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CONVOLUTION WITH AFFINE ARCLENGTH MEASURES IN THE PLANE

DANIEL M. OBERLIN

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ABSTRACT. We obtain an estimate for the $L^{3/2,1}(\mathbb{R}^2) - L^3(\mathbb{R}^2)$ norm of a certain convolution operator.

Let ϕ be real-valued and smooth on an interval $(a,b) \subseteq \mathbb{R}$. Define the measure λ on $\mathbb{R}^2$ by

$$\int_{\mathbb{R}^2} f \; d\lambda = \int_a^b f(t,\phi(t)) \; |\phi''(t)|^{1/3} dt.$$

We are interested in the $L^p(\mathbb{R}^2) - L^q(\mathbb{R}^2)$ mapping properties of the operator given by convolution with λ. The study of this operator was initiated by Drury ([D]), who used complex interpolation and certain integral estimates to obtain the optimal result

$$\|\lambda * f\|_3 \leq C \; \|f\|_{3/2}$$

when ϕ satisfies certain regularity conditions. Drury's approach was refined and extended by Choi in [C]. The purpose of this note is to prove an almost-sharp estimate for a large class of functions ϕ:

Theorem. *Suppose $\phi'' > 0$ and $\phi^{(3)} \geq 0$ on (a,b). Then*

$$\|\lambda * \chi_E\|_3 \leq 12^{1/3} \; \|\chi_E\|_{3/2} \tag{1}$$

for any measurable $E \subseteq \mathbb{R}^2$.

Proof. Define $\gamma(s) = \big(s, \phi(s)\big)$. Inequality (1) is equivalent to

$$\int_{\mathbb{R}^2} \int_a^b \int_a^b \int_a^b \prod_1^3 \Big[\chi_E\big(x - \gamma(s_i)\big)\phi''(s_i)^{1/3}\Big] \; ds_1 \; ds_2 \; ds_3 \; dx \leq 12 \; |E|^2$$

and so will follow from

$$\int_{\mathbb{R}^2} \int_a^b \int_{s_3}^b \int_{s_3}^b \prod_1^3 \Big[\chi_E\big(x - \gamma(s_i)\big)\phi''(s_i)^{1/3}\Big] \; ds_1 \; ds_2 \; ds_3 \; dx \leq 4 \; |E|^2.$$

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By Fubini's theorem this is equivalent to

$$\int_{\mathbb{R}^2} \chi_E(x) \int_a^b \Big(\int_t^b \chi_E\big(x+\gamma(t)-\gamma(s)\big)\ \phi''(s)^{1/3} ds \Big)^2 \phi''(t)^{1/3} dt\ dx \le 4\ |E|^2.$$

Thus, it is enough to establish the inequality

$$\int_a^b \Big(\int_t^b \chi_E\big(\gamma(t)-\gamma(s)\big)\ \phi''(s)^{1/3} ds \Big)^2 \phi''(t)^{1/3} dt \le 4\ |E|. \tag{2}$$

The convexity of the graph of ϕ shows that the change of variables

$$(s,t) \to \gamma(s)-\gamma(t) = \big(s-t, \phi(s)-\phi(t)\big)$$

is one-to-one. Thus

$$\int_a^b \int_a^b \chi_E\big(\gamma(t)-\gamma(s)\big)\ |\phi'(s)-\phi'(t)|\ ds\ dt \le |E|$$

and (2) will follow from the inequality

$$\phi''(t)^{1/3} \Big(\int_t^b \chi_A(s)\ \phi''(s)^{1/3} ds \Big)^2 \le 4 \int_t^b \chi_A(s) \big(\phi'(s)-\phi'(t)\big)\ ds \tag{3}$$

if $A \subseteq (t,b)$. To prove (3) we let $|A_u|$ stand for the (one-dimensional) Lebesgue measure of $A \cap (u,b)$ whenever $t \le u \le b$. Then

$$\int_t^b \chi_A(s)\big(\phi'(s)-\phi'(t)\big) ds = \int_t^b \chi_A(s) \int_t^s \phi''(u)\ du\ ds = \int_t^b \phi''(u) |A_u|\ du. \tag{4}$$

Also

$$\int_t^b \chi_A(s)\ \phi''(s)^{1/3} ds = \int_t^b \chi_A(s)\ \phi''(s)^{1/3} |A_s|^{1/3} |A_s|^{-1/3} ds$$

$$\le \Big(\int_t^b \chi_A(s)\ \phi''(s)\ |A_s|\ ds \Big)^{1/3} \Big(\int_t^b \chi_A(s) |A_s|^{-1/2} ds \Big)^{2/3}.$$

Thus, it follows from (4) that

(5)

$$\Big(\int_t^b \chi_A(s) \phi''(s)^{1/3} ds \Big)^3 \le \Big(\int_t^b \chi_A(s)\ \big(\phi'(s)-\phi'(t)\big) ds \Big) \Big(\int_t^b \chi_A(s) |A_s|^{-1/2} ds \Big)^2.$$

If $0 \le \rho \le |A|$, then $|\{s \in A :\ |A_s| \le \rho\}| = \rho$, and so

$$\int_t^b \chi_A(s) |A_s|^{-1/2} ds = \int_0^{|A|} y^{-1/2} dy = 2\ |A|^{1/2}.$$

With this and the fact that ϕ'' is nondecreasing, (5) yields (3) to complete the proof.

References

[C] Y. Choi, Convolution operators with affine arclength measures on plane curves, J. Korean Math. Soc. 36 (1999), 193–207. CMP 99:09

[D] S. W. Drury, Degenerate curves and harmonic analysis, *Math. Proc. Camb. Phil. Soc.* 108 (1990), 89-96. MR **91h:**42021

Department of Mathematics, Florida State University, Tallahassee, Florida 32306-4510

E-mail address: oberlin@math.fsu.edu