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AN INFINITE FAMILY OF MANIFOLDS WITH BOUNDED TOTAL CURVATURE

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Abstract. The negative answer to the following problem of V. I. Arnold is given: Is the number of topologically different k-manifolds of bounded total curvature finite?

§1. Introduction

In [Ar] V. I. Arnold defined the total curvature $\alpha(g)$ of an immersion $g : M^k \to \mathbf{R}^m$ of a k-manifold in the m-dimensional euclidean space as the k-volume $vol_k(g\Delta\bar{g})(M^k)$ of a submanifold $(g\Delta\bar{g})(M^k) \subset \mathbf{R}^m \times G_{m,k}$ where $\bar{g} : M^k \to G_{m,k}$ is the induced map to the Grassmann manifold. He proved that for any two submanifolds $X, Y \subset \mathbf{R}^m$ and for every diffeomorphism $A : \mathbf{R}^m \to \mathbf{R}^m$ generically the total curvature $\alpha(A^n(X) \cap Y)$ is bounded by $Ce^{\lambda n}$ where $A^n = A \circ \ldots \circ A$ and C and λ do not depend on n. Then the following question of Arnold appeared naturally: *Is the number of topologically different k-manifolds M lying in $\mathbf{R}^m$ with the total curvature $\alpha(M)$ bounded from above finite?*

We note that Arnold's total curvature $\alpha(g)$ differs from the classic Chern-Lashof total curvature $\tau(g)$ [N-K]. Recall that $\tau(g) = \frac{1}{vol(S^{m-1})} \int_{SM^k} \nu^*(d\sigma)$ where SM^k is the unit sphere bundle of the normal bundle of an immersion $g : M^k \to \mathbf{R}^m$, $\nu : SM^k \to S^{m-1}$ is the Gauss map and $d\sigma$ is the volume element on S^{m-1}. Arnold's total curvature is always greater than Chern-Lashof's and the last can be estimated from below by the Morse number $m(M^k)$= the minimal number of critical points of Morse functions.

It is easy to show that the answer to Arnold's question is affirmative for 2-manifolds. The purpose of this paper is to show that for 3-manifolds the answer is negative.

Theorem 1. *There exist a number $C > 0$ and an infinite family $\{M_i\}$ of pairwise nonhomeomorphic 3-dimensional compact submanifolds of $\mathbf{R}^4$ such that the total curvature $\alpha(M_i) < C$ for all i.*

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Theorem 2. *There exist a number $C > 0$ and an infinite family $\{\Sigma_i\}$ of pairwise nonhomeomorphic homology 3-spheres with embeddings $\{\eta_i : \Sigma_i \to \mathbf{R}^7\}$ such that $\alpha(\eta_i) < C$ for all i.*

The construction of both families of manifolds is basically due to the presence of the fundamental group. This situation suggests the following:

Conjecture. *The answer to Arnold's problem is positive for simply connected manifolds.*

§2. The construction of the first family

Let $K(n,m) \subset S^3$ be an (n,m)-torus knot in S^3 and let $N(n,m)$ be its regular neighborhood. Let $\{p_i\}$ be a sequence of all odd numbers. For every i let M_i denote the result of doubling the compact 3-manifold $S^3 - IntN(2,p_i)$ along its boundary $\partial N(2,p_i)$.

Lemma 1. *For every $i \neq j$ manifolds M_i and M_j are not homeomorphic.*

Proof. An S^1-action on $S^3 = S^1 * S^1$ defined by the formula

$$(\theta,(\phi,\psi)) \to (\phi + 2\theta, \psi + p_i\theta)$$

has two exceptional orbits with the orbit space homeomorphic to S^2. We may assume that $N(2,p_i)$ is a preimage of a 2-disk under the projection to the orbit space. Then it easy to see that each M_i is a Seifert manifold [Or] with the orbit space S^2 with 4 exceptional orbits of types $(2,1),(2,1),(p_i,2),(p_i,2)$. Since these orbit invariants are different for $i \neq j$, Seifert bundles on M_i and M_j are different. Note that M_i are large Seifert manifolds for all i. Hence M_i and M_j are not homeomorphic for $i \neq j$ [Or, Theorem 6]. □

