ON THE MULTIPLICITIES OF THE ZEROS OF LAGUERRE–PÓLYA FUNCTIONS

JOE KAMIMOTO, HASEO KI, AND YOUNG–ONE KIM

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Abstract. We show that all the zeros of the Fourier transforms of the functions $\exp(-x^{2m})$, $m = 1, 2, \ldots$, are real and simple. Then, using this result, we show that there are infinitely many polynomials $p(x_1, \ldots, x_n)$ such that for each $(m_1, \ldots, m_n) \in (\mathbb{N} \setminus \{0\})^n$ the translates of the function

$$p(x_1, \ldots, x_n) \exp\left(-\sum_{j=1}^n x_j^{2m_j}\right)$$

generate $L^1(\mathbb{R}^n)$. Finally, we discuss the problem of finding the minimum number of monomials $p_{\alpha}(x_1, \ldots, x_n)$, $\alpha \in A$, which have the property that the translates of the functions $p_{\alpha}(x_1, \ldots, x_n) \exp(-\sum_{j=1}^n x_j^{2m_j})$, $\alpha \in A$, generate $L^1(\mathbb{R}^n)$, for a given $(m_1, \ldots, m_n) \in (\mathbb{N} \setminus \{0\})^n$.

1. Introduction

This paper is concerned with the zeros of real entire functions. Recall that a real entire function is an entire function which assumes only real values on the real axis. In [P2], G. Pólya proved that for each $m = 1, 2, \ldots$ the function $\psi_m(z)$ defined by

$$(1) \quad \psi_m(z) = \int_{-\infty}^{\infty} \exp(-x^{2m})e^{izx} \, dx \quad (z \in \mathbb{C})$$

is a real entire function of order $\frac{2m}{2m-1}$ and has real zeros only;

$$\psi_1(z) = \sqrt{\pi} \exp(-z^2/4)$$

has no zeros, and for $m \geq 2$, $\psi_m(z)$ has infinitely many zeros all of which are real. For generalizations of this result, see [B]. In [K], the first author of this paper proved that all but a finite number of the zeros of $\psi_m(z)$ are simple, and conjectured that all the zeros of $\psi_m(z)$ are simple for $m = 2, 3, \ldots$.

In this paper, we will prove the following generalization of the conjecture.
**Theorem 1.** For all $k = 0, 1, 2, \ldots$ and $m = 1, 2, \ldots$ all the zeros of $\psi^{(k)}_m(z)$ are real and simple.

We will prove Theorem 1 in Section 2. In fact, we will prove a slightly more general one (Theorem 2). Our proof is based on special properties of the Laguerre–Pólya functions and the fact that each $\psi_m(z)$ satisfies a differential equation. Finally, in Section 3, we apply Theorem 1 to show that for each $n$–tuple $(m_1, \ldots, m_n)$ of positive integers and for each $n$–tuple of nonnegative integers $(k_1, \ldots, k_n)$ the translates of the function

$$x_1^{k_1} \cdots x_n^{k_n}(1 + x_1) \cdots (1 + x_n) \exp \left( -\sum_{j=1}^{n} x_j^{2m_j} \right)$$

generate $L^1(\mathbb{R}^n)$, and conclude this paper with a discussion on the minimum number of monomials $p_\alpha(x_1, \ldots, x_n), \alpha \in A$, which have the property that the translates of the functions

$$p_\alpha(x_1, \ldots, x_n) \exp \left( -\sum_{j=1}^{n} x_j^{2m_j} \right) \quad (\alpha \in A)$$

generate $L^1(\mathbb{R}^n)$, for a given $n$–tuple $(m_1, \ldots, m_n)$ of positive integers.

