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ON THE MULTIPLICITIES OF THE ZEROS OF LAGUERRE–PÓLYA FUNCTIONS

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ABSTRACT. We show that all the zeros of the Fourier transforms of the functions $\exp(-x^{2m})$, $m = 1, 2, \dots$, are real and simple. Then, using this result, we show that there are infinitely many polynomials $p(x_1, \dots, x_n)$ such that for each $(m_1, \dots, m_n) \in (\mathbb{N} \setminus \{0\})^n$ the translates of the function

$$p(x_1, \dots, x_n) \exp\left(-\sum_{j=1}^{n} x_j^{2m_j}\right)$$

generate $L^1(\mathbb{R}^n)$. Finally, we discuss the problem of finding the minimum number of monomials $p_\alpha(x_1, \dots, x_n)$, $\alpha \in A$, which have the property that the translates of the functions $p_\alpha(x_1, \dots, x_n) \exp(-\sum_{j=1}^{n} x_j^{2m_j})$, $\alpha \in A$, generate $L^1(\mathbb{R}^n)$, for a given $(m_1, \dots, m_n) \in (\mathbb{N} \setminus \{0\})^n$.

1. INTRODUCTION

This paper is concerned with the zeros of real entire functions. Recall that a real entire function is an entire function which assumes only real values on the real axis. In [P2], G. Pólya proved that for each $m = 1, 2, \dots$ the function $\psi_m(z)$ defined by

$$\psi_m(z) = \int_{-\infty}^{\infty} \exp(-x^{2m}) e^{izx} dx \qquad (z \in \mathbb{C}) \tag{1}$$

is a real entire function of order $\frac{2m}{2m-1}$ and has real zeros only;

$$\psi_1(z) = \sqrt{\pi} \exp(-z^2/4)$$

has no zeros, and for $m \geq 2$, $\psi_m(z)$ has infinitely many zeros all of which are real. For generalizations of this result, see [B]. In [K], the first author of this paper proved that all but a finite number of the zeros of $\psi_m(z)$ are simple, and conjectured that all the zeros of $\psi_m(z)$ are simple for $m = 2, 3, \dots$.

In this paper, we will prove the following generalization of the conjecture.

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Theorem 1. *For all $k = 0, 1, 2, \dots$ and $m = 1, 2, \dots$ all the zeros of $\psi_m^{(k)}(z)$ are real and simple.*

We will prove Theorem 1 in Section 2. In fact, we will prove a slightly more general one (Theorem 2). Our proof is based on special properties of the Laguerre–Pólya functions and the fact that each $\psi_m(z)$ satisfies a differential equation. Finally, in Section 3, we apply Theorem 1 to show that for each n–tuple $(m_1, \dots, m_n)$ of positive integers and for each n–tuple of nonnegative integers $(k_1, \dots, k_n)$ the translates of the function

$$x_1^{k_1} \cdots x_n^{k_n} (1 + x_1) \cdots (1 + x_n) \exp\left(-\sum_{j=1}^{n} x_j^{2m_j}\right)$$

generate $L^1(\mathbb{R}^n)$, and conclude this paper with a discussion on the minimum number of monomials $p_\alpha(x_1, \dots, x_n)$, $\alpha \in A$, which have the property that the translates of the functions

$$p_\alpha(x_1, \dots, x_n) \exp\left(-\sum_{j=1}^{n} x_j^{2m_j}\right) \qquad (\alpha \in A)$$

generate $L^1(\mathbb{R}^n)$, for a given n–tuple $(m_1, \dots, m_n)$ of positive integers.

2. The Laguerre–Pólya functions (proof of Theorem 1)

We start this section with a brief introduction to the Laguerre–Pólya functions. An entire function $\psi(z)$ is said to be a *Laguerre–Pólya function* if it can be expressed in the form

$$\psi(z) = cz^n e^{-\alpha z^2 + \beta z} \prod_j \left(1 - \frac{z}{a_j}\right) e^{z/a_j}, \tag{2}$$

where c, β, a_j are real, $\alpha \geq 0$, n is a nonnegative integer and $\sum_j |a_j|^{-2} < \infty$. By a classical result of Laguerre [L] and Pólya [P1], an entire function $\psi(z)$ can be expressed in the form (2) if and only if there is a sequence of real polynomials with real zeros only which converges to $\psi(z)$ uniformly in compact sets in the complex plane. For a modern proof of this theorem, see Levin [Le, Chapter 8]. Therefore if $\psi(z)$ is a Laguerre–Pólya function, then all the derivatives of $\psi(z)$ are also Laguerre–Pólya functions. If $\psi(z)$ is given by (2), then the logarithmic derivative of $\psi(z)$ is given by

$$\frac{\psi'(z)}{\psi(z)} = \frac{n}{z} - 2\alpha z + \beta + \sum_j \left(\frac{1}{z - a_j} + \frac{1}{a_j}\right)$$

and therefore

$$\frac{d}{dz}\left(\frac{\psi'}{\psi}\right)(z) < 0 \qquad (z \in \mathbb{R},\ \psi(z) \neq 0), \tag{3}$$

provided $\psi(z)$ is not of the form $\psi(z) = c\, e^{\beta z}$.

