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ENTROPY ESTIMATES FOR SOME C*-ENDOMORPHISMS

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Abstract. In this paper we compute the non-commutative topological entropy in the sense of Voiculescu for some endomorphisms of stationary inductive limits of circle algebras. These algebras are groupoid C*-algebras, and the endomorphisms restricted to the canonical diagonal are induced by some expansive maps, whose entropies provide a lower bound. For the upper bound, we use a result of Voiculescu, similar to the classical Kolmogorov-Sinai theorem. The same technique is used to compute the entropy of a non-commutative Markov shift.

Introduction

The notion of entropy plays an important role in ergodic theory. In recent years, this notion has been extended to automorphisms of operator algebras, using different approaches. For an introduction to this field, see the expository article of Størmer, [St].

In this paper, we compute the topological entropy in the sense of Voiculescu for some non-commutative shifts on stationary inductive limits of circle algebras. Although the definitions and the main results in section 4 of [Vo] are stated for the case of a unital automorphism of a nuclear C*-algebra, these could be applied (as Voiculescu mentions in 9.4 of his paper) to endomorphisms as well. We will recall the statements that we need, and we will sketch the proofs adapted to the case of endomorphisms.

This notion of entropy is based on the idea of growth. It coincides with the "classical" topological entropy of Adler, Konheim, and McAndrew in the case of abelian C*-algebras. The C*-algebras we consider are groupoid C*-algebras, and the endomorphisms restricted to the canonical diagonal are induced by some expansive maps of the unit space for which the topological entropy is known. This provides a lower bound for the entropy of the endomorphism. For the upper bound, we use a result of Voiculescu similar to the Kolmogorov-Sinai theorem which, in the classical case, computes the entropy by using a generator (see Theorem 4.17 in [Wa]).

The same technique is used to compute the entropy of a non-commutative Markov shift.

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1. Topological entropy

For a nuclear unital C*-algebra A, the set of completely positive approximations is defined as

$$cpa(A) := \{(\varphi, \psi, D) \mid \ D \text{ a finite dimensional C*-algebra}, \\ \varphi : A \to D,\ \psi : D \to A \text{ unital completely positive maps}\}.$$

For $\omega \in \mathcal{P}f(A)$ (the finite parts of A) and $\delta > 0$, the completely positive rank of the pair (ω, δ) is defined as

$$rcp(\omega, \delta) = \inf\{rankD \mid \ (\varphi, \psi, D) \in cpa(A), \| (\psi \circ \varphi)(a) - a \| < \delta \ \forall a \in \omega\}.$$

Here, by $rankD$ the dimension of a maximal abelian subalgebra of D is understood.

1.1. Definition. For a nuclear C*-algebra A and for a unital endomorphism $\alpha \in End(A)$, we define the topological entropy $ht(\alpha)$ as follows:

$$ht(\alpha, \omega; \delta) = \lim \sup_{n\to\infty} \frac{1}{n} \log rcp(\bigcup_{j=0}^{n-1} \alpha^j(\omega), \delta),$$

$$ht(\alpha, \omega) = \sup_{\delta>0} ht(\alpha, \omega; \delta),$$

$$ht(\alpha) = \sup_{\omega} ht(\alpha, \omega).$$

Note that $ht(\alpha, \omega; \delta)$ is the growth rate of the "rank" of the "finite piece of orbit" $\omega \cup \alpha(\omega) \cup ... \cup \alpha^{n-1}(\omega)$ with respect to δ.

1.2. *Remark.* This definition makes sense also for a non-unital endomorphism, or just for a completely positive map $\alpha : A \to A$.

The main ingredients for computing the topological entropy are the following three propositions. The first one is similar to the classical Kolmogorov-Sinai theorem.

