha)x^2]\Big\|_{\widetilde{X}} \leq t.$$

It follows that if $[w_n] \in \bigcup_k W_k$, then (3) holds. The proof for $[w_n] \in \widetilde{W_t} = \overline{\bigcup_k W_k}$ follows from the following fact.

Let $\{[w_n^m]\}_{m=1}^{\infty}$ and $\{x^m\}_{m=1}^{\infty}$ be two sequences in $\widetilde{K}$ and K such that the sequence $\big\{[w_n^m]\big\}_{m=1}^{\infty}$ converges to $[w_n]$ in $\widetilde{X}$ and for all $m \in \mathbb{N}$,

$$\Big\|[w_n^m] - [x^m]\Big\|_{\widetilde{X}} \leq t.$$

Since K is weakly compact, there is a weak cluster point x of $\{x^m\}_{m=1}^{\infty}$ in K. By the Hahn-Banach Theorem, we have

$$\Big\|[w_n] - [x]\Big\|_{\widetilde{X}} \leq t.$$

Assume that (b) is not true. Then there exists a subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ of $\{w_n\}_{n=1}^{\infty}$ with

$$\lim_{k\to\infty} |w_{n_k} - x| = \alpha < t.$$

Hence

$$\begin{aligned}\limsup_{k\to\infty} |w_{n_k} - x_{n_k}| &\geq \limsup\Big[|x_{n_k} - x| - |x - w_{n_k}|\Big] \\ &\geq 1 - \alpha \qquad\qquad \text{by Theorem 1} \\ &> 1 - t,\end{aligned}$$

which contradicts (1). The same argument shows that (a) and (c) must hold.

The first part of (d) follows from (2). Let $\{w_{n_k}\}$ be a subsequence of $\{w_n\}$ which converges weakly to $w \in K$. Then (by (2) and (c) of Lemma 2)

$$\begin{aligned} t &\geq \limsup_{m\to\infty} \limsup_{k\to\infty} |w_m - w_{n_k}| \\ &\geq \liminf_{m\to\infty} \limsup_{k\to\infty} |w_m - w_{n_k}| \\ &\geq \liminf_{m\to\infty} |w_m - w| \geq t. \end{aligned}$$

Hence,

$$\lim_{m\to\infty} \limsup_{k\to\infty} |w_m - w_{n_k}| = \lim_{m\to\infty} |w_m - w| = t, \tag{4.a}$$

$$\limsup_{k\to\infty} |x_{n_k} - w_{n_k} + w| \geq \lim_{k\to\infty} |x_{n_k}| - \lim_{k\to\infty} |w_{n_k} - w| = 1 - t. \tag{4.b}$$

The proof is complete. □

Lemma 3. *Suppose that X is a Banach space such that $\|x\|_p \leq |x| \leq B\|x\|_p$ for some $B > 1$. Let $[w_n]$ be an element in $\widetilde{W}_t$. If $\{n_k\}$ is an increasing sequence such that $\{w_{n_k}\}_{k=1}^{\infty}$ converges weakly to w and if all the following limits*

$$\lim_{k\to\infty} \|w_{n_k} - w\|_p, \qquad \lim_{k\to\infty} \|w_{n_k} - w - x_{n_k}\|_p,$$
$$\lim_{k\to\infty} \|x_{n_k} - w_{n_k}\|_p, \qquad \lim_{k\to\infty} \|w_{n_k}\|_p$$

exist, then

$$\begin{aligned}(1-t)^p &\geq \Big(\frac{1-t}{B}\Big)^p + \|w\|_p^p; \\ \lim_{k\to\infty} \|w_{n_k}\|_p^p &\leq t^p/2 + \|w\|_p^p.\end{aligned}$$

Proof. It is known that if $\{z_n\}$ is weakly null sequence in ℓ_p and $z \in \ell_p$, then

$$\limsup_{n\to\infty} \|z_n - z\|_p^p = \|z\|_p^p + \limsup_{n\to\infty} \|z_n\|_p^p.$$