Let $\psi(z)$ be a transcendental Laguerre–Pólya function. If $a \in \mathbb{R}$, $\psi(a) \neq 0$ and $\psi'(a) = 0$, then (3) implies that

$$\psi(a)\psi''(a) < 0;$$

in particular, $z = a$ is a simple zero of $\psi'(z)$. Since all the derivatives of $\psi(z)$ are Laguerre–Pólya functions, we have the following.

Proposition. *Let $\psi(z)$ be a transcendental Laguerre–Pólya function. If k is a positive integer and if $\psi^{(k)}(z)$ has a multiple zero at $z = a \in \mathbb{R}$, then*

$$\psi(a) = \psi'(a) = \cdots = \psi^{(k)}(a) = 0.$$

In particular, if $\psi(z)$ has simple zeros only, then all the derivatives of $\psi(z)$ have simple zeros only.

Remark 1. This proposition is closely related to the fact that a Laguerre–Pólya function has no Fourier critical points. For the definition of the Fourier critical points of real entire functions and related results, see [CCS], [KK], [Km2], [Km3], [P3]. It may also be remarked that if $\psi(z)$ is a transcendental Laguerre–Pólya function, then for each positive real number B there is a positive integer N such that $\psi^{(n)}(z)$ has only simple zeros in the interval $[-B\sqrt{n}, B\sqrt{n}]$ whenever $n \geq N$. For a proof of this fact, see [Km1].

Now, consider the functions $\psi_m(z)$, $m = 1, 2, \ldots$, defined by (1). It is clear that $\psi_1(z) = \sqrt{\pi}\exp(-z^2/4)$ is a Laguerre–Pólya function. For $m \geq 2$, $\psi_m(z)$ is a real entire function of order less than 2 with real zeros only, and therefore Hadamard's theorem implies that $\psi_m(z)$ can be expressed in the form (2) with $\alpha = 0$. Hence, to prove Theorem 1, it is enough to show that for each $m = 1, 2, \ldots$, $\psi_m(z)$ has simple zeros only, because of the proposition.

From an integration by parts, it can easily be shown that for each $m = 1, 2, \ldots$, $\psi_m(z)$ satisfies the differential equation

$$\psi_m^{(2m-1)}(z) - \frac{(-1)^m}{2m} z\psi_m(z) = 0 \qquad (z \in \mathbb{C}). \tag{4}$$

Then Theorem 1 is a consequence of the following.

Theorem 2. *Let $\psi(z)$ be a transcendental Laguerre–Pólya function, and assume that $\psi(z)$ satisfies the differential equation*

$$\psi^{(l)}(z) = A(z)\psi(z) \qquad (z \in \mathbb{R}) \tag{5}$$

for some positive integer l and some function $A(z)$ which is analytic in the whole real axis. Then all the (*real*) *zeros of $\psi(z)$ are simple.*

Proof. Assume, to get a contradiction, that $\psi(z)$ has a multiple zero, say at $z = a \in \mathbb{R}$. Then $\psi(a) = \psi'(a) = 0$. From (5), we obtain $\psi^{(l)}(a) = 0$. By differentiating both sides of (5), we obtain

$$\psi^{(l+1)}(z) = A'(z)\psi(z) + A(z)\psi'(z),$$

so that $\psi^{(l+1)}(a) = 0$. Then the proposition implies that

$$\psi(a) = \psi'(a) = \cdots = \psi^{(l)}(a) = \psi^{(l+1)}(a) = 0. \tag{6}$$

By differentiating both sides of (5) $k(> 0)$ times, we obtain

$$\psi^{(l+k)}(z) = \sum_{\lambda=0}^{k} \binom{k}{\lambda} A^{(k-\lambda)}(z)\psi^{(\lambda)}(z). \tag{7}$$

Then (6), (7) and an inductive argument shows that

$$\psi^{(k)}(a) = 0$$

for all $k = 0, 1, 2, \ldots$, and this is the desired contradiction. □

Remark 2. Let k and l be integers with $0 \leq k < l$. If a transcendental Laguerre–Pólya function $\psi(z)$ satisfies the differential equation

$$\psi^{(l)}(z) = A_0(z)\psi(z) + A_1(z)\psi'(z) + \cdots + A_k(z)\psi^{(k)}(z) \qquad (z \in \mathbb{R})$$

for some functions $A_0(z), \ldots, A_k(z)$ which are analytic in the whole real axis, then a similar argument as in the above proof shows that every zero of $\psi(z)$ has multiplicity $\leq k+1$, or equivalently, all the zeros of $\psi^{(k)}(z)$ are simple.