2. The Laguerre–Pólya functions (proof of Theorem 1)

We start this section with a brief introduction to the Laguerre–Pólya functions. An entire function $\psi(z)$ is said to be a Laguerre–Pólya function if it can be expressed in the form

$$\psi(z) = cz^n e^{-\alpha z^2 + \beta z} \prod_j \left( 1 - \frac{z}{a_j} \right) e^{z/a_j},$$

where $c, \beta, a_j$ are real, $\alpha \geq 0$, $n$ is a nonnegative integer and $\sum_j |a_j|^{-2} < \infty$. By a classical result of Laguerre [L] and Pólya [P1], an entire function $\psi(z)$ can be expressed in the form (2) if and only if there is a sequence of real polynomials with real zeros only which converges to $\psi(z)$ uniformly in compact sets in the complex plane. For a modern proof of this theorem, see Levin [Le, Chapter 8]. Therefore if $\psi(z)$ is a Laguerre–Pólya function, then all the derivatives of $\psi(z)$ are also Laguerre–Pólya functions. If $\psi(z)$ is given by (2), then the logarithmic derivative of $\psi(z)$ is given by

$$\frac{\psi'(z)}{\psi(z)} = \frac{n}{z} - 2\alpha z + \beta + \sum_j \left( \frac{1}{z - a_j} + \frac{1}{a_j} \right)$$

and therefore

$$\frac{d}{dz} \left( \frac{\psi'}{\psi} \right)(z) < 0 \quad (z \in \mathbb{R}, \; \psi(z) \neq 0),$$

provided $\psi(z)$ is not of the form $\psi(z) = c e^{\beta z}$.

Let $\psi(z)$ be a transcendental Laguerre–Pólya function. If $a \in \mathbb{R}, \; \psi(a) \neq 0$ and $\psi'(a) = 0$, then (3) implies that

$$\psi(a)\psi''(a) < 0;$$

in particular, $z = a$ is a simple zero of $\psi'(z)$. Since all the derivatives of $\psi(z)$ are Laguerre–Pólya functions, we have the following.
Proposition. Let \( \psi(z) \) be a transcendental Laguerre–Pólya function. If \( k \) is a positive integer and if \( \psi^{(k)}(z) \) has a multiple zero at \( z = a \in \mathbb{R} \), then
\[
\psi(a) = \psi'(a) = \cdots = \psi^{(k)}(a) = 0.
\]
In particular, if \( \psi(z) \) has simple zeros only, then all the derivatives of \( \psi(z) \) have simple zeros only.

Remark 1. This proposition is closely related to the fact that a Laguerre–Pólya function has no Fourier critical points. For the definition of the Fourier critical points of real entire functions and related results, see [CCS], [KK], [Km2], [Km3], [P3]. It may also be remarked that if \( \psi(z) \) is a transcendental Laguerre–Pólya function, then for each positive real number \( B \) there is a positive integer \( N \) such that \( \psi^{(n)}(z) \) has only simple zeros in the interval \([-B\sqrt{n}, B\sqrt{n}]\) whenever \( n \geq N \). For a proof of this fact, see [Km1].

Now, consider the functions \( \psi_m(z) \), \( m = 1, 2, \ldots \), defined by (1). It is clear that \( \psi_1(z) = \sqrt{\pi} \exp(-z^2/4) \) is a Laguerre–Pólya function. For \( m \geq 2 \), \( \psi_m(z) \) is a real entire function of order less than 2 with real zeros only, and therefore Hadamard’s theorem implies that \( \psi_m(z) \) can be expressed in the form (2) with \( \alpha = 0 \). Hence, to prove Theorem 1, it is enough to show that for each \( m = 1, 2, \ldots \), \( \psi_m(z) \) has simple zeros only, because of the proposition.

From an integration by parts, it can easily be shown that for each \( m \geq 2 \),
\[
\psi_m(z) = \frac{(-1)^m}{2m} z \psi_m(z) = 0 \quad (z \in \mathbb{C}).
\]

Then Theorem 1 is a consequence of the following.

Theorem 2. Let \( \psi(z) \) be a transcendental Laguerre–Pólya function, and assume that \( \psi(z) \) satisfies the differential equation
\[
\psi^{(l)}(z) = A(z) \psi(z) \quad (z \in \mathbb{R})
\]
for some positive integer \( l \) and some function \( A(z) \) which is analytic in the whole real axis. Then all the (real) zeros of \( \psi(z) \) are simple.

