

IRREDUCIBLE REPRESENTATIONS OF THE CUNTZ ALGEBRA \mathcal{O}_N

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ABSTRACT. In this paper, we establish formulas for the configuration of a special class of irreducible representations of the Cuntz algebra \mathcal{O}_N , $N = 2, 3, \dots, \infty$. These irreducible representations arise as subrepresentations of naturally occurring representations of \mathcal{O}_N acting in $L^2(\mathbb{T})$ and arise from consideration of multiresolution wavelet filters.

1. INTRODUCTION

Recent papers (e.g., [BraJo97a], [BraJo97b], [Jor]) show that decomposition of representations of finitely generated C^* -algebras has applications to

1. filter functions for multiresolutions from wavelet theory;
2. limit problems in analytic number theory;
3. multiplicity problems from noncommutative harmonic analysis.

In this paper, we introduce a special class of representations related to 1–3 above. They can be described on the Hilbert space $\ell^2(S)$, where S is a discrete index set, or more specifically on $L^2(\mathbb{T}^n) \simeq \ell^2(\mathbb{Z}^n)$ where $S = \mathbb{Z}^n$, $n = 1, 2, \dots$. Recall that the Cuntz algebra \mathcal{O}_N , $N = 2, 3, \dots$, is the C^* -algebra generated by isometries s_1, \dots, s_N satisfying

$$s_i^* s_j = \delta_{ij} \mathbb{1} \quad \text{and} \quad \sum_{i=1}^N s_i s_i^* = \mathbb{1}$$

for $i, j \in \{0, 1, \dots, N-1\}$ [Cun77]. The C^* -algebra \mathcal{O}_∞ is the one generated by isometries s_i , $i \in \mathbb{Z}$, satisfying $s_i^* s_j = \delta_{ij} \mathbb{1}$. We will say that φ is a nondegenerate representation of \mathcal{O}_∞ , with a slight abuse of terminology, if φ is a representation with $\sum_{i \in \mathbb{Z}} \varphi(s_i s_i^*) = \mathbb{1}$, where the sum is in the strong operator topology.

2. REPRESENTATIONS OF \mathcal{O}_N ARISING IN MULTIREOLUTION WAVELETS IN SCALE N

Let \mathcal{H} be the Hilbert space $L^2(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the torus, and let $\{z^n\}_{n=-\infty}^\infty$ be the usual orthonormal basis of Fourier analysis: the convention is $z = e^{it}$, $t \in \mathbb{R}$.

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The Haar measure on \mathbb{T} will be normalized. Let functions $\{m_i\}_{i=0}^{N-1} \subset L^2(\mathbb{T})$ be given such that the corresponding $N \times N$ matrix

$$(2.1) \quad \left(m_k \left(e^{i\frac{2\pi l}{N}} z\right)\right)_{k,l=0}^{N-1}$$

is unitary. The representation of \mathcal{O}_N in question is defined from the function m_i as follows:

$$S_i \xi(z) = \sqrt{N} m_i(z) \xi(z^N),$$

where $S_i = \varphi(s_i)$. These representations were introduced in [Jor], and arise in the study of filter-banks in wavelet theory: see [Dau92], [CoRy], [JoPe94], and [JoPe96]. While not wavelets, the representations coming from (2.2) are closely related to so-called “tight frame” wavelet bases. For the wavelets, we must require the added so-called “low pass” condition $|m_0(1)| = 1$ [Dau92] which is of course not satisfied for $m_i(z) = N^{-\frac{1}{2}} z^{d_i}$. We review a simple case. Let $N \in \{2, 3, \dots\}$ and let $D = \{d_0, d_1, \dots, d_{N-1}\} \subset \mathbb{Z}$ be a set of integers such that any two distinct members are mutually incongruent modulo N . Then the operators

$$(2.2) \quad S_i: z^n \mapsto z^{d_i + Nn}$$

satisfy the Cuntz relations

$$(2.3) \quad S_i^* S_j = \delta_{ij} \mathbb{1} \quad \text{and} \quad \sum_{i=0}^{N-1} S_i S_i^* = \mathbb{1}.$$