Since $\{x_n\}_{n=1}^{\infty}$ and $\{w_{n_k} - w\}_{k=1}^{\infty}$ are weakly null sequences,

$$\begin{aligned}\lim_{k\to\infty} 2\|w_{n_k} - w\|_p^p &= \lim_{k\to\infty} \|w_{n_k} - w\|_p^p + \lim_{r\to\infty} \|w_{n_r} - w\|_p^p \\ &= \lim_{k\to\infty} \lim_{r\to\infty} \|w_{n_k} - w_{n_r}\|_p^p.\end{aligned} \tag{5}$$

$$\lim_{k\to\infty} \|x_{n_k} - w_{n_k}\|_p^p = \|w\|_p^p + \lim_{k\to\infty} \|x_{n_k} - w_{n_k} + w\|_p^p, \tag{6}$$

$$\lim_{k\to\infty} \|w_k\|_p^p = \|w\|_p^p + \lim_{k\to\infty} \|w_{n_k} - w\|_p^p. \tag{7}$$

By assumption $\|\cdot\|_p \le |\cdot| \le B\|\cdot\|_p$, we have

(8)

$$\lim_{k\to\infty} \|w_{n_k} - w\|_p^p = \lim_{k\to\infty}\lim_{r\to\infty} \frac{\|w_{n_k} - w_{n_r}\|_p^p}{2} \le \frac{t^p}{2}, \quad \text{by (5) and Lemma 2(d),}$$

$$(1-t)^p \ge \lim_{k\to\infty} \|x_{n_k} - w_{n_k}\|_p^p, \quad \text{by Lemma 2(a),} \tag{9}$$

$$\left(\frac{1-t}{B}\right)^p \le \lim_{k\to\infty} \|x_{n_k} - w_{n_k} + w\|_p^p, \quad \text{by (4.b).} \tag{10}$$

By (5)–(10), we get

$$\begin{aligned}(1-t)^p &\overset{(9)}{\ge} \lim_{k\to\infty} \|x_{n_k} - w_{n_k}\|_p^p \\ &\overset{(6)}{=} \lim_{k\to\infty} \|x_{n_k} - w_{n_k} + w\|_p^p + \|w\|_p^p \overset{(10)}{\ge} \left(\frac{1-t}{B}\right)^p + \|w\|_p^p,\end{aligned}$$

$$\lim_{k\to\infty} \|w_{n_k}\|_p^p \overset{(7)}{=} \lim_{k\to\infty} \|w_{n_k} - w\|_p^p + \|w\|_p^p \overset{(8)}{\le} t^p/2 + \|w\|_p^p.$$

□

Theorem 4 (Jiménez-Melado and Llorens-Fuster)**.** *For $1 < p < \infty$, let $C_p > 1$ be the smallest positive solution of the equation*

$$C(C-1) = [C^{1/(p-1)} + (2C-2)^{1/(p-1)}]^{p-1}.$$

If the Banach-Mazur distance from X to ℓ_p is less than $(C_p)^{\frac{1}{p}}$, then X has the fixed point property.

Proof. Suppose that $(X, |\cdot|)$ does not have the fixed point property. For any $B < d(X, \ell_p)$, we may assume that $\|x\|_p \le |x| \le B\|x\|_p$.

For $0 < t < 1$, let $\widetilde{W}_t$ be the set defined in Lemma 2. Since $\widetilde{W}_t$ is an invariant closed convex subset for $\widetilde{T}$, it contains an apfs. By Theorem 1, for any $\epsilon > 0$, there is $[w_n] \in \widetilde{W}_t$ such that

$$\liminf_{n\to\infty} |w_n| > 1 - \epsilon. \tag{11}$$

For $\epsilon > 0$, let (w_n) be an element in $\widetilde{W}_t$ such that $\big\|[w_n]\big\|_{\widetilde{X}} > 1 - \epsilon$. Since K is weakly compact, there is a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ which converges weakly to $w \in K$. By passing to further subsequences of $\{w_n\}$ and $\{x_n\}$, we can assume that $|w_{n_k}| \ge 1 - \epsilon$ for all k and all the following limits

$$\lim_{k\to\infty} \|w_{n_k} - w\|_p, \qquad \lim_{k\to\infty} \|w_{n_k} - w - x_{n_k}\|_p,$$
$$\lim_{k\to\infty} \|x_{n_k} - w_{n_k}\|_p, \qquad \lim_{k\to\infty} \|w_{n_k}\|_p$$

exist. By (11) and Lemma 3, we have

$$\frac{(1-\epsilon)^p}{B^p} \le \lim_{k\to\infty} \|w_{n_k}\|_p^p \le t^p/2 + \|w\|_p^p \le \frac{t^p}{2} + (1-t)^p - \frac{(1-t)^p}{B^p}.$$

