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ON SEMISIMPLE HOPF ALGEBRAS OF DIMENSION pq

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ABSTRACT. We consider the problem of the classification of semisimple Hopf algebras A of dimension pq where $p < q$ are two prime numbers. First we prove that the order of the group of grouplike elements of A is not q, and that if it is p, then $q = 1 \pmod p$. We use it to prove that if A and its dual Hopf algebra A^* are of Frobenius type, then A is either a group algebra or a dual of a group algebra. Finally, we give a complete classification in dimension $3p$, and a partial classification in dimensions $5p$ and $7p$.

In this paper we consider semisimple Hopf algebras of dimension pq over an algebraically closed field k of characteristic 0, where p and q are distinct prime numbers. Masuoka has proved that a semisimple Hopf algebra of dimension $2p$ over k, where p is an odd prime, is *trivial* (i.e. is either a group algebra or a dual of a group algebra) [Ma1]. Izumi and Kasaki have proved that Kac algebras (i.e. semisimple Hopf algebras over the field of complex numbers, with an additional condition on the existence of an involution), of dimension $3p$ over k, where p is prime, are trivial [IK]. Thus, a natural conjecture is:

Conjecture 1. Any semisimple Hopf algebra of dimension pq over k, where p and q are distinct prime numbers, is trivial.

A well known property of A, a finite dimensional semisimple group algebra or a dual of a group algebra, is that it is of *Frobenius type*; that is, the dimension of any irreducible representation of A divides the dimension of A (the definition is due to Montgomery [Mo]). A special case of Kaplansky's 6th conjecture [K] is:

Conjecture 2. Any semisimple Hopf algebra of dimension pq over k, where p and q are distinct prime numbers, is of Frobenius type.

In this paper we prove among the rest that Conjecture 1 is equivalent to Conjecture 2 (see Theorem 3.5).

A major role in the analysis is played by $G(A)$ (where $G(A)$ denotes the group of grouplike elements of A). By [NZ], $|G(A)|$ is either $1, p, q$ or pq. We prove in Theorem 2.1 that if $p < q$, then $|G(A)| \neq q$, and if $|G(A)| = p$, then $q = 1 \pmod p$. Consequently, we prove in Theorem 2.2 that if $|G(A)| \neq 1$ and $q \neq 1 \pmod p$, then A is a commutative group algebra.

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Thus Theorem 2.2 suggests the following question: When is $|G(A)| \neq 1$? In Proposition 3.1 we prove that this is guaranteed when A^* is of Frobenius type, and in Theorem 3.2 we prove that if moreover $q \neq 1 \pmod p$, then A is a commutative group algebra. In Theorem 3.4 we prove that if $|G(A^*)| \neq 1$ and A^* is of Frobenius type, then A is trivial, and $|G(A)| = p < q$ or pq. The equivalence of Conjectures 1 and 2 is thus a consequence of Proposition 3.1 and Theorem 3.4.

A complete classification of semisimple Hopf algebras of dimension $3p$ is then given in Theorem 4.3. Indeed, they are all trivial.

We conclude by using Theorem 2.2 to prove in Theorem 4.5 that if A is a semisimple Hopf algebra of dimension $5p$, p an odd prime, and if $p = 2$ or $4 \pmod 5$ or $p \in \{13, 23\}$, then A is a commutative group algebra. Moreover, we obtain in Theorem 4.6 the same result for semisimple A of dimension $7p$, p a prime, and $p = 6 \pmod 7$ or $p \in \{17, 31\}$.

1. Preliminaries

In this paper k will always denote an algebraically closed field of characteristic 0.

Recall that a finite dimensional Hopf algebra over k is semisimple if and only if it is cosemisimple [LR].