Lemma 2. *Suppose that $g_i : M \to N$ is a sequence of embeddings of a Riemannian manifold M into a Riemannian manifold N, converging in the C^1-topology to an embedding $g : M \to N$. Then $\lim vol(g_i(M)) = vol(g(M))$, where $vol(L)$ for $L \subset N$ means the volume of submanifold L in N.*

The proof follows for instance from [D-N-F], v.1.

Let D^m denote the unit ball in $\mathbf{R}^m$ and let D^m_r be a concentric ball of radius r. The *normal regular neighborhood of radius* r of immersed k-manifold $f : M^k \to \mathbf{R}^m$ is an immersion $F : M^k \times D^{m-k}_r \to \mathbf{R}^m$ such that 1) $F\mid_{M^k\times\{0\}} = f$ where 0 is the center of the disk D^{m-k}_r, 2) for every $t \in M^k$ the restriction $F\mid_{\{t\}\times D^{n-k}_r}$ is an isometrical embedding and the disk $F(\{t\} \times D^{m-k}_r)$ is orthogonal to the manifold $f(M^k)$ at the point $f(t)$.

Every C^1-manifold has locally a normal regular neighborhood for some r [H], and hence, every closed manifold with trivial normal bundle has a normal regular neighborhood. Therefore every smooth enough immersion of the circle in $\mathbf{R}^3$ has a normal regular neighborhood. Here is a parameterized version of that statement.

Lemma 3. *Suppose that $\{f_i : S^1 \to \mathbf{R}^3\}$ is a sequence of immersions of S^1 into $\mathbf{R}^3$, converging in C^2-metric to an immersion f_∞. Then there exist $r > 0$ and normal regular neighborhoods $F_i : S^1 \times D_r \to \mathbf{R}^3$, $i = 1,2,...,\infty$, such that $\{F_i\}$ converges in C^2-metric to F_∞.*

Let $C_k(\ , \ )$ be a space of mappings in C^k-topology. Denote by S^1 the boundary of the unit disk D^2 and denote by $w_r : S^1 \to \partial D_r$ the natural projection. In this notation the center of D^2 is ∂D_0. Suppose that $f : S^1 \to \mathbf{R}^3$ is a C^2-immersion with a normal regular neighborhood F of radius r. We define a map $R_f : [0,r] \to C_2(S^1 \times S^1, \mathbf{R}^3)$ by setting $R_f(x) = F \circ (id_{S^1} \times w_r) : S^1 \times S^1 \to \mathbf{R}^3$.

Lemma 4. *Suppose that a sequence $\{f_i : S^1 \to \mathbf{R}^3\}$ is converging to f_∞ in C^2-metric and assume that r is given by Lemma 3. Then R_{f_i} uniformly converges to R_{f_∞}.*

We omit the proof since it is straightforward.

We define the 'Gauss map' $G_f : [0,r] \to C_1(S^1 \times S^1, G_{3,2})$ for R_f in the following way. For $x > 0$ the value $G_f(x)$ is the tangent bundle map $\nu : S^1 \times S^1 \to G_{3,2}$ for $S^1 \times \partial D_x$. For $x = 0$ for every $(t,\theta) \in S^1 \times S^1$ the value $G_f(0)(t,\theta)$ is the span of the derivative $f'(t)$ and the vector orthogonal to $F(t,(r,\theta)) - F(t,0) = b(t,\theta)$. Here $(r,\theta) \in D_r$ means a point written in polar coordinates. Note that for all $x \in [0,r]$ the Gauss map is defined as $G_f(x)(t,\theta) = span\{\frac{\partial F(t,(x,\theta))}{\partial t}, b(t,\theta)^\perp\}$.