3. An application to the harmonic analysis on $L^1(\mathbb{R}^n)$

For a subset $\mathfrak{M}$ of the complex Banach algebra $L^1(\mathbb{R}^n)$ let $I(\mathfrak{M})$ denote the ideal generated by $\mathfrak{M}$, i.e., $I(\mathfrak{M})$ is the smallest closed linear subspace of $L^1(\mathbb{R}^n)$ which contains $\mathfrak{M}$, and has the property that

$$f \in I(\mathfrak{M}) \text{ and } g \in L^1(\mathbb{R}^n) \Rightarrow f * g \in I(\mathfrak{M}).$$

It is well known (see [R, Chapter 7]) that a closed linear subspace V of $L^1(\mathbb{R}^n)$ is an ideal if and only if it is translation–invariant, i.e.,

$$f \in V \Rightarrow f_{\mathbf{y}} \in V \qquad (\mathbf{y} \in \mathbb{R}^n),$$

where $f_{\mathbf{y}}$ is defined by

$$f_{\mathbf{y}}(\mathbf{x}) = f(\mathbf{x}+\mathbf{y}) \qquad (\mathbf{x} \in \mathbb{R}^n)$$

for $\mathbf{y} \in \mathbb{R}^n$. Therefore $I(\mathfrak{M})$ is the closed linear subspace of $L^1(\mathbb{R}^n)$ which is generated by the translates of the functions in the set $\mathfrak{M}$. The following theorem of N. Wiener, whose proof can be found in [R, Chapter 7] or in [W], gives a necessary and sufficient condition for a subset $\mathfrak{M}$ of $L^1(\mathbb{R}^n)$ to satisfy $I(\mathfrak{M}) = L^1(\mathbb{R}^n)$ (or equivalently, the translates of all the functions in $\mathfrak{M}$ generate $L^1(\mathbb{R}^n)$).

Wiener's Theorem. *Let $\mathfrak{M} \subset L^1(\mathbb{R}^n)$. Then $I(\mathfrak{M}) = L^1(\mathbb{R}^n)$ if and only if there does not exist a point in $\mathbb{R}^n$ at which the Fourier transforms of all the functions in $\mathfrak{M}$ vanish simultaneously.*

For $\mathbf{m} = (m_1, \ldots, m_n) \in (\mathbb{N} \setminus \{0\})^n$ let $\phi_{\mathbf{m}} \in L^1(\mathbb{R}^n)$ be defined by

$$\phi_{\mathbf{m}}(\mathbf{x}) = \exp\left(-\sum_{j=1}^{n} x_j^{2m_j}\right) \qquad (\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n).$$

Then Theorem 1 and Wiener's theorem imply that if $p(\mathbf{x}) = x_1^{k_1} \cdots x_n^{k_n}(1+x_1)\cdots(1+x_n)$, where $k_1, \ldots, k_n$ are nonnegative integers, then

$$I\left(\{p(\mathbf{x})\phi_{\mathbf{m}}(\mathbf{x})\}\right) = L^1(\mathbb{R}^n)$$

for each $\mathbf{m} \in (\mathbb{N} \setminus \{0\})^n$.

Let $\mathbf{m} = (m_1, \ldots, m_n) \in (\mathbb{N} \setminus \{0\})^n$ be given. In the remainder of this paper, we will be interested in the minimum number $M(\mathbf{m})$ of monomials $p_\alpha(\mathbf{x})$, $\alpha \in A$, which satisfy

$$I\left(\{p_\alpha(\mathbf{x})\phi_{\mathbf{m}}(\mathbf{x}) : \alpha \in A\}\right) = L^1(\mathbb{R}^n).$$