Let A be semisimple Hopf algebra over k, and let $\rho_V : A \to End_k(V)$ be a finite dimensional representation of A. The associated character χ_V is given by $\chi_V(a) = tr(\rho_V(a))$ for all $a \in A$. A character χ_V is called irreducible if the representation V is irreducible. Let $R(A)$ denote the character ring of A; that is, the k-subalgebra of A^* generated by the characters χ_V of finite dimensional A-modules V. The set of all irreducible characters forms a basis of $R(A)$ [La]. Zhu has proved that $R(A)$ is semisimple and if $e_{A^*}, e_1, \ldots, e_k$ are the primitive idempotents of $R(A)$, where e_{A^*} is an integral of A^*, then

$$\dim A = 1 + \sum_{i=1}^{k} \dim(e_i A^*) \tag{1}$$

and the dimension of each $e_i A^*$ divides the dimension of A [Z]. Note that $\dim R(A) \geq k + 1$, and equality holds if and only if $R(A)$ is commutative.

Let $f : A \to A^*$ be the map given by $f(a) = a \rightharpoonup \lambda = \sum \langle a, \lambda_{(2)} \rangle \lambda_{(1)}$ for all $a \in A$, where λ is a non-zero integral of A^*. Recall that f gives a linear isomorphism between $kG(A)$ and the sum of the 1-dimensional ideals of A^*, and a linear isomorphism between the center $Z(A)$ of A and $R(A)$. Therefore, using the notation of (1), $\dim(e_i A^*) = 1$ for some i if and only if $G(A) \cap Z(A) \neq \{1\}$.

Let A be a semisimple Hopf algebra over k. Any simple subcoalgebra C_l of A^* has a basis $\{x^l_{ij} | 1 \leq i, j \leq n_l\}$, where $\Delta(x^l_{ij}) = \sum_{k=1}^{n_l} x^l_{ik} \otimes x^l_{kj}$ and $\varepsilon(x^l_{ij}) = \delta_{i,j}$. Note that $n_l = 1$ if and only of $C_l = \{g\}$ for some $g \in G(A)$. Nichols and Richmond have proved that if $\dim A$ is odd, then A does not have a 2-dimensional irreducible module [NR], hence

$$\dim A = |G(A)| + \sum_l n_l^2, \qquad n_l \geq 3. \tag{2}$$

Now, L is an irreducible left coideal of C_l if and only if

$$L = L^l_j = sp\{x^l_{kj} | 1 \leq k \leq n_l\} \tag{3}$$

for some $1 \leq j \leq n_l$. Similarly, R is an irreducible right coideal of C_l if and only if

$$R = R_k^l = sp\{x_{kj}^l | 1 \leq j \leq n_l\} \tag{4}$$

for some $1 \leq k \leq n_l$. Note that

$$\dim(L_j^l \cap R_k^l) = 1 \tag{5}$$

for any $1 \leq j, k \leq n_l$.

In what follows we recall some of the properties of a Hopf algebra with a projection, which we shall use in the sequel.

Theorem 1.1 ([R]). *If $H \overset{i}{\hookrightarrow} A \overset{\pi}{\hookrightarrow} H$ is a sequence of finite dimensional Hopf algebra maps where i is injective, π is surjective and $\pi \circ i = id_H$, then there exists $B \subseteq A$ so that*:

(i) *B is a left H-module algebra and coalgebra via the adjoint action.*

(ii) *B is a left H-comodule algebra and coalgebra via $\rho(b) = \sum b^{(1)} \otimes b^{(2)} = \sum \pi(b_{(1)}) \otimes b_{(2)}$.*

(iii) *$B \cong A/AH^+$ as a coalgebra, via the map $b \times h \mapsto b\varepsilon(h)$.*

(iv) *B is a left coideal subalgebra of A.*

(v) *As an algebra $A = B \times H$ is a smash product.*

(vi) *As a coalgebra $A = B \times H$ is a smash coproduct, that is*: $\Delta(b \times h) = \sum b_{(1)} \times b_{(2)}^{(1)} h_{(1)} \otimes b_{(2)}^{(2)} \times h_{(2)}$.

(vii) *The map $B \times H \to A(b \times h \mapsto bi(h))$ is an isomorphism of bialgebras.*

2. On the order of $G(A)$ and $G(A^*)$

In this section we prove some results concerning the group of grouplike elements of semisimple Hopf algebras of dimension pq.