Proof. Assume, to get a contradiction, that \( \psi(z) \) has a multiple zero, say at \( z = a \in \mathbb{R} \). Then \( \psi(a) = \psi'(a) = 0 \). From (5), we obtain \( \psi^{(l)}(a) = 0 \). By differentiating both sides of (5), we obtain
\[
\psi^{(l+1)}(z) = A'(z) \psi(z) + A(z) \psi'(z),
\]
so that \( \psi^{(l+1)}(a) = 0 \). Then the proposition implies that
\[
\psi(a) = \psi'(a) = \cdots = \psi^{(l)}(a) = \psi^{(l+1)}(a) = 0.
\]

By differentiating both sides of (5) \( k(> 0) \) times, we obtain
\[
\psi^{(l+k)}(z) = \sum_{\lambda=0}^{k} \binom{k}{\lambda} A^{(k-\lambda)}(z) \psi^{(\lambda)}(z).
\]

Then (6), (7) and an inductive argument shows that
\[
\psi^{(k)}(a) = 0
\]
for all \( k = 0, 1, 2, \ldots \), and this is the desired contradiction. \( \Box \)
Remark 2. Let $k$ and $l$ be integers with $0 \leq k < l$. If a transcendental Laguerre–Pólya function $\psi(z)$ satisfies the differential equation
\[
\psi^{(l)}(z) = A_0(z)\psi(z) + A_1(z)\psi'(z) + \cdots + A_k(z)\psi^{(k)}(z) \quad (z \in \mathbb{R})
\]
for some functions $A_0(z), \ldots, A_k(z)$ which are analytic in the whole real axis, then a similar argument as in the above proof shows that every zero of $\psi(z)$ has multiplicity $\leq k + 1$, or equivalently, all the zeros of $\psi^{(k)}(z)$ are simple.

3. An Application to the Harmonic Analysis on $L^1(\mathbb{R}^n)$

For a subset $\mathcal{M}$ of the complex Banach algebra $L^1(\mathbb{R}^n)$ let $I(\mathcal{M})$ denote the ideal generated by $\mathcal{M}$, i.e., $I(\mathcal{M})$ is the smallest closed linear subspace of $L^1(\mathbb{R}^n)$ which contains $\mathcal{M}$, and has the property that
\[
f \in I(\mathcal{M}) \text{ and } g \in L^1(\mathbb{R}^n) \Rightarrow fg \in I(\mathcal{M}).
\]
It is well known (see [R, Chapter 7]) that a closed linear subspace $V$ of $L^1(\mathbb{R}^n)$ is an ideal if and only if it is translation–invariant, i.e.,
\[
f \in V \Rightarrow f_y \in V \quad (y \in \mathbb{R}^n),
\]
where $f_y$ is defined by
\[
f_y(x) = f(x + y) \quad (x \in \mathbb{R}^n)
\]
for $y \in \mathbb{R}^n$. Therefore $I(\mathcal{M})$ is the closed linear subspace of $L^1(\mathbb{R}^n)$ which is generated by the translates of the functions in the set $\mathcal{M}$. The following theorem of N. Wiener, whose proof can be found in [R, Chapter 7] or in [W], gives a necessary and sufficient condition for a subset $\mathcal{M}$ of $L^1(\mathbb{R}^n)$ to satisfy $I(\mathcal{M}) = L^1(\mathbb{R}^n)$ (or equivalently, the translates of all the functions in $\mathcal{M}$ generate $L^1(\mathbb{R}^n)$).

Wiener’s Theorem. Let $\mathcal{M} \subset L^1(\mathbb{R}^n)$. Then $I(\mathcal{M}) = L^1(\mathbb{R}^n)$ if and only if there does not exist a point in $\mathbb{R}^n$ at which the Fourier transforms of all the functions in $\mathcal{M}$ vanish simultaneously.

For $\mathbf{m} = (m_1, \ldots, m_n) \in (\mathbb{N} \setminus \{0\})^n$ let $\phi_{\mathbf{m}} \in L^1(\mathbb{R}^n)$ be defined by
\[
\phi_{\mathbf{m}}(x) = \exp \left( - \sum_{j=1}^{n} x_j^{2m_j} \right) \quad (x = (x_1, \ldots, x_n) \in \mathbb{R}^n).
\]
Then Theorem 1 and Wiener’s theorem imply that if $p(x) = x_1^{k_1} \cdots x_n^{k_n}(1 + x_1) \cdots (1 + x_n)$, where $k_1, \ldots, k_n$ are nonnegative integers, then
\[
I \{ p(x)\phi_{\mathbf{m}}(x) \} = L^1(\mathbb{R}^n)
\]
for each $\mathbf{m} \in (\mathbb{N} \setminus \{0\})^n$.