1.3. Proposition. *Let $\omega_j \in \mathcal{P}f(A), j \geq 1,\ \omega_1 \subset \omega_2 \subset ...,$ be such that the linear span of $\bigcup_{j,k} \alpha^k(\omega_j)$ is dense in A. Then*

$$ht(\alpha) = \sup_j ht(\alpha, \omega_j).$$

Proof. Let $\omega \in \mathcal{P}f(A), \omega = \{a_1, ..., a_m\}$, and $\delta > 0$. By hypothesis, there are $N \geq 1$ and $p \geq 1$ so that if $\bigcup_{k\leq p} \alpha^k(\omega_N) = \{x_1, ..., x_n\}$, then

$$\| a_i - \sum_{1\leq j\leq n} \lambda_{ij} x_j \| < \delta,\ 1 \leq i \leq m,$$

for some $\lambda_{ij} \in \mathbf{C}$. With $K = \max_i \sum_j | \lambda_{ij} |$, it follows from the triangle inequality that

$$rcp(\omega \cup ... \cup \alpha^q(\omega); 3\delta) \leq rcp(\bigcup_{s\leq q+p} \alpha^s(\omega_N); K^{-1}\delta),$$

which implies

$$ht(\alpha, \omega; 3\delta) \leq ht(\alpha, \omega_N; K^{-1}\delta), \quad ht(\alpha) \leq \sup_j ht(\alpha, \omega_j).$$

The opposite inequality is obvious from the definition.

1.4. Proposition. *Let $B \subset A$ be a unital C*-subalgebra with $\alpha(B) \subset B$ and such that there is a conditional expectation $E : A \to B$. Then $ht(\alpha\mid_B) \leq ht(\alpha)$.*

Proof. For $\omega \in \mathcal{P}f(B), rcp(\omega;\delta)$ is the same with respect to A or B. Indeed, if $(\varphi, \psi, D) \in cpa(A)$, then $(\varphi\mid_B, E\circ\psi, D) \in cpa(B)$ and

$$\| (E\circ\psi)(\varphi\mid_B)(a) - a \| = \| E((\psi\circ\varphi)(a) - a) \| \leq \| (\psi\circ\varphi)(a) - a \|$$

for $a \in B$.

1.5. Proposition. *If $T : X \to X$ is continuous, X is a compact metric space, $A = C(X)$ and $\alpha(f) = f\circ T$, then*

$$ht(\alpha) = h_{top}(T).$$

Proof. The proof is the same as for Proposition 4.8 in [Vo], modulo the fact that the Connes-Narnhofer-Thirring entropy could be defined for endomorphisms, and it coincides with the classical one for abelian C*-algebras.

1.6. Example (The non-commutative Bernoulli shift)**.** Let $A = UHF(2^\infty) = \varinjlim \mathbf{M}_{2^n}$, and let $\alpha : A \to A$,

$$\alpha(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

be the (non-commutative) unilateral Bernoulli shift. Then $ht(\alpha) = \log 2$.

Indeed, let $B = C(X) \subset A$, $X = \{1,2\}^{\mathbf{N}}$ the Cantor set. Then $\alpha\mid_B$ is induced by the usual Bernoulli shift, which has the topological entropy $\log 2$. Hence, $ht(\alpha) \geq \log 2$. On the other hand, let τ_j be the set of matrix units in $A_j = \mathbf{M}_{2^j} \subset A$, and let $\omega_j = \tau_1 \cup ... \cup \tau_j$. It is clear that the linear span of $\bigcup_{k,j} \alpha^k(\omega_j)$ is dense in A. For fixed j and $\delta > 0$, we have $rcp(\omega_j \cup \alpha(\omega_j) \cup ... \cup \alpha^{m-1}(\omega_j);\delta) \leq 2^{j+m-1}$, since $\alpha(A_j) \subset A_{j+1}$. Here we use some conditional expectations $A \to A_j$ and the inclusions $A_j \hookrightarrow A$. Hence $ht(\alpha) \leq \log 2$, and we have equality. For the bilateral version, see Proposition 4.7 in [Vo].

1.7. Example (The non-commutative Markov shift)**.** Let Λ be an irreducible $0-1$ square matrix and let $A = AF(\Lambda)$ be the stationary AF-algebra defined by Λ. Consider the non-commutative unilateral Markov shift $\alpha : A \to A$. Then $ht(\alpha) = \log\rho(\Lambda)$, where $\rho(\Lambda)$ is the spectral radius of Λ.