These representations induce naturally a function system on the index set \mathbb{Z} of the orthonormal basis:

$$\begin{aligned} \sigma: \mathbb{Z} &\longrightarrow \mathbb{Z} & N \text{ to } 1 \text{ with} \\ \sigma_i: \mathbb{Z} &\longrightarrow \mathbb{Z} & \text{satisfying } \sigma \cdot \sigma_i = \text{id}_{\mathbb{Z}} \end{aligned}$$

for $i = 0, 1, \dots, N - 1$, or

$$(2.4) \quad S_i(z^n) = z^{\sigma_i(n)}$$

and

$$S_i^*(z^n) = z^{\sigma(n)}$$

for $z \in \mathbb{T}$. (See [BraJo97b], [JeongI].)

From (2.2), $\sigma_i, i = 0, 1, \dots, N - 1$, and σ , can be described as follows:

$$\begin{aligned} \sigma_i(n) &= Nn + d_i, \quad i = 0, 1, \dots, N - 1, \\ \sigma(n) &= m, \end{aligned}$$

where $m = \frac{n-d_{i_0}}{N}$ with unique $d_{i_0} \in \{d_0, d_1, \dots, d_{N-1}\}$ satisfying $n = d_{i_0}$ modulo N .

Following [BraJo97b], we define two equivalence relations \sim and \approx on \mathbb{Z} :

$n \sim m$ if there exist k_1 and k_2 such that $n = \sigma_{i_{k_1}} \cdots \sigma_{i_2} \sigma_{i_1} \sigma^{k_2} m$ for some $n, m \in \mathbb{Z}$ (equivalently $z^n = S_{i_{k_1}} \cdots S_{i_2} S_{i_1} S_{j_{k_2}}^* \cdots S_{j_2}^* S_{j_1}^* z^m$, $z \in \tilde{\mathbb{T}}$, where $\tilde{\mathbb{T}} = \{t \in \mathbb{T} \mid t = 0 \text{ modulo } 2\pi\}$); $n \approx m$ if furthermore $k_1 = k_2$.

Consider the gauge group (“the automorphism group”) of \mathcal{O}_N , denoted by $(r_z)_{z \in \mathbb{T}}$, which is determined by $r_z(S_i) = zS_i$, $z \in \mathbb{T}$. The subalgebra UHF_N of gauge-invariant elements is

$$\text{UHF}_N := \{a \in \mathcal{O}_N \mid r_z(a) = a \ \forall z \in \mathbb{T}\}.$$

It is a UHF algebra of Glimm type M_{n^∞} (see, e.g., [Cun77], [BJP], and [BraJo]). For general representations $\pi \in \text{Rep}(\mathcal{O}_N, \mathcal{H})$, the case when (UHF_N) is weakly*-dense in $\pi(\mathcal{O}_N)$ is studied in [BEEK] and [Pow88]. In fact, if we consider a separable Hilbert space \mathcal{H} with an orthonormal basis $\{e_n \mid n \in \mathbb{Z}\}$, then the corresponding representation in (2.2) or (2.4) can be written:

$$\begin{aligned} S_i(e_n) &= e_{Nn+d_i} \\ &= e_{\sigma_i(n)} \end{aligned}$$

and

$$S_i^*(e_n) = e_{\sigma(n)}.$$

Theorem 2.1 ([BraJo, Theorem 2.7]). *The closure of any subspace of \mathcal{H} spanned by vectors e_n , where n runs through a \sim -equivalence (\approx -equivalence) class, is an irreducible \mathcal{O}_N -module (UHF $_N$ -module, respectively).*

Proof. [BraJo]. □

Therefore, φ splits any representations given by (2.2) into a direct sum of irreducible mutually inequivalent representations of \mathcal{O}_N ,

$$\varphi = \sum_i^\oplus \varphi_i.$$

Theorem 2.2. *Suppose S is the index set of an orthonormal basis of a separable Hilbert space. If S has a radix-representation in base N with set $\{a_0, a_1, \dots, a_{N-1}\}$ of coefficients, i.e., for every $s \in S$ there exists a unique polynomial $P_s(N) = a_0 + a_1N + \dots + a_lN^l$ such that $P_s(N) = s$, then the representation coming from (2.2) splits into finitely many irreducible subrepresentations with $N < \infty$.*