Lemma 5. *Suppose that a sequence $\{f_i : S^1 \to \mathbf{R}^3\}$ is converging to f_∞ in C^2-metric and assume that the number r is provided by Lemma 3. Then G_{f_i} converges uniformly to G_{f_∞}.*

Proof. According to Lemma 3 a sequence F_i converges uniformly to F_∞ in C^2-metric. It implies a uniform convergence of $\frac{\partial F_i}{\partial t}$ to $\frac{\partial F_\infty}{\partial t}$ in C^1-metric. The second vector $b_i(t,\theta)^\perp$ does not depend on x, and therefore by virtue of Lemma 3 the sequence $b_i(t,\theta)^\perp$ converges to $b_\infty(t,\theta)^\perp$ in C^1-metric as a sequence of functions in t and θ. This implies the lemma. □

The proof of Theorem 1. We construct manifolds M_i in $\mathbf{R}^4$ symmetrically with respect to the hyperplane $\{0\} \times \mathbf{R}^3$. Let us consider two symmetric 3-dimensional spheres $S^3_+ \subset [1,\infty) \times \mathbf{R}^3$ and $S^3_- \subset (-\infty,-1] \times \mathbf{R}^3$ such that the intersection $S^3_+ \cap (\{1\} \times \mathbf{R}^3)$ is a 3-disk D^3_+ and similarly $S^3_- \cap (\{-1\} \times \mathbf{R}^3) = D^3_-$. Let $S^1 \subset D^3_+$ be a smoothly embedded circle and let $f : S^1 \to S^1 \subset D^3_+$ be a double winding around that circle ($f \in C^2$). For every i we consider the boundary of the normal regular neighborhood of S^1 of radius ϵ_i and realize a torus knot $K(2,p_i)$ on it. For small enough ϵ_i it is possible to realize the knot $K(2,p_i)$ by a map $f : S^1 \to \mathbf{R}^3$ which is $1/i$-close to f in C^2-metric. By virtue of Lemma 3 there exist a number r and the normal regular neighborhoods F_i of radius r. For every i there exists $r_i < r$ such that the restriction of F_i on $S^1 \times D^2_{r_i}$ is an embedding. We denote $N(2,p_i) = F_i(S^1 \times D^2_{r_i})$. Let C_i be the cylinder over the boundary $\partial N(2,p_i)$ that connects S^3_+ with S^3_-. We define $M_i = (S^3_+ - Int N(2,p_i)) \cup C_i \cup (S^3_- - Int N(2,p_i))$.

We will write $\alpha(M)$ if the immersion of M is fixed. Since

$$\alpha(M_i) \leq \alpha(S^3_+) + \alpha(S^3_-) + \alpha(\partial N(2,p_i) \times [-1,1]),$$

it suffices to find a common estimate for $\alpha(\partial N(2,p_i) \times [-1,1])$. By an obvious version of the Kuiper theorem [N-K] for the Arnold total curvature it follows that $\alpha(\partial N(2,p_i) \times [-1,1]) = 2\alpha(\partial N(2,p_i))$.

Lemmas 4, 5 imply that $\lim_{i\to\infty} R_{f_i}(r_i) = R_{f_\infty}(0)$ and $\lim_{i\to\infty} G_{f_i}(r_i) = G_{f_\infty}(0)$. Denote by A_i the diagonal product of R_i and G_i, $i = 1,2,\ldots,\infty$. By Lemma 2 it follows that $\lim_{i\to\infty} vol(A_i(r_i)(S^1 \times S^1)) = vol(A_\infty(0)(S^1 \times S^1))$. Since $vol(A_i(r_i)(S^1 \times S^1)) = \alpha(\partial N(2,p_i))$, it follows that the sequence $\alpha(\partial N(2,p_i))$