The inequality $ht(\alpha) \geq \log\rho(\Lambda)$ follows from the fact that $\alpha\mid_{C(X)}$ is induced by the Markov shift $\sigma : X \to X$, which has topological entropy $\log\rho(\Lambda)$. Here X denotes the usual Cantor set associated with the $0-1$ matrix Λ. For the other inequality, let $A = \varinjlim A_j$, where A_j are the canonical finite dimensional C*-algebras. Using again a set τ_j of matrix units in A_j, $\omega_j = \tau_1 \cup ... \cup \tau_j$, the inclusions $A_j \hookrightarrow A$, and some conditional expectations $A \to A_j$, we have

$$rcp(\omega_j \cup \alpha(\omega_j) \cup ... \cup \alpha^{m-1}(\omega_j);\delta) \leq rank(A_{j+m-1}),$$

for fixed j and $\delta > 0$. But $rank(A_{j+m-1}) = (1,...,1)\Lambda^{j+m-1}(1,...,1)^t$, and it is known that

$$\lim_{m\to\infty} \frac{1}{m} \log((1,...,1)\Lambda^m(1,...,1)^t) = \log\rho(\Lambda).$$

2. Main result

In this section we will compute the topological entropy of an endomorphism of a generalized Bunce-Deddens algebra. This endomorphism could be viewed as a "continuous" version of the non-commutative Bernoulli shift, since the UHF-algebra is replaced by a stationary A**T**-algebra, and the Cantor set by a Cartesian product between a circle and a Cantor set.

More precisely, let $p_1, ..., p_k$ be some integers with $|p_j| \geq 2$, let $p = |p_1| + ... + |p_k|$, and consider the particular embedding

$$\Phi : C(\mathbf{T}) \to C(\mathbf{T}) \otimes \mathbf{M}_p, \Phi = \begin{pmatrix} \hat{\sigma}_{p_1} & & \\ & \ddots & \\ & & \hat{\sigma}_{p_k} \end{pmatrix}.$$

Here $\hat{\sigma}_q$ denotes the q-times around embedding,

$$\hat{\sigma}_q : C(\mathbf{T}) \to C(\mathbf{T}) \otimes \mathbf{M}_{|q|}, \hat{\sigma}_q(z) = \begin{pmatrix} 0 & 0 & ... & z \\ 1 & 0 & ... & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & ... & 1 & 0 \end{pmatrix},$$

for q positive, with z replaced by $\bar{z}$ for q negative.

2.1. Theorem. *Let $A = \varinjlim A_n$ be the stationary A**T**-algebra defined by the particular embedding Φ. Let $\alpha : A \to A$ be the non-commutative shift induced by the maps $\Phi : A_n \to A_{n+1}$. Then*

$$ht(\alpha) = \log p.$$

In particular, for $k = 1$, we obtain the topological entropy of the unilateral shift on a Bunce-Deddens algebra.

Proof. The C*-algebra A has a canonical diagonal $C(X)$, where $X = \mathbf{T} \times \{1, 2, ..., k\}^{\mathbf{N}}$ could be viewed as the space of infinite paths in a diagram (see [De]). The endomorphism $\alpha : A \to A$ restricted to $C(X)$ is induced by the unilateral shift

$$\sigma : X \to X, \sigma(z, x_1 x_2 ...) = (z^{p_{x_1}}, x_2 x_3 ...).$$

The compact space X has a natural metric

$$d((z, x), (w, y)) = d_e(z, w) + \sum_n \frac{1}{2^n} \mid x_n - y_n \mid,$$

where d_e is the usual metric on the unit circle. The map σ is expansive with respect to this metric, and by Theorem 8.16 in [Wa],

$$h_{top}(\sigma) \geq \lim_{n \to \infty} \frac{1}{n} \log \theta_n(\sigma),$$

where

$$\theta_n(\sigma) = \#\{x \in X \mid \sigma^n(x) = x\} = \sum_{1 \leq i_l \leq k} (\mid p_{i_1} ... p_{i_n} \mid -1).$$

It follows that

$$h_{top}(\sigma) \geq \lim_{n \to \infty} \frac{1}{n} \log(p^n - k^n) = \log p.$$