Proof. Let the set $D = \{d_0, d_1, \dots, d_{N-1}\}$ be a residue set modulo N in S and representations be defined by

$$S_i(e_n) = e_{nN+d_i},$$

$i = 0, 1, \dots, N - 1$. For $s \in S$, define $\text{deg}(s) :=$ degree of s in the form of the polynomials in N . Note that $\sigma(s) = \frac{s-d_{i_0}}{N}$ for some $i_0 \in \{0, 1, \dots, N - 1\}$ such that $s = d_{i_0}$ modulo N . Thus $\text{deg}(\sigma(s)) < \text{deg}(s)$ if $\text{deg}(d_{i_0}) \leq \text{deg}(s)$, and $\text{deg}(s) < \text{deg}(\sigma(s))$ if $\text{deg}(s) < \text{deg}(d_{i_0}) - 1$ for every $i = 0, 1, \dots, N - 1$. Thus, for every $s \in S$ there exists a positive integer n_s such that

$$m - 1 \text{ (or } 0 \text{ if } m = 0) \leq \text{deg}(\sigma^k(s)) \leq M - 1,$$

for every integer $k \geq n_s$, where $m = \min \{\text{deg}(d_i) \mid i = 0, 1, \dots, N - 1\}$ and $M = \max \{\text{deg}(d_i) \mid i = 0, 1, \dots, N - 1\}$. Therefore, for every $s \in S$ we have an element $x \in B(p) := \{a_0 + a_1N + \dots + a_lN^l \mid a_l \neq 0, m - 1 \leq l \leq M - 1\}$, which implies the representation coming from (2.2) splits into at most $\#(B(p))$ irreducible subrepresentations. Since the cardinality of the set $B(p)$ is finite, this completes the proof. □

Corollary 2.3. *The representation of \mathcal{O}_N , $N < \infty$, on $\ell^2(\mathbb{Z})$ coming from (2.2) splits into finitely many irreducible subrepresentations.*

Proof. From Theorem 2.2 above, we need to build a radix-representation of \mathbb{Z} in base N . If $2 < N < \infty$, then take a digit set $\{a_0, a_1, \dots, a_{N-1}\} := \mathbb{Z} \cap \left(-\frac{N}{2}, \frac{N}{2}\right]$. With this digit set, any $s \in \mathbb{Z}$ has a unique radix-representation in N . (For details see [JeongII].) Now suppose $N = 2$. If we take a digit set $\{0, 1\}$, the argument in the proof of Theorem 2.2 holds for the index set $\mathbb{Z}_+ \cup \{0\}$. Thus the number of irreducible subrepresentations is at most $2 \times \#(B(p))$. \square

Remark 2.4. Not every representation of \mathcal{O}_N , either $N < \infty$ or $N = \infty$, splits into finitely many irreducible subrepresentations. For $N < \infty$, representations of \mathcal{O}_N on a separable Hilbert space $\ell^2(\mathbb{Z}^n)$ for $n > 1$ can have infinitely many irreducible subrepresentations. (See Chapter 3, Example 3.)

3. IRREDUCIBLE REPRESENTATIONS OF \mathcal{O}_N , $N < \infty$

We first consider representations of \mathcal{O}_N , $N < \infty$, on the Hilbert space $\ell^2(\mathbb{Z})$. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ ($\lceil x \rceil$) denotes the largest (smallest) integer such that $\lfloor x \rfloor \leq x$ ($\lceil x \rceil \geq x$, respectively). Extending a theorem in [BraJo], we have

Theorem 3.1. *For $N < \infty$, and the digit set $\{d_0, d_1, \dots, d_{N-1}\}$, the number of irreducible subrepresentations of the representation of \mathcal{O}_N resulting from (2.2) is at most*

$$1 + \left\lfloor -\frac{d_m}{N-1} \right\rfloor - \left\lceil -\frac{d_M}{N-1} \right\rceil,$$

where $d_m = \min \{d_i \mid i = 0, \dots, N-1\}$ and $d_M = \max \{d_i \mid i = 0, \dots, N-1\}$.