Now, let ϵ approach to 0. We have

$$(1-t)^p - \Big(\frac{1-t}{B}\Big)^p - \frac{1}{B^p} + \frac{t^p}{2} \geq 0, \tag{12}$$

which yields

$$(B^p-1)(1-t)^p - 1 + \frac{(Bt)^p}{2} \geq 0. \tag{13}$$

Let $C = B^p$ and $t = \frac{(2C-2)^{\frac{1}{p-1}}}{C^{\frac{1}{p-1}}+(2C-2)^{\frac{1}{p-1}}}$. Then

$$(C-1)\Big(\frac{C^{\frac{1}{p-1}}}{C^{\frac{1}{p-1}}+(2C-2)^{\frac{1}{p-1}}}\Big)^p - 1 + \frac{C}{2}\Big(\frac{(2C-2)^{\frac{1}{p-1}}}{C^{\frac{1}{p-1}}+(2C-2)^{\frac{1}{p-1}}}\Big)^p \geq 0,$$

$$C(C-1)\left(\frac{C^{\frac{1}{p-1}}+(2C-2)^{\frac{1}{p-1}}}{\Big(C^{\frac{1}{p-1}}+(2C-2)^{\frac{1}{p-1}}\Big)^p}\right) - 1 \geq 0.$$

Hence we have

$$C(C-1) \geq [C^{1/(p-1)} + (2C-2)^{1/(p-1)}]^{p-1}$$
$$B^p \geq C_p.$$

The proof is complete. □

One can easily get the following corollary.

Corollary 5. *Let C_p be the number defined in Theorem 4. Let X_i be a sequence of finite dimensional normed spaces and let X be a Banach space. If the Banach-Mazur distance from X to $(\sum \bigoplus X_i)_p$ is less than $(C_p)^{1/p}$, then X has the fixed point property.*

First, we note that $\widetilde{W}_t$ has no fixed point for $\widetilde{T}$ (by Theorem 1 and Lemma 2(d)). Let $\{x_n\}$ be an fixed approximate sequence. For $0 < t < 1$, $\widetilde{W}_t$ is defined as above. Suppose that there is B such that for any $x \in X$, $\|x\|_p \leq |x| \leq B\|x\|_p$. We are interested in the number

$$E_t = \sup\Big\{\limsup_{n\to\infty} \|w_n\|_p : [w_n] \in \widetilde{W}_t\Big\}.$$

In (11), we used the trivial estimate i.e. $E_t \geq \frac{1}{B}$. The following theorem shows that we can have a better estimate if $p = 2$. Using this result, we prove that if $d(X, \ell_2) < \sqrt{\frac{5+\sqrt{13}}{2}}$, then X has the fixed point property.

Theorem 6. *If the Banach-Mazur distance from X to ℓ_2 is less than $\sqrt{\frac{5+\sqrt{13}}{2}}$, then X has the fixed point property.*

Proof. Let

$$D = \inf\{\liminf_{n\to\infty} \|y_n - y\|_p : \{y_n\} \text{ is an afps and } \{y_n\} \text{ converges to } y \text{ weakly}\}.$$

We claim that if $p = 2$ and if $\|x\|_2 \leq |x| \leq B\|x\|_2$ for all $x \in X$, then $D^2 \geq \frac{1}{B^2-1}$.

Assume that the claim were proved. Subclaim: for any $0 < t < 1$ and any $\epsilon > 0$, there is $[w_n] \in \widetilde{W}_t$ such that $\|[w_n]\|_{\widetilde{X}} > D - \epsilon$.

Suppose that the subclaim is not true. Then there is an afps $\{[w_n^m]\}_{m=1}^\infty$ in $\widetilde{W}_t$ such that $\|[w_n^m]\|_{\widetilde{X}} < D-\epsilon$. By the diagonal method, one can construct an afps $\{z_k\}_{k=1}^\infty$ of T from $\{w_n^m : n,m \in \mathbb{N}\}$ such that $|z_k| < D-\epsilon$. This contradicts the definition of D.