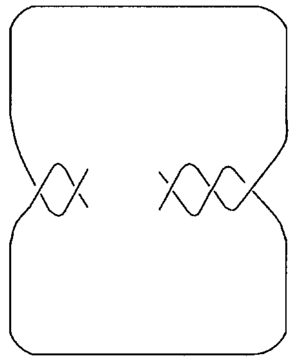

FIGURE 1

has an upper bound. So, it remains to note that by Lemma 1 the family $\{M_i\}$ is pairwise nonhomeomorphic. □

Remark. Manifolds M_i are not smoothly embedded in $\mathbf{R}^4$ because at the points where spheres S^3_+ and S^3_- are attached to the cylinder C_i we have singularities. If we smooth these angles by a small perturbation near $\partial N(2, p_i)$, then $\alpha(M_i)$ will increase by approximately $vol(\partial N(2, p_i)) \times \pi/2$. Since $vol(\partial N(2, p_i)) \to 0$, then the sequence $\alpha(M_i)$ for resulting smooth manifolds M_i also will be bounded.

§3. THE CONSTRUCTION OF THE SECOND FAMILY

For any knot $K \subset S^3$ we denote by K_1 the 3-manifold obtained from S^3 by Dehn surgery on the knot K with the homology class $(1, 1)$ [R]. Let K^m be the following knot with $2m + 1$-crossings shown in Figure 1:

Lemma 6. *The manifolds K_1^m and K_1^n are homeomorphic if and only if $m = n$.*

Proof. The direct computation of the Casson invariant $\lambda(K_1^k)$ shows that $\lambda(K_1^m)$ and $\lambda(K_1^n)$ are different for $m \neq n$ [A-M]. □

By the definition a normal regular neighborhood induces a framing of the normal vector bundle. Let $\xi : S^1 \to \mathbf{R}^3$ be an immersion of the circle in $\mathbf{R}^3$ and let $F^k : S^1 \times D_r^2 \to \mathbf{R}^3$ be a normal regular neighborhood of ξ of radius r such that the induced framing has Hopf invariant equal to k. We consider the embedding of $\mathbf{R}^3$ in $\mathbf{R}^7$ as a factor $\mathbf{R}^3 \times 0 \times ... \times 0 \subset \mathbf{R}^7$. A map $\phi : (M, N) \to (\mathbf{R}^7, \mathbf{R}^3)$ is said to be a relative embedding if the restriction of ϕ on N is an immersion into $\mathbf{R}^3$ and the restriction of ϕ on $M - N$ is an embedding in $\mathbf{R}^7 - \mathbf{R}^3$, transversal to $\mathbf{R}^3$.

Lemma 7. *Let $f : (D^2, \partial D^2) \to (\mathbf{R}^7, \mathbf{R}^3)$ be a relative embedding of class C^2. Then there exist $r > 0$ and a relative embedding of class C^2, $F : (D^2 \times D_r^2, \partial D^2 \times D_r^2) \to (\mathbf{R}^7, \mathbf{R}^3)$, such that*

1) $F|_{D^2 \times \{0\}} = f$.

2) $f|_{\partial D^2 \times D_r^2} = F^1$ is a normal regular neighborhood in $\mathbf{R}^3$, with Hopf invariant equal to one.

3) For every $x \in IntD^2$ the restriction $F|_{\{x\} \times D_r^2}$ imbeds the disk D_r^2 isometrically into the 5-plane, orthogonal to $f(D^2)$ at the point $f(x)$.