Theorem 2.1. *Let A be a semisimple Hopf algebra of dimension pq over k, where $p < q$ are two prime numbers. Then*:

1. *$|G(A)| \neq q$.*
2. *If $|G(A)| = p$, then $q = 1 \pmod p$.*

Proof. 1. Suppose to the contrary that $|G(A)| = q$. If $G(A) \cap Z(A) = G(A)$, then $H = kG(A)$ is central in A, hence is a normal sub-Hopf algebra of A. Since A/AH^+ is a Hopf algebra of dimension p it follows by [Z] that $A/AH^+ \cong kC_p$. An elementary argument which follows from [Ma2, Section 2], shows that A is isomorphic as an algebra to the twisted group ring $kC_q^t[C_p]$ of the cyclic group C_p over the commutative algebra kC_q, and hence must be commutative. Thus, A^* is a group algebra and hence of Frobenius type. By (2), $pq = q + ap^2 + bq^2$ for some integers $a, b \geq 0$. But $p < q$, hence $b = 0$, which yields a contradiction. We conclude that $G(A) \cap Z(A) = \{1\}$. Therefore, using the notation of (1), $\dim(e_i A^*) \in \{p, q\}$ for all i. Let E_0 be the integral of $H = kG(A)$ with $\varepsilon(E_0) = 1$. Since $\dim(A/AH^+) = p$, it follows that $AH^+ = A(1 - E_0)$ has dimension $(q-1)p$ and thus $\dim(AE_0) = p$. Moreover, $E_0 e_A = e_A$, hence $E_0 = e_A + \sum_j e_{i_j}$. But $p < q$, hence counting dimensions yields a contradiction and the result follows.

2. If $G(A) \cap Z(A) = G(A)$, then $H = kG(A)$ is central in A, and hence A is commutative. Therefore, A^* is a group algebra and hence of Frobenius type. By (2), $pq = p + ap^2 + bq^2$ for some integers $a, b \geq 0$. Clearly, $b = 0$ and hence $q = 1 + ap$. If $G(A) \cap Z(A) = \{1\}$, then using the notation of (1), $\dim(e_i A^*) \in \{p, q\}$ for all i. Since $\dim(A/AH^+) = q$, it follows that $AH^+ = A(1 - E_0)$ has dimension $(p-1)q$

and thus $\dim(AE_0) = q$. Hence, $E_0 = e_A + \sum_j e_{i_j}$. But, counting dimensions yields that $\dim(e_{i_j}A^*) = p$ for all j, and the result follows in this case as well. □

As a direct consequence of Theorem 2.1 we have:

Theorem 2.2. *Let A be a semisimple Hopf algebra of dimension pq over k, where $p < q$ are two prime numbers satisfying $q \neq 1 \pmod p$. If $|G(A)| \neq 1$, then A is a commutative group algebra.*

3. The main result

In this section we consider semisimple Hopf algebras A of dimension pq such that A^* is of Frobenius type. First we find out when $|G(A)| \neq 1$ is guaranteed.

Proposition 3.1. *Let A be a semisimple Hopf algebra of dimension pq over k, where $p < q$ are two prime numbers. If A^* is of Frobenius type, then either $|G(A)| = p$ and $q = 1 \pmod p$, or $|G(A)| = pq$.*

Proof. If A is cocommutative, then $|G(A)| = pq$. Otherwise, $|G(A)| \neq pq$, and by Theorem 2.1, $|G(A)| \neq q$. If $|G(A)| = 1$, then by (1), $pq = 1 + ap^2 + bq^2$ for some integers $a, b \geq 0$, as A^* is of Frobenius type. But, $q^2 > pq$ hence $b = 0$ which yields a contradiction. □

As a corollary of Proposition 3.1 we have:

Theorem 3.2. *Let A be a semisimple Hopf algebra of dimension pq over k, where $p < q$ are two prime numbers satisfying $q \neq 1 \pmod p$. If A^* is of Frobenius type, then A is a commutative group algebra.*

In the following proposition we determine the coalgebra structure of A.