Corollary 3.2. *Using the digit set $\{d_0, d_1, \dots, d_{N-1}\}$, the representation of \mathcal{O}_N , $N < \infty$, from (2.2) is irreducible if one of the following conditions is fulfilled:*

1. $\left\lfloor -\frac{d_m}{N-1} \right\rfloor = \left\lceil -\frac{d_M}{N-1} \right\rceil$.
2. *There exists an integer n_0 such that*

$$\max_i |d_i - n_0| < N - 1.$$

3. $d_0 \neq 0$ modulo N and $d_i = d_0 + i$, $i = 0, 1, \dots, N-1$.

Proof. It is easy to see that if any one of the three conditions 1–3 holds, then $1 + \left\lfloor -\frac{d_m}{N-1} \right\rfloor - \left\lceil -\frac{d_M}{N-1} \right\rceil$ is equal to 1. Thus the representation coming from (2.2) is itself irreducible. \square

Proof of Theorem 3.1. If $x > -\frac{d_m}{N-1}$, $x \in \mathbb{R}$, then

$$Nx - x + d_k \geq (N-1) \left(x - \frac{-d_m}{N-1}\right) > 0, \quad k = 0, 1, \dots, N-1.$$

If $n > \left\lfloor -\frac{d_m}{N-1} \right\rfloor$, $n \in \mathbb{Z}$, then $n > -\frac{d_m}{N-1}$ and

$$\sigma_k(n) - n = Nn - n + d_k > 0, \quad k = 0, 1, \dots, N-1.$$

Thus $\sigma_k(n) > n$, $k = 0, 1, \dots, N-1$, for every integer $n > \left\lfloor -\frac{d_m}{N-1} \right\rfloor$. If we take a joint left inverse σ , we have

$$n = \sigma(\sigma_k(n)) > \sigma(n).$$

Similarly, if $n < \left\lceil -\frac{d_M}{N-1} \right\rceil$, then $n < \sigma(n)$. We have shown that for each $n \in \mathbb{Z}$ there is a nonnegative integer m_0 such that

$$\left\lceil -\frac{d_M}{N-1} \right\rceil \leq \sigma^k(n) \leq \left\lfloor -\frac{d_m}{N-1} \right\rfloor$$

for all $k \geq m_0$. For any $n \in \mathbb{Z}$ the sequence $\{\sigma^k(n)\}_{k=0}^\infty$ is eventually in the set $F := \left\{ \left\lceil -\frac{d_M}{N-1} \right\rceil, \dots, \left\lfloor -\frac{d_m}{N-1} \right\rfloor \right\}$. In other words, for $n \in \mathbb{Z}$, there is an element $f_0 \in F$ such that $n \sim f_0$. Hence the number of \sim -equivalence classes in \mathbb{Z} , or equivalently the number of irreducible subrepresentations of the representation of \mathcal{O}_N , $N < \infty$, coming from (2.2), is at most the cardinality of the set F , which is $1 + \left\lfloor -\frac{d_m}{N-1} \right\rfloor - \left\lceil -\frac{d_M}{N-1} \right\rceil$. \square

Thus every representation of \mathcal{O}_N , $N < \infty$, on $\ell^2(\mathbb{Z})$ defined by (2.2) decomposes into finitely many irreducible subrepresentations. In contrast, the representations of \mathcal{O}_N , $N < \infty$, on $\ell^2(\mathbb{Z}^n)$ have examples which have infinitely many irreducible subrepresentations. Let \mathbf{M} be an $n \times n$ integer matrix with $|\det \mathbf{M}| = N > 1$. Consider a digit set $\{d_0, d_1, \dots, d_{N-1}\}$, a complete residue set modulo $\mathbf{M}\mathbb{Z}^n$ in \mathbb{Z}^n . The representation of \mathcal{O}_N , $N < \infty$, on $\ell^2(\mathbb{Z}^n)$ that is defined by