Assume, for a moment, that $m_j \geq 2$ for all $j = 1, \ldots, n$. If $p(\mathbf{x}) = x_1^{k_1} \cdots x_n^{k_n}$ is a monomial, then the Fourier transform of $p(\mathbf{x})\phi_{\mathbf{m}}(\mathbf{x})$ is

$$\prod_{j=1}^{n} (-i)^{k_j} \psi_{m_j}^{(k_j)}(z_j).$$

Therefore if we are given a set $\{p_\alpha(\mathbf{x}) : \alpha \in A\}$ of monomials, then the set of Fourier transforms of the functions $p_\alpha(\mathbf{x})\phi_{\mathbf{m}}(\mathbf{x})$, $\alpha \in A$, can be written in the form

$$\{f_{\alpha,1}(z_1)f_{\alpha,2}(z_2)\cdots f_{\alpha,n}(z_n) : \alpha \in A\},$$

where each $f_{\alpha,j}$ has infinitely many real zeros for $\alpha \in A$ and $j = 1, \dots, n$. If $\#A \le n$, then it is clear that there is a point in $\mathbb{R}^n$ at which all the functions in the above set vanish simultaneously. Consequently, we have

$$n + 1 \le M(\mathbf{m}).$$

For $\mathbf{k} = (k_1, \dots, k_n) \in \{0,1\}^n$ let $p_{\mathbf{k}}(\mathbf{x}) = x_1^{k_1} \cdots x_n^{k_n}$. Then it is easy to see that there does not exist a point in $\mathbb{R}^n$ at which the Fourier transforms of all the functions in the set

$$\{p_{\mathbf{k}}(\mathbf{x})\phi_{\mathbf{m}}(\mathbf{x}) : \mathbf{k} \in \{0,1\}^n\}$$

vanish simultaneously, so that

$$M(\mathbf{m}) \le 2^n. \tag{8}$$

In the general case, we have

$$r(\mathbf{m}) + 1 \le M(\mathbf{m}) \le 2^{r(\mathbf{m})} \qquad (\mathbf{m} = (m_1, \dots, m_n) \in (\mathbb{N} \setminus \{0\})^n), \tag{9}$$

where $r(\mathbf{m})$ denotes the number of the indices j for which $m_j \ge 2$.

Finally, we consider an improvement of the inequality (8). The inequality follows from the fact that for each $m = 1, 2, \dots$ there are nonnegative integers k_1 and k_2, namely 0 and 1, such that the zero sets of $\psi_m^{(k_1)}(z)$ and $\psi_m^{(k_2)}(z)$ are disjoint. In fact, for each $m = 1, 2, \dots$ the zero sets of the functions $\psi_m(z)$, $\psi_m'(z)$ and $\psi_m^{(2m)}(z)$ are mutually disjoint, because of the differential equation

$$\psi_m^{(2m)}(z) - \frac{(-1)^m}{2m} z\psi_m'(z) - \frac{(-1)^m}{2m}\psi_m(z) = 0 \qquad (z \in \mathbb{C}),$$

which is obtained by differentiating both sides of (4). Hence there does not exist a point in $\mathbb{R}^n$ at which the Fourier transforms of all the functions in the set

$$\{p(x_1,x_2)x_3^{k_3}\cdots x_n^{k_n}\phi_{\mathbf{m}}(\mathbf{x}) : p(x_1,x_2) = 1, x_1x_2, \text{ or } x_1^{2m_1}x_2^{2m_2}, \text{ and } k_3, \dots, k_n \in \{0,1\}\}$$

vanish simultaneously, so that $M(\mathbf{m}) \le 3 \cdot 2^{n-2}$ for $n \ge 2$. Therefore we can replace the inequality $M(\mathbf{m}) \le 2^{r(\mathbf{m})}$ in (9) by $M(\mathbf{m}) \le 3 \cdot 2^{r(\mathbf{m})-2}$ whenever $r(\mathbf{m}) \ge 2$.

Let l be an integer greater than 3. If it were true that for each $m = 1, 2, \dots$ there are nonnegative integers $k_1, \dots, k_l$ such that the zero sets of $\psi_m^{(k_1)}(z), \dots, \psi_m^{(k_l)}(z)$ are mutually disjoint, then a similar argument as above would imply that $M(\mathbf{m}) = r(\mathbf{m}) + 1$ if $r(\mathbf{m}) \le l - 1$, and $M(\mathbf{m}) \le l \cdot 2^{r(\mathbf{m})-l+1}$ if $r(\mathbf{m}) > l - 1$. We do not know whether or not such nonnegative integers exist, so we conclude this paper with the following question.

Question. *Let $l \ge 4$ and $m \ge 1$ be integers. Do there exist nonnegative integers $k_1, \dots, k_l$ such that the zero sets of $\psi_m^{(k_1)}(z), \dots, \psi_m^{(k_l)}(z)$ are mutually disjoint?*

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