Proposition 3.3. *Let A be a non-cocommutative and non-commutative semisimple Hopf algebra of dimension pq over k, where $p < q$ are prime numbers. Let $R(A^*)$ be the character ring of A^*. If A^* is of Frobenius type, then*:

1. *$R(A^*)$ is commutative.*
2. *As a coalgebra $A = k1 \oplus kg \oplus \cdots \oplus kg^{p-1} \oplus C_1 \oplus \cdots \oplus C_a$, where $a = \frac{q-1}{p}$, g is a grouplike element and C_i is a simple subcoalgebra of A of dimension p^2 for all $1 \leq i \leq a$.*
3. *$gC_i = C_i = C_ig$ for all $1 \leq i \leq a$.*

Proof. Set $H = kG(A)$. By Theorem 2.1 and Proposition 3.1, $\dim H = p$. If H is central in A, then (as in the proof of Theorem 2.1) A must be commutative. Therefore, we conclude that $G(A) \cap Z(A) = \{1\}$.

Set $n = \dim R(A^*) - 1$. Then, by (1) there exist two natural numbers a and b such that:

$$pq = 1 + ap + bq \tag{6}$$

and

$$n \geq a + b. \tag{7}$$

Clearly, $a \geq 1$ and $b < p$. Moreover, A^* is of Frobenius type and $p < q$, hence by (2)

$$pq = p^2(n + 1 - p) + p. \tag{8}$$

Substituting (6) and (7) in (8) yields

$$pq \geq p^2(a+b+1-p)+p = p^2\left(\frac{(p-b)q-1}{p}+b+1-p\right)+p$$

and hence $(1-p+b)q \geq (1-p+b)p$. Since $p<q$ and $b<p$, this is possible if and only if $b=p-1$ and equality holds in (7). This implies that $R(A^*)$ is commutative and that $pq = 1+ap+(p-1)q$, and hence $a=\frac{q-1}{p}$. Let

$$e_A, e_1, \ldots, e_a, e_{a+1}, \ldots, e_{a+p-1}$$

be the primitive idempotents of $R(A^*)$, where e_A is the integral of A with $\varepsilon(e_A) = 1, \dim(Ae_i) = p$ for $1 \leq i \leq a$ and $\dim(Ae_{a+j}) = q$ for $1 \leq j \leq p-1$. Let $E_0 = \frac{1}{p}\sum_{i=0}^{p-1} g^i$ be an integral of H where g is a generator of $G(A)$. Since $\dim(A/AH^+) = q$, it follows that $AH^+ = A(1-E_0)$ has dimension $(p-1)q$ and thus $\dim(AE_0)=q$. Moreover, $E_0e_A = e_A$, hence counting dimensions yields that

$$E_0 = e_A + e_1 + \cdots + e_a.$$

Since $R(A^*)$ is commutative, $\dim(R(A^*)e_A) = \dim(R(A^*)e_i) = 1$ for all $1 \leq i \leq a$, and hence

$$\dim(R(A^*)E_0) = a+1. \tag{9}$$

Since the set of all the irreducible left A^*-modules consists of p 1-dimensional modules and $a=\frac{q-1}{p}$ p-dimensional modules, it follows that

$$A = k1 \oplus kg \oplus \cdots \oplus kg^{p-1} \oplus C_1 \oplus \cdots \oplus C_a$$

as a coalgebra, where C_i is a simple subcoalgebra of A of dimension p^2 for all $1 \leq i \leq a$. Let

$$\{1, g, \ldots, g^{p-1}, \chi_1, \ldots, \chi_a\}$$

be the set of irreducible characters of A^*, where χ_i corresponds to C_i. This set clearly forms a basis of $R(A^*)$. Then

$$R(A^*)E_0 = sp\{E_0, E_0\chi_1, \ldots, E_0\chi_a\}$$

which implies by (9) that $\{E_0, E_0\chi_1, \ldots, E_0\chi_a\}$ forms a basis of $R(A^*)E_0$. If $g\chi_i = \chi_j$ for $i \neq j$, then $E_0\chi_i = E_0\chi_j$ which is a contradiction. Therefore, $g\chi_i = \chi_i$, and hence

$$gC_i = C_i = C_ig$$

for all i. This concludes the proof of the proposition. □

Theorem 3.4. *Let A be a semisimple Hopf algebra of dimension pq over k, where $p<q$ are prime numbers. If A^* is of Frobenius type and $|G(A^*)| \neq 1$, then A is trivial, and $|G(A)| = p < q$ or pq.*

Proof. If A is either cocommutative or commutative, then A is either a group algebra or a dual of a group algebra respectively. In any event A^* is of Frobenius type, hence by Proposition 3.1, $|G(A)| = p < q$ or pq and we are done.