$$(3.1) \quad S_i(e_x) = e_{\mathbf{M}x+d_i}, \quad i = 0, 1, \dots, N-1,$$

for $x \in \mathbb{Z}^n$ has a corresponding function system $\sigma_i(x) = \mathbf{M}x + d_i$ and $\sigma(x) = \mathbf{M}^{-1}(x - d_{i_0})$ for unique $d_{i_0} \in \{d_0, d_1, \dots, d_{N-1}\}$ such that $x = d_{i_0}$ modulo $\mathbf{M}\mathbb{Z}^n$ in \mathbb{Z}^n . Let U be the open closed hypercube $\prod_{i=1}^n (-\frac{1}{2}, \frac{1}{2}]$ in \mathbb{R}^n and $C = \{0\} \cup \{(0, \dots, 0, x_i, 0, \dots, 0) \mid x_i = -1 \text{ or } 1 \text{ for } i = 1, \dots, n\}$.

Theorem 3.3. *If an $n \times n$ matrix \mathbf{M} with integer entries satisfies $\lim_{k \rightarrow \infty} \mathbf{M}^k U = \mathbb{R}^n$ and $C \subset \mathbf{M}U \cap \mathbb{Z}^n$, then the representation of \mathcal{O}_N , $N < \infty$, on $\ell^2(\mathbb{Z}^n)$ defined by (3.1) using the digit set $D := \mathbf{M}y + (\mathbf{M}U \cap \mathbb{Z}^n)$, $y \in \mathbb{Z}^n$, is an irreducible representation.*

Proof. We first prove the case $y = 0 = (0, \dots, 0)$ in \mathbb{Z}^n . To this end, we construct a number system in \mathbb{Z}^n . Let $D_0^* := \mathbf{M}U \cap \mathbb{Z}^n$ and $D_1^* = \bigcup_{x \in D_0^*} \{\mathbf{M}x + d \mid d \in D_0^*\}$, and inductively

$$D_k^* = \bigcup_{x \in D_{k-1}^*} \{\mathbf{M}x + d \mid d \in D_0^*\}$$

for $k = 2, 3, \dots$. We have a strictly increasing sequence $\{D_k^*\}_{k=0}^\infty$, and furthermore $\mathbf{M}^k U \cap \mathbb{Z}^n$ is a subset of D_{k+1}^* for every $k = 0, 1, 2, \dots$. Since $\lim_{k \rightarrow \infty} \mathbf{M}^k U = \mathbb{R}^n$, we have $\lim_{k \rightarrow \infty} \mathbf{M}^k U \cap \mathbb{Z}^n = \mathbb{Z}^n$ and $\lim_{k \rightarrow \infty} D_k^* = \mathbb{Z}^n$. Now, for any given $x \in \mathbb{Z}^n$, there exists a unique positive integer l such that $x \in D_l^*$ and $x \notin D_{l-1}^*$. From the construction of the sequence $\{D_k^*\}_{k=0}^\infty$, there exist a unique $x_1 \in D_{l-1}^*$ and $d_l \in D_0^*$ such that $x = \mathbf{M}x_1 + d_l$. By induction on l , we can establish that

$$x = \mathbf{M}^l d_l + \mathbf{M}^{l-1} d_{l-1} + \dots + \mathbf{M}d_1 + d_0,$$

with $d_0, d_1, \dots, d_l \in D_0^*$. Thus every $x \in \mathbb{Z}^n$ has a polynomial form in base \mathbf{M} , where \mathbf{M} is an $n \times n$ integer matrix. Note that $\sigma(x) = \mathbf{M}^{l-1} d_l + \dots + \mathbf{M}d_2 + d_1$, and by induction, $\sigma^l(x) = d_l$. Therefore $\sigma^k(x) = 0$ for $k \geq l + 1$, where $0 = (0, \dots, 0)$ in \mathbb{Z}^n , i.e., every $x \in \mathbb{Z}^n$ is \sim -equivalent to 0. By Theorem 2.1, the representation of \mathcal{O}_N , $N < \infty$, on $\ell^2(\mathbb{Z}^n)$ is irreducible if $y = 0$.