Since there is a conditional expectation $E : A \to C(X)$, and $\alpha(C(X)) \subset C(X)$, we have

$$ht(\alpha) \geq ht(\alpha \mid_{C(X)}) = h_{top}(\sigma).$$

For the other inequality, let $\{e_{ij}^n\}_{ij}$ be matrix units in $\mathbf{M}_{p^n}$, and

$$\omega_n = \{z^q \mid -n \leq q \leq n\} \otimes \bigcup_{k \leq n} \{e_{ij}^k\}_{ij},$$

viewed as a finite subset of $A_n = C(\mathbf{T}) \otimes \mathbf{M}_{p^n}$. It follows that $\omega_n \subset \omega_{n+1}$ and $\bigcup_{k,n \geq 0} \alpha^k(\omega_n)$ spans a dense set in A. Let's fix some conditional expectations $E_n : A \to A_n$ and let's denote by i_n the inclusions $A_n \hookrightarrow A$. For fixed $n \geq 1$ and $\delta > 0$, let $z_1, ..., z_m \in \mathbf{T}$ be some points dividing $\mathbf{T}$ in m equal parts of length $< \delta/n$, and let $Z_m = \{z_1, ..., z_m\}$. Consider $\varphi_n : A \to C(Z_m) \otimes \mathbf{M}_{p^n}$ to be the completely positive map obtained by composing the restriction $C(\mathbf{T}) \otimes \mathbf{M}_{p^n} \to C(Z_m) \otimes \mathbf{M}_{p^n}$ with E_n. Let $\{\chi_j\}_{1 \leq j \leq m}$ be a partition of unity with $supp \chi_j$ contained in an open arc V_j of length $2\pi/m$ centered at z_j, and let $\psi_n : C(Z_m) \otimes \mathbf{M}_{p^n} \to A$ be the completely positive map given by

$$\psi_n(g) = i_n(\sum_j g(z_j)\chi_j).$$

Now $\| (\psi_n \circ \varphi_n)(a) - a \| < \delta$ for all $a \in \omega_n$. Indeed, let $a = z^q \otimes e_{ij}^k$ with $-n \leq q \leq n$ and some fixed i, j. Then

$$\| (\psi_n \circ \varphi_n)(z^q \otimes e_{ij}^k) - z^q \otimes e_{ij}^k \| = \| \sum_l z_l^q \chi_l \otimes e_{ij}^k - z^q \otimes e_{ij}^k \| < \sup_{z \in V_l} \mid z_l^q - z^q \mid$$
$$< \mid q \mid \delta/n \leq \delta.$$

Hence $rcp(\omega_n; \delta) \leq mp^n$. Since $\alpha(A_n) \subset A_{n+1}$ and for $a \in \omega_n$, the degree of z appearing in $\alpha(a)$, in absolute value, does not exceed n, we have

$$rcp(\bigcup_{l=0}^{q-1} \alpha^l(\omega_n); \delta) \leq mp^{q+n-1},$$
$$ht(\alpha, \omega_n; \delta) \leq \log p,$$
$$ht(\alpha) \leq \log p.$$

2.1. **Comments.** It is interesting to note that, in our example, the topological entropy coincides with the logarithm of the map induced by Φ on K_0-theory. It is known that, in the commutative settings, the topological entropy of a map is related to the maps induced on homotopy, homology, etc., so that a natural question is how the non-commutative entropy could be related to K-theory.

Another question is: when the topological entropy of an endomorphism is finite? It is desirable to have a notion of *expansive endomorphism*, like in the abelian situation. Recall that a map $T : X \to X$ is expansive iff it has a generator, i.e. a finite open cover $\mathcal{A}$ of X such that for every bisequence $\{A_n\}_{-\infty}^{\infty}$ of members of $\mathcal{A}$, the set

$$\bigcap_{n=-\infty}^{\infty} T^{-n} \bar{A}_n$$

contains at most one point of X (see [Wa], p. 139). Proposition 1.3 suggests that a possible definition for an expansive endomorphism $\alpha : A \to A$ should take into account the case when there is a finite subset ω of A such that the linear span of $\bigcup_k \alpha^k(\omega)$ is dense in A.

Recently we learned that S. Friedland studied the entropy of a shift on a space associated to a continuous graph (see [Fr]). For example, if the continuous graph is the union of the graphs of the maps $z \mapsto z^p$ and $z \mapsto z^q$ on the unit circle, he proved that the entropy is $\leq \log(p+q)$, and conjectured that the equality holds.

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