Suppose that A is not cocommutative and not commutative. Then Proposition 3.3 is applicable. Set $H = kG(A)$. By Proposition 3.1, $|G(A)| = p$. Let g be a generator of $G(A)$. By Theorem 2.1, $|G(A^*)| \neq q$, hence $|G(A^*)| = p$ too. Thus we have the following sequence of maps:

$$H \overset{i}{\hookrightarrow} A \overset{\pi}{\hookrightarrow} H$$

where i is the inclusion map and π is a surjection homomorphism of Hopf algebras. If $\pi \circ i = \varepsilon$, then $H \subseteq K = A^{coH}$. Since K is a left coideal of A, it is a direct sum of irreducible left coideals $K = k1 \oplus kg \oplus \cdots \oplus kg^{p-1} \oplus V_1 \oplus \cdots \oplus V_n$. Since A^* is of Frobenius type it follows that $\dim V_i = p$ for all i. But, this is a contradiction since p does not divide $\dim K = q$. Therefore $\pi \circ i \neq \varepsilon$ and we may assume that $\pi \circ i = id_H$. Therefore by Theorem 1.1, there exists $B \subset A$ so that $A \cong B \times H$. By Theorem 1.1(iv), B is a left coideal of A, hence a direct sum of irreducible left coideals of A. By Proposition 3.3(2), the dimensions of these irreducible left coideals are either 1 or p. Since $\dim B = q$, it follows that B contains an irreducible left coideal V of A, of dimension p. Since $V \subset C$ for some p^2-dimensional simple subcoalgebra C, it follows by Theorem 1.1(vii) and Proposition 3.3(3), that $V \times H \subseteq C$. But, $\dim(V \times H) = p^2 = \dim C$, hence $V \times H = C$. By Theorem 1.1(iii), $A/AH^+ \cong B$ as coalgebras, and V is the image of $C = V \times H$ under this isomorphism, hence

$$V \text{ is a subcoalgebra of } B.$$

We wish to prove that V is a simple subcoalgebra of B and thus to reach a contradiction. Note that since V is an irreducible left coideal of A it follows that $V \times g^i$ is also an irreducible left coideal of A for all $0 \leq i \leq p-1$. By Proposition 3.3(3), it follows that

$$\{V \times g^i | 0 \leq i \leq p-1\}$$

is the set of all the irreducible left coideals of A contained in C. Since V is a left coideal of A, it follows from Theorem 1.1(ii) that V is an H subcomodule of B. Let $\rho : B \to H \otimes B$ be the comodule structure map, and write $V = \bigoplus_{i=0}^{p-1} V_i$, where $V_i = \rho^{-1}(g^i \otimes V)$. We claim that $\dim V_i = 1$ for all i. Indeed, let $\{v_0, \ldots, v_{p-1}\}$ be a basis of V consisting of homogeneous elements; that is, $\rho(v_i) = g^{m_i} \otimes v_i$ for some $0 \leq m_i \leq p-1$. Let $0 \neq v \in V$ and write $\Delta_B(v) = \sum_{i=0}^{p-1} b_i \otimes v_i$. Then by Theorem 1.1(vi),

$$\Delta_A(v \times 1) = \sum_{i=0}^{p-1} b_i \times g^{m_i} \otimes v_i \times 1.$$