Now we assume $y \neq 0$ and $\mathbf{M}y = \mathbf{M}^t a_t + \cdots + \mathbf{M}a_1$, so that every $d_i \in D$ can be expressed as

$$d_i = \mathbf{M}^t a_t + \cdots + \mathbf{M}a_1 + a_{0i},$$

where $a_{0i} \in D_0^*$, $i = 0, 1, \dots, N - 1$. Since

$$M = m = t$$

with

$$M = \max \{ \deg(d_i) \mid d_i \in D, i = 0, 1, \dots, N - 1 \},$$

$$m = \min \{ \deg(d_i) \mid d_i \in D, i = 0, 1, \dots, N - 1 \},$$

for every $x \in \mathbb{Z}^n$, there exists a positive integer n_x such that $\deg(\sigma^k(x)) = t - 1$ for every $k \geq n_x$. For $x \in \mathbb{Z}^n$ with $\deg(x) = t - 1$, we may express x as $x = \mathbf{M}^{t-1}b_{t-1} + \cdots + \mathbf{M}b_1 + b_0$, where $b_0, b_1, \dots, b_{t-1} \in D_0^*$. Then

$$\begin{aligned} \sigma(x) &= \mathbf{M}^{-1}(x - d_{i_0}) \\ &= -\mathbf{M}^{t-1}a_t + \mathbf{M}^{t-2}(b_{t-1} - a_{t-1}) + \cdots + b_1 - a_1, \end{aligned}$$

where $d_{i_0} = \mathbf{M}^t a_t + \cdots + \mathbf{M}a_1 + a_0$ with $a_0 = b_0$. Therefore $\sigma^k(x) = -\mathbf{M}^{t-1}a_t - \mathbf{M}^{t-2}a_t - \cdots - \mathbf{M}a_t - a_t$ for large k . As a consequence, every $x \in \mathbb{Z}^n$ is \sim -equivalent to the point $-\mathbf{M}^{t-1}a_t - \mathbf{M}^{t-2}a_t - \cdots - \mathbf{M}a_t - a_t$, which implies that every representation of \mathcal{O}_N , $N < \infty$, of the type described is an irreducible representation. \square

Corollary 3.4. *If an $n \times n$ matrix \mathbf{M} with integer entries satisfies $\lim_{k \rightarrow \infty} \mathbf{M}^k U = \mathbb{R}^n$ and $C \subset \mathbf{M}U \cap \mathbb{Z}^n$, then the representation defined by (3.1) using any digit set D which is a residue set modulo $\mathbf{M}\mathbb{Z}^n$ in \mathbb{Z}^n decomposes into finitely many irreducible subrepresentations.*

Proof. The proof follows from Theorem 2.2 and the first half of the proof of Theorem 3.3. \square

Remark 3.5. If the set C is not a subset of $\mathbf{M}U \cap \mathbb{Z}^n$, the element x_1 is not uniquely determined in the proof of Theorem 3.3, but the representation coming from (3.1) still has a finite irreducible decomposition.

Theorem 3.6. *If an $n \times n$ integer matrix \mathbf{M} satisfies the condition $\lim_{k \rightarrow \infty} \mathbf{M}^k U = \mathbb{R}^n$, then the representation coming from (3.1) with digit set $D := \mathbf{M}U \cap \mathbb{Z}^n$ decomposes into finitely many irreducible subrepresentations. In fact, there are at most $2n + 1$ such irreducible subrepresentations. Loosely speaking, if $\lim_{k \rightarrow \infty} \mathbf{M}^k U = \mathbb{R}^n$, the corresponding representation with any residue set D modulo $\mathbf{M}\mathbb{Z}^n$ as a digit set decomposes into finitely many irreducible subrepresentations.*

