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OPENNESS AND MONOTONEITY OF INDUCED MAPPINGS

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Abstract. It is shown that for locally connected continuum X if the induced mapping $C(f) : C(X) \to C(Y)$ is open, then f is monotone. As a corollary it follows that if the continuum X is hereditarily locally connected and $C(f)$ is open, then f is a homeomorphism. An example is given to show that local connectedness is essential in the result.

All spaces considered in this paper are assumed to be metric. A *mapping* means a continuous function. We denote by $\mathbb{N}$ the set of all positive integers, and by $\mathbb{C}$ the complex plane. Given a space S, a point $c \in S$ and a number $\varepsilon > 0$, we denote by $B_S(c, \varepsilon)$ the open ball in S with center c and radius ε.

A *continuum* means a compact connected space. Given a continuum X with a metric d, we let 2^X denote the hyperspace of all nonempty closed subsets of X equipped with the Hausdorff metric H defined by

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\}$$

(see, e.g., [5, (0.1), p. 1 and (0.12), p. 10]). Further, we denote by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X, and by $F_1(X)$ the hyperspace of singletons. The reader is referred to Nadler's book [5] for needed information on the structure of hyperspaces.

Given a mapping $f : X \to Y$ between continua X and Y, we consider mappings (called the *induced* ones)

$$2^f : 2^X \to 2^Y \quad \text{and} \quad C(f) : C(X) \to C(Y)$$

defined by

$$2^f(A) = f(A) \text{ for every } A \in 2^X \quad \text{and} \quad C(f)(A) = f(A) \text{ for every } A \in C(X).$$

A mapping between continua is said to be:

— *open* provided the image of an open subset of the domain is open in the range;
— *monotone* provided the point-inverses are connected;
— *light* provided the point-inverses are zero-dimensional.

The following theorem is the main result of this paper.

1. Theorem. *Let a continuum X be locally connected, and a mapping $f : X \to Y$ be such that the induced mapping $C(f) : C(X) \to C(Y)$ is open. Then f is monotone.*

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Proof. Assume f satisfies the assumptions of the theorem and that it is not monotone. Let p and q be two points of X such that $f(p) = f(q)$ that belong to different components of $f^{-1}(f(p))$. By continuity of f there is a positive ε such that for every continuum $L \subset Y$ such that $f(p) \in L$ and $H(L, \{f(p)\}) < \varepsilon$ the components of $f^{-1}(L)$ containing p and q respectively are distinct. By local connectedness of Y there is a continuum V such that $f(p) \in \operatorname{int} V$ and $H(V, \{f(p)\}) < \varepsilon$, i.e., $V \subset B_Y(f(p), \varepsilon)$. Let U_p and U_q be components of $f^{-1}(V)$ containing p and q respectively. Since in locally connected continua components of open sets are open [4, §49, II, Theorem 4, p. 230], we conclude that $p \in \operatorname{int} U_p$ and $q \in \operatorname{int} U_q$. Let $\delta > 0$ be such that $B_X(p, \delta) \subset U_p$ and $B_X(q, \delta) \subset U_q$.

Let $\mathcal{B}$ be an order arc in $C(Y)$ from $\{f(p)\}$ to Y through V. Define $\mathcal{A}$ as a subset of $\mathcal{B}$ composed of all elements $L \in \mathcal{B}$ such that the component of $f^{-1}(L)$ containing p is distinct from the component of $f^{-1}(L)$ containing q. Note that $V \in \mathcal{A}$ and that if $L, L' \in \mathcal{B}$, $L \in \mathcal{A}$ and $L' \subset L$, then $L' \in \mathcal{A}$. Thus $\mathcal{A}$ is a connected subset of $\mathcal{B}$ containing $\{f(p)\}$ and V. Since $\mathcal{B} \setminus \mathcal{A}$ is closed, we see that $\mathcal{A}$ is an open subset of $\mathcal{B}$. Let $Q = \sup \mathcal{A} = \inf(\mathcal{B} \setminus \mathcal{A})$. Then $Q \in \operatorname{cl} \mathcal{A} \setminus \mathcal{A}$. Denote by P the component of $f^{-1}(Q)$ containing both p and q. Openness of $C(f)$ implies that f is open (see [3, Theorem 4.3, p. 243]; compare also [2, Theorem 3.2]), so $f(P) = Q$ [6, (7.5), p. 148]. We will show that $C(f)(B_{C(X)}(P, \delta))$ is not open in $C(Y)$. So, assume the contrary. Then there is a continuum $K \in B_{C(X)}(P, \delta)$ with $f(K) \in \mathcal{A}$. Since $p, q \in P$ and $H(P, K) < \delta$, we have $K \cap U_p \neq \emptyset \neq K \cap U_q$. Then $U_p \cup K \cup U_q$ is a continuum containing both p and q, whose image $f(U_p \cup K \cup U_q) = f(K)$ is in $\mathcal{A}$, contrary to the definition of $\mathcal{A}$. The proof is finished. □