Therefore, using Kaplansky's notation [K], $L(v \times 1) = sp\{b_i \times g^{m_i} | 0 \leq i \leq p-1\} \subset C$ is a *right* coideal of A of dimension $\leq p$. Since C is a simple subcoalgebra of A of dimension p^2, it follows that $\dim(L(v \times 1)) = p$ and $L(v \times 1)$ is irreducible. Therefore by (5), $\dim(L(v \times 1) \cap (V \times g^i)) = 1$ for all i, hence $\{m_i | 0 \leq i \leq p-1\} = \{0, 1, \ldots, p-1\}$. Thus V has a basis $\{v_i | 0 \leq i \leq p-1\}$, where $\rho(v_i) = g^i \otimes v_i$. Since V is an H-comodule coalgebra it follows that $\Delta_B(V_i) \subseteq \sum_{j=0}^{p-1} V_j \otimes V_{i-j}$, hence $\Delta_B(v_i) = \sum_{j=0}^{p-1} \alpha_{ij} v_j \otimes v_{i-j}$ for all i, for some $\alpha_{ij} \in k$. Computing $\Delta_A(v_i \times 1)$ yields that $R_i = L(v_i \times 1) = sp\{\alpha_{ij} v_j \times g^{i-j} | 0 \leq j \leq p-1\} \subset C$ is a right coideal of A of dimension $\leq p$, for all i. Hence $\dim R_i = p$ and

$$R_i = sp\{v_j \times g^{i-j} | 0 \leq j \leq p-1\} \tag{10}$$

is irreducible. It is straightforward to verify that $R_i \neq R_t$ for $i \neq t$, and hence the set $\{R_i | 0 \leq i \leq p-1\}$ is the set of *all* the irreducible right coideals of A which are contained in C.

Finally, let $D \subseteq V$ be a subcoalgebra. By Theorem 1.1(vi), $D \times H \subseteq C$ is a *right* coideal of A and hence $D \times H = \bigoplus_l R_{i_l}$, where R_{i_l} is as in (10). But, the image of $D \times H$ under the map $id \otimes \varepsilon : A \to B$ equals D, while the image of $\bigoplus_l R_{i_l}$ under this map equals V. Therefore $D = V$, and hence V is a simple coalgebra. But, this is a contradiction since $\dim V = p$ is not a square. $\square$

As a corollary we obtain the following:

Theorem 3.5. *Let A be a semisimple Hopf algebra of dimension pq over k, where $p < q$ are prime numbers. If both A and A^* are of Frobenius type, then A is trivial.*

Proof. Follows from Proposition 3.1 and Theorem 3.4. $\square$

4. The dimensions $3p, 5p$ and $7p$

We start this section with a complete classification of semisimple Hopf algebras of dimension $3p$.

Proposition 4.1. *Let A be a non-cocommutative semisimple Hopf algebra of dimension $3p$ over k, where $p > 3$ is prime. Then $|G(A)| = 3$.*

Proof. By Theorem 2.1, $|G(A)| \neq p$. Since A is non-cocommutative, $|G(A)| \neq 3p$. Assume $|G(A)| = 1$ and let $R(A^*) \subseteq A$ be the ring of characters of A^*. Set $n = \dim R(A^*) - 1$. Then by (1), there exist two natural numbers a and b such that

$$3p = 1 + 3a + bp \quad \text{and} \quad n \geq a + b.$$

Note that $a \geq 1$ and hence $b = 1$ or 2. Since 2 does not divide $3p$, and A^* is semisimple we have by [NR] that A^* does not have a 2-dimensional irreducible module and hence the following two inequalities hold:

$$n \geq a + 1 = \frac{(3-b)p + 2}{3} \quad \text{and} \quad 3p \geq 9n + 1.$$

But these two inequalities are incompatible since they imply that $(-6 + 3b)p \geq 7$ which is impossible. This concludes the proof of the proposition. $\square$

Proposition 4.2. *Every semisimple Hopf algebra A of dimension $3p$ over k, where $p > 3$ is prime, is of Frobenius type.*

Proof. If A is a group algebra or a dual of a group algebra, then it is known that A is of Frobenius type. Otherwise, by Proposition 4.1, $|G(A)| = 3$. Since A is non-commutative we must have $G(A) \cap Z(A) = \{1\}$.