Proof. Let us define two sequences $\{G_k\}_{k=0}^\infty$ and $\{H_k\}_{k=0}^\infty$ of sets in \mathbb{R}^n and \mathbb{Z}^n , respectively. Let

$$G_0 = 0 = H_0 \quad \text{in } \mathbb{R}^n,$$

$$G_1 = \left(\bigcup_{x \in D} (x + U) \right) \cup \left(\bigcup_{x \in C - D} (x + U) \right) \quad \text{and} \quad H_1 = G_1 \cap \mathbb{Z}^n,$$

and then, inductively,

$$G_k = \bigcup_{x \in H_{k-1}} \{ \sigma_i(x) + U \mid i = 0, 1, \dots, N - 1 \} \quad \text{and} \quad H_k = G_k \cap \mathbb{Z}^n,$$

for $k = 2, 3, \dots$. We then see that $\mathbf{M}^k U$ is a subset of G_k for every k . Since $\lim_{k \rightarrow \infty} \mathbf{M}^k U = \mathbb{R}^n$, it follows that $\lim_{k \rightarrow \infty} G_k = \mathbb{R}^n$ and $\lim_{k \rightarrow \infty} H_k = \mathbb{Z}^n$. For any $x \in \mathbb{Z}^n$, there is a positive integer k such that $x \in H_l$ for $l \geq k$ and $x \notin H_{k-1}$. Because the sequence $\{H_k\}_{k=0}^\infty$ has been constructed by an iteration starting with the set H_1 , the sequence $\sigma^t(x)$, $x \in \mathbb{Z}^n$, eventually converges in the set H_1 . However, every $x \in D \subset H_1$ is \sim -equivalent to 0 by construction of H_1 , so D is a subset of the 0 \sim -equivalence class. Thus the number of irreducible subrepresentations is at most $1 + \#(C - D)$. Since $\#C = 2n + 1$ and $\#D = N$, if $D \subset C$, then $1 + \#(C - D) \leq 1 + (2n + 1) - N = 2n + 2 - N$. In general, the number of irreducible subrepresentations is at most $1 + \#(C - D) \leq 2n + 1$. The last statement follows from the proof of Theorem 2.2 using the set H_1 instead of the set $\{a_0, a_1, \dots, a_{N-1}\}$, with the necessary changes. This completes the proof. \square

Example 1 (of Theorem 3.3). Let $\mathbf{M} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ with $N = 5$ and $D := \mathbf{M}U \cap \mathbb{Z}^2 = \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. Here $D = C$ and $x \sim 0$ for every $x \in \mathbb{Z}^2$. The representation of \mathcal{O}_5 defined by (3.1) on $\ell^2(\mathbb{Z}^2)$ is irreducible.

Example 2 (of Theorem 3.6). Let $\mathbf{M} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ with $N = 3$ and $D := \mathbf{M}U \cap \mathbb{Z}^2 = \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$. Note that both eigenvalues of \mathbf{M} have modulus greater than 1, so $\lim_{k \rightarrow \infty} \mathbf{M}^k U = \mathbb{R}^2$, but $C \subsetneq D$. In fact, for any $x \in \mathbb{Z}^2$ only one of the following is true:

$$\text{either } x \sim 0 \text{ or } x \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ or } x \sim \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

The number of irreducible subrepresentations is $3 = 2n + 2 - N$ with $n = 2$ and $N = 3$.

Example 3 (of having infinitely many irreducible subrepresentations). Let $\mathbf{M} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$ with $N = 4$ and $D := \mathbf{M}U \cap \mathbb{Z}^2 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$. Note that the eigenvalues of \mathbf{M} are 4 and 1, which implies that $\lim_{k \rightarrow \infty} \mathbf{M}^k U$ is a proper subset of \mathbb{R}^2 . Since $\begin{pmatrix} z \\ -z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ modulo $\mathbf{M}\mathbb{Z}^2$ in \mathbb{Z}^2 , $\sigma \begin{pmatrix} z \\ -z \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}^{-1} \left(\begin{pmatrix} z \\ -z \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} z \\ -z \end{pmatrix}$. Thus the \sim -equivalence classes coming from $\begin{pmatrix} z \\ -z \end{pmatrix}$, $z \in \mathbb{Z}$, are mutually disjoint. Thus, the representation of \mathcal{O}_4 defined by (3.1) on $\ell^2(\mathbb{Z}^2)$ has countably many irreducible subrepresentations.

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