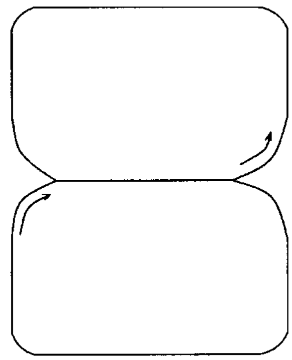

FIGURE 2

Proof. The map f induces a map $\nu(f) : D^2 \to G_{7,5}$ via the normal bundle. Let us consider a fibration $\eta : E \to G_{7,5}$ with the Stiefel manifold $V_{5,2}$ as a fiber. Here E is a subset of $G_{7,5} \times V_{7,2}$ defined in the following way: $L \times (a, b) \in E$ if and only if $a, b \in L$ and $\eta(L \times (a, b)) = L$. First we choose a normal regular neighborhood F^1 of $f\mid_{\partial D^2}$ of some radius r and with Hopf invariant equal to 1. The map F^1 induces a lifting $\beta : \partial D^2 \to E$ of a map $\nu(f)\mid_{\partial D^2} : \partial D^2 \to G_{7,5}$. Since the fundamental group $\pi_1(V_{5,2})$ is trivial, there is a lifting $\gamma : D^2 \to E$ extending β. The map γ defines a map $\Phi : D^2 \times \mathbf{R}^2 \to \mathbf{R}^7$ such that (1) $\Phi\mid_{D^2\times\{0\}} = f$, (2) $\Phi\mid_{\partial D^2 \times D^2_{r_0}} = F^1$ and (3) for every $x \in IntD^2$, the restriction $\Phi\mid_{\{x\}\times\mathbf{R}^2}$ is an isometric imbedding into a normal plane $\nu(f)(x)$ at the point $f(x) \in \mathbf{R}^7$. In order to complete the proof we choose $r < r_0$ such that the restriction $F = \Phi\mid_{D^2\times D^2_r}$ is a relative embedding. □

We note that the following parameterized version of Lemma 7 is valid.

Lemma 8. *Suppose that a sequence $f_i : (D^2, \partial D^2) \to (\mathbf{R}^7, \mathbf{R}^3)$ converges in C^2-metric to f. Then there exists $r > 0$ and a sequence of maps $F_i : (D^2 \times D^2_r, \partial D^2 \times D^2_r) \to (\mathbf{R}^7, \mathbf{R}^3)$ with the properties (1)-(3) of Lemma 6 for f_i for every $i = 1, 2, ..., \infty$, such that F_i converges in C^2-metric to F_∞.*

Proof of Theorem 2. Consider the 3-sphere S^3 in $\mathbf{R}^7$ with a flat part in $\mathbf{R}^3 \subset \mathbf{R}^7$. We define a sequence of embeddings $f_i : (D^2, \partial D) \to (\mathbf{R}^7, \mathbf{R}^3)$, C^2-converging to a relative embedding f_∞ with $f_i(\partial D^2) \subset S^3$ and $f_i(IntD^2) \cap S^3 = \emptyset$ for $i = 1, 2, ..., \infty$. Let $g : S^1 \to \mathbf{R}^3 \cap S^3$ be an immersion of the circle with the self-intersection along the interval as shown in Figure 2.

Let $f : D^2 \to \mathbf{R}^7$ be an extension of g to a relative C^2-embedding with $f(IntD^2) \cap (S^3) = \emptyset$. For every i we realize the knot K^i by a map $g_i : S^1 \to \mathbf{R}^3 \cap S^3$ such that the distance between g and $g_i : S^1 \to \mathbf{R}^3$ in C^2-metric is less than $1/i$. For every i there is an extension f_i of g_i which is $2/i$-close to f and moreover f_i is an embedding. Apply Lemma 8 to the sequence f_i to obtain F_i, $i = 1, 2, ..., \infty$. For every $i < \infty$ there exists a small number $\epsilon_i > 0$ such that the restriction $R_i = F_i\mid_{D^2\times\partial D^2_{\epsilon_i}}$ is an embedding. We define $\Sigma_i = S^3 - (S^1 \times D^2_{\epsilon_i}) \cup ImR_i$ for all i. It easy to check that Σ_i is homeomorphic to K^i_1.

In order to complete the proof it is sufficient to find a common upper bound for $\alpha(ImR_i)$. For large i there is a rough estimate for $\alpha(ImR_i)$ as $2\pi\alpha(f(D^2))$. That implies an upper bound for the sequence $\{\alpha(ImR_i)\}$. Thus, Lemma 6 completes the proof. □

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