Set $n = \dim R(A^*) - 1$. Then, by (1) there exist two natural numbers a and b such that:

$$3p = 1 + 3a + bp \quad \text{and} \quad n \geq a + b.$$

Clearly, $a \geq 1$ and hence $b = 1$ or 2. Since 2 does not divide $3p$ we have by [NR] that A^* does not have a 2-dimensional irreducible module and hence that the following two inequalities hold:

$$n \geq a + b \quad \text{and} \quad 3p \geq 9(n-2) + 3.$$

Therefore, $3p \geq 9(\frac{(3-b)p-1}{3} + b - 2) + 3$ and hence $(-2 + b)p \geq 3b - 6$. Clearly, this is possible if and only if the equalities above hold, and $b = 2$. Therefore, $3p = 1 + 3a + 2p$ and $a = \frac{p-1}{3}$. This implies that $R(A^*)$ is commutative and that

$$A = k1 \oplus kg \oplus kg^2 \oplus C_1 \oplus \cdots \oplus C_a$$

as a coalgebra where C_i is a simple subcoalgebra of A of dimension 9 for all $1 \leq i \leq a$. Hence A^* is of Frobenius type. Replacing A by A^* yields the same result for A. $\square$

As a corollary of the above and of Theorem 3.5 we have:

Theorem 4.3. *A semisimple Hopf algebra of dimension $3p$ over k, where $p > 3$ is prime, is trivial.*

We conclude the paper by considering semisimple Hopf algebras of dimensions $5p$ and $7p$.

Lemma 4.4. *Let A be a semisimple Hopf algebra of odd dimension over k. If $|G(A)| = 1$, then there exists an irreducible A^*-module V with $\dim V \geq 4$.*

Proof. Suppose on the contrary that for any non-trivial A^*-irreducible module V, $\dim V \leq 3$. Then by [NR], $\dim V = 3$. Hence, $\dim(V \otimes V^*) = 9$ and by [La], $V \otimes V^* = k \oplus V_1 \oplus \cdots \oplus V_i$, where $V_j \neq k$ is an A^*-irreducible module for all j. Since $\dim V_j = 3$, this is a contradiction. $\square$

Theorem 4.5. *Let A be a semisimple Hopf algebra over k. If $\dim A = 5p$, p an odd prime, and if $p = 2$ or $4 \pmod 5$ or $p \in \{13, 23\}$, then A is a commutative group algebra.*

Proof. We wish to show that $|G(A)| \neq 1$. Suppose on the contrary that $|G(A)| = 1$. Set $n = \dim R(A^*) - 1$. By (1), there exist two natural numbers $1 \leq a$ and $1 \leq b \leq 4$ such that

$$\begin{aligned} 5p &= 1 + 5a + bp, \\ n &\geq a + b \quad \text{and} \\ 5p &\geq 9(n-1) + 16 + 1 \end{aligned}$$

where the last inequality follows from (2) and Lemma 4.4. Hence $(-20 + 9b)p \geq 45b + 31$. But, if $p = 2$ or $4 \pmod 5$, then $b = 2$ or 1 respectively and if $p \in \{13, 23\}$, then $b = 3$. In any event this is impossible and we have proved that $|G(A)| \neq 1$. The result follows now from Theorem 2.2. $\square$

Theorem 4.6. *Let A be a semisimple Hopf algebra over k. If $\dim A = 7p$, p a prime, and if $p = 6 \pmod 7$ or $p \in \{17, 31\}$, then A is a commutative group algebra.*

Proof. Suppose $|G(A)| = 1$ and set $n = \dim R(A^*) - 1$. By (1), there exist two natural numbers $1 \leq a$ and $1 \leq b \leq 6$ so that

$$\begin{aligned} 7p &= 1 + 7a + bp, \\ n &\geq a + b \quad \text{and} \\ 7p &\geq 9(n-1) + 16 + 1 \end{aligned}$$

where the last inequality follows from (2) and Lemma 4.4. Thus, $(-14 + 9b)p \geq 63 + 47$. But, if $p = 6 \pmod 7$ or $p \in \{17, 31\}$, then this is impossible. The result follows now from Theorem 2.2. $\square$

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