2. Corollary. *Let a continuum X be hereditarily locally connected, and a mapping $f : X \to Y$ be such that the induced mapping $C(f) : C(X) \to C(Y)$ is open. Then f is a homeomorphism.*

Proof. It is enough to show that monotone open mappings on hereditarily locally connected continua are homeomorphisms. Assume the contrary, and let $y \in Y$ be such that $f^{-1}(y)$ is a nondegenerate continuum in X. Let $\{y_n\}$ be an arbitrary sequence converging to y. Then continua $f^{-1}(y_n)$ tend to $f^{-1}(y)$, so $f^{-1}(y)$ is a nondegenerate continuum of convergence, contrary to hereditary local connectedness of X. □

3. Example. There are a continuum X and a mapping $f : X \to X$ such that $C(f) : C(X) \to C(X)$ is light and open, but not monotone.

Proof. Let $S = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. For $n \in \mathbb{N}$ put $X_n = S$, and let $\varphi_n : X_{n+1} \to X_n$ be defined by $\varphi_n(z) = z^3$. Then $X = \varprojlim(X_n, \varphi_n)$ is the triodic solenoid. Define $f : X \to X$ by $f(\{z_1, z_2, \dots\}) = \{z_1^2, z_2^2, \dots\}$, and note that f is well-defined. It has been proved in [1, Example 4.5] that the restriction $C(f)|(C(X) \setminus \{X\})$ is two-to-one and $C(f)^{-1}(X)$ is a singleton. Thus $C(f)$ is light and it is not a homeomorphism. We will prove that $C(f)$ is open. To this aim it is enough to show that the mapping is interior at each point of its domain [6, p. 149], i.e., that for each $P \in C(X)$ and for each open neighborhood $\mathcal{U}$ of P in $C(X)$ we have $C(f)(P) \in \operatorname{int} C(f)(\mathcal{U})$. For each $n \in \mathbb{N}$ let $f_n : X_n \to X_n$ be defined by $f_n(z) = z^2$ (and thus $f = \varprojlim f_n$), and let $\pi_n : X \to X_n$ be the projection. Let $P \in C(X)$ be a proper subcontinuum of X. Then there exists an index $n \in \mathbb{N}$ such that $\pi_{n-1}(P)$ is a proper subcontinuum of X_{n-1}, so $\pi_n(P)$ is an arc of length less than $2\pi/3$. Let U_n be an open arc in X_n containing $\pi_n(P)$ and having its length still less than $2\pi/3$. Then the set $\mathcal{V} = \{A \in C(X) : \pi_n(A) \in U_n\}$ is an open neighborhood of P in X such that the restriction $C(f)|\mathcal{V} : \mathcal{V} \to C(f)(\mathcal{V})$ is a homeomorphism onto the open set $C(f)(\mathcal{U}) = \{A \in C(X) : \pi_n(A) \in f_n(U_n)\}$ containing $C(f)(P)$. So interiority of $C(f)$ at P is shown in the case $P \neq X$. To prove that $C(f)$ is interior at X consider, for $n \in \mathbb{N}$, the sets $\mathcal{V}_n = \{A \in C(X) : \pi_n(A) = X_n\}$ and note that the family $\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a local base of (closed) neighborhoods of X on $C(X)$. So, it is enough to prove that $C(f)(\mathcal{V}_n) \supset \mathcal{V}_{n+1}$. To this end take $A \in \mathcal{V}_{n+1}$, and let $B \in X$ be such that $f(B) = A$. Since

$$f_{n+1}(\pi_{n+1}(B)) = \pi_{n+1}(f(B)) = \pi_{n+1}(A) = X_{n+1},$$

we see that $\pi_{n+1}(B)$ is an arc in X_{n+1} of length at least π. Thus $\pi_n(B) = \varphi_n(\pi_{n+1}(B)) = X_n$, i.e., $B \in \mathcal{V}_n$, whence it follows that $A = f(B) \in C(f)(\mathcal{V}_n)$. The proof is then complete. □

In connection with Theorem 1 and Example 3 it would be interesting to know if a stronger result is true, namely whether or not the conclusion of Theorem 1 can be deduced from local connectedness of Y only (without assuming local connectedness of X). In other words we have the following question.

4. Question. Can the assumption of local connectedness of the domain continuum X be relaxed to that of the range continuum Y in Theorem 1?

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