Let $\mathbf{m} = (m_1, \ldots, m_n) \in (\mathbb{N} \setminus \{0\})^n$ be given. In the remainder of this paper, we will be interested in the minimum number $M(\mathbf{m})$ of monomials $p_\alpha(x)$, $\alpha \in A$, which satisfy
\[
I \{ p_\alpha(x)\phi_{\mathbf{m}}(x) : \alpha \in A \} = L^1(\mathbb{R}^n).
\]
Assume, for a moment, that $m_j \geq 2$ for all $j = 1, \ldots, n$. If $p(x) = x_1^{k_1} \cdots x_n^{k_n}$ is a monomial, then the Fourier transform of $p(x)\phi_{\mathbf{m}}(x)$ is
\[
\prod_{j=1}^{n} (-i)^{k_j} \psi^{(k_j)}(z_j).
\]
Therefore if we are given a set \( \{ p_\alpha(x) : \alpha \in A \} \) of monomials, then the set of Fourier transforms of the functions \( p_\alpha(x)\varphi_m(x) \), \( \alpha \in A \), can be written in the form

\[
\{ f_{\alpha,1}(z_1)f_{\alpha,2}(z_2)\cdots f_{\alpha,n}(z_n) : \alpha \in A \},
\]

where each \( f_{\alpha,j} \) has infinitely many real zeros for \( \alpha \in A \) and \( j = 1, \ldots, n \). If \( \#A \leq n \), then it is clear that there is a point in \( \mathbb{R}^n \) at which all the functions in the above set vanish simultaneously. Consequently, we have

\[
n + 1 \leq M(m).
\]

For \( \mathbf{k} = (k_1, \ldots, k_n) \in \{0,1\}^n \) let \( p_\mathbf{k}(x) = x_1^{k_1} \cdots x_n^{k_n} \). Then it is easy to see that there does not exist a point in \( \mathbb{R}^n \) at which the Fourier transforms of all the functions in the set

\[
\{ p_\mathbf{k}(x)\varphi_m(x) : \mathbf{k} \in \{0,1\}^n \}
\]

vanish simultaneously, so that

\[
(8) \quad M(m) \leq 2^n.
\]

In the general case, we have

\[
(9) \quad r(\mathbf{m}) + 1 \leq M(\mathbf{m}) \leq 2^{r(\mathbf{m})} \quad (\mathbf{m} = (m_1, \ldots, m_n) \in (\mathbb{N} \setminus \{0\})^n),
\]

where \( r(\mathbf{m}) \) denotes the number of the indices \( j \) for which \( m_j \geq 2 \).

Finally, we consider an improvement of the inequality (8). The inequality follows from the fact that for each \( m = 1, 2, \ldots \) there are nonnegative integers \( k_1 \) and \( k_2 \), namely 0 and 1, such that the zero sets of \( \psi_m^{(k_1)}(z) \) and \( \psi_m^{(k_2)}(z) \) are disjoint. In fact, for each \( m = 1, 2, \ldots \) the zero sets of the functions \( \psi_m(z) \), \( \psi_m'(z) \) and \( \psi_m^{(2m)}(z) \) are mutually disjoint, because of the differential equation

\[
\psi_m^{(2m)}(z) - \frac{(-1)^m}{2m} z\psi_m'(z) - \frac{(-1)^m}{2m} \psi_m(z) = 0 \quad (z \in \mathbb{C}),
\]

which is obtained by differentiating both sides of (4). Hence there does not exist a point in \( \mathbb{R}^n \) at which the Fourier transforms of all the functions in the set

\[
\{ p(x_1,x_2)x_3^{k_3} \cdots x_n^{k_n} \varphi_m(x) : p(x_1,x_2) = 1, x_1x_2, \text{ or } x_1^{2m_