By (12) (replacing $\frac{1}{B^2}$ by $\frac{1}{B^2-1}$), we have

$$(1-t)^2 - \Big(\frac{1-t}{B}\Big)^2 - \frac{1}{B^2-1} + \frac{t^2}{2} \geq 0,$$
$$(3B^2-2)t^2 - 4(B^2-1)t + 2(B^2-1) - \frac{2B^2}{B^2-1} \geq 0.$$

Let $t = \frac{2B^2-2}{3B^2-2}$. We have

$$-\frac{4(B^2-1)^2}{3B^2-2} + 2(B^2-1) - \frac{2B^2}{B^2-1} \geq 0.$$

This implies (note: $B>1$) that

$$2(-2B^2+2+3B^2-2)(B^2-1)^2 - 2B^2(3B^2-2) = 2B^2(B^4-5B^2+3) \geq 0,$$
$$B^2 \geq \frac{5+\sqrt{13}}{2}.$$

Hence we only need to prove our claim.

Proof of claim. First, let us pretend that we have

(a) there exists an apfs $\{x_n\}$ such that, for all $n \in \mathbb{N}$, $\|x_n\|_2 = D$.

(b) there is a vector $\widetilde{w}^t = [w_n^t]$ in $\widetilde{W}_t$ such that $\|w_n^t\|_2 \geq D$ for all $n \in \mathbb{N}$.

Fix a t and let $O=0$, $P=x_n$, $Q_t = t\,x_n$ and $R_t = w_n^t$. Consider the triangle $\triangle PQ_tR_t$. Let $\alpha_t,\ \beta_t,\ \gamma_t$ denote the three angles $\angle PQ_tR_t,\ \angle Q_tR_tP,\ \angle R_tPQ_t = \angle R_tPO$, respectively. We would like to estimate the least upper bound of $\cos\gamma_t$. It is easy to see that the worst case is $\|w_n^t\| = D$. Hence if $t \uparrow 1$, R_t approaches to P and $\liminf_{t\uparrow 1}\gamma_t \geq \frac{\pi}{2}$. In other word, we have

$$\lim_{t\uparrow 1}\cos\gamma_t \leq 0.$$

Now, let us do the estimate of $\cos\gamma_t$. By approximation, we may assume that (at least for n large enough)

$$\|(1-t)x_n\|_2 = (1-t)D,$$
$$\|x_n - w_n\|_2 \leq 1-t.$$

By Lemma 2(a), the distance in $|\cdot|$ norm from P to any point on the segment $\overline{Q_tR_t}$ is $1-t$. Hence the distance in $|\cdot|$ norm from P to any point on the line which contains Q_t and R_t is at least $1-t$. This implies the distance in $\|\cdot\|_2$ norm from P to any point on the line which contains Q_t and R_t is at least $\frac{1-t}{B}$. Hence

$$\begin{aligned}\sin\alpha_t \geq& \frac{1}{BD},\\ \sin\beta_t \geq& \frac{1}{B},\\ \cos\gamma_t =& \sin\alpha_t\sin\beta_t - \cos\alpha_t\cos\beta_t\\ \geq& \frac{1}{B^2D} - \sqrt{\big(1-\frac{1}{B^2}\big)\big(1-\frac{1}{B^2D^2}\big)}.\end{aligned}$$

Since $\liminf_{t\uparrow 1}\gamma_t \geq \frac{\pi}{2}$, we have

$$\frac{1}{B^2D} - \sqrt{\left(1-\frac{1}{B^2}\right)\left(1-\frac{1}{B^2D^2}\right)} \leq \limsup_{t\uparrow 1}\cos\gamma_t \leq \cos\frac{\pi}{2} = 0.$$

Move one term to other side and then square both sides. We get

$$(B^2-1)(B^2D^2-1) \geq 1,$$

which yields

$$B^4D^2 - B^2 - B^2D^2 \geq 0.$$

Since $B > 1$, we have

$$D^2 \geq \frac{1}{B^2-1}.$$

This is the idea of proof. Now let us do the computation.

To avoid taking limits, we will use ultraproduct of X instead of the quotient space $\ell_\infty(X)/c_o(X)$ (for definiton of ultraproduct, see [ELOS] or [GK]). It is known that the ultraproduct of an L_p-space is an L_p-space and the ultraproduct of a Hilbert space is a Hilbert space. Here, we still use the notation $\widetilde{X}$ for the ultraproduct of X.

Assume that the conclusion of Theorem 6 does not hold. Then there is a Banach space X such that the distance from X to ℓ_2 is less than some $B < \sqrt{\frac{5+\sqrt{13}}{2}}$ and $(X, |\cdot|)$ does not have the fixed point property. Without loss of generality, we can assume that, for any $x \in X$,

$$\|x\|_2 \leq |x| \leq B\|x\|_2.$$

For any $t < 1$, let $\epsilon_t = (1-t)^2$. By the definition of D, there are a weakly null afps $\widetilde{x}^t = \{x_n^t\}$ (after translation of K), $\widetilde{W}_t$ (dependent on $\widetilde{x}^t$) and $\widetilde{w}^t \in \widetilde{W}_t$ such that

$$D \leq \|\widetilde{x}^t\|_2 \leq D + \epsilon_t \quad \text{and} \quad \|\widetilde{w}^t\|_2 \geq D - \epsilon_t.$$

Let $P_t = \widetilde{x}^t$, $Q_t = t\widetilde{x}^t$ and $R_t = \widetilde{w}^t$. Then we have the following estimate.

$$\begin{aligned}
\sin\alpha_t \geq& \frac{1}{B(D+\epsilon_t)},\\
\sin\beta_t \geq& \frac{1}{B},\\
\cos\gamma_t =& \sin\alpha_t\sin\beta_t - \cos\alpha_t\cos\beta_t\\
\geq& \frac{1}{B^2(D+\epsilon_t)} - \sqrt{\left(1-\frac{1}{B^2}\right)\left(1-\frac{1}{B^2(D+\epsilon_t)^2}\right)}.
\end{aligned}$$

By the law of cosines on the triangle $\triangle OP_tR_t$ (where $O = 0$), we have

$$\begin{aligned}
&4\epsilon_tD + \left\|\widetilde{w}^t - \widetilde{x}^t\right\|_2^2\\
\geq& \left\|\widetilde{x}^t\right\|_2^2 - \left\|\widetilde{w}^t\right\|_2^2 + \left\|\widetilde{w}^t - \widetilde{x}^t\right\|_2^2\\
=& 2\left\|\widetilde{x}^t\right\|_2\left\|\widetilde{w}^t - \widetilde{x}^t\right\|_2\cos\gamma_t\\
\geq& 2D\left\|\widetilde{w}^t - \widetilde{x}^t\right\|_2\left(\frac{1}{B^2(D+\epsilon_t)} - \sqrt{\left(1-\frac{1}{B^2}\right)\left(1-\frac{1}{B^2(D+\epsilon_t)^2}\right)}\right).
\end{aligned}$$

Since $1-t \geq \|\widetilde{w}^t - \widetilde{x}^t\|_2 \geq \frac{1-t}{B}$ and $\lim_{t\uparrow 1} \frac{\epsilon_t}{1-t} = 0$,

$$\begin{aligned} 0 =& \lim_{t\uparrow 1} \frac{4\epsilon_t D}{\|\widetilde{w}^t + \widetilde{x}^t\|_2} + \|\widetilde{w}^t - \widetilde{x}^t\|_2 \\ \geq& \lim_{t\uparrow 1} 2D \left(\frac{1}{B^2(D+\epsilon_t)} - \sqrt{\left(1-\frac{1}{B^2}\right)\left(1-\frac{1}{B^2(D+\epsilon_t)^2}\right)} \right) \\ =& 2D \left(\frac{1}{B^2(D+\epsilon_t)} - \sqrt{\left(1-\frac{1}{B^2}\right)\left(1-\frac{1}{B^2(D+\epsilon_t)^2}\right)} \right). \end{aligned}$$

So we have $D^2 \geq \frac{1}{B^2-1}$. Using the same equivalent norm, and Lemma 2 and Lemma 3, we have

$$(1-t)^2 - \left(\frac{1-t}{B}\right)^2 - \frac{1}{B^2-1} + \frac{t^2}{2} \geq 0,$$

for all $0 < t < 1$ (cf. (12)). But this is impossible if $t = \frac{2B^2-2}{3B^2-2}$ (and $B^2 < \frac{5+\sqrt{13}}{2}$). The proof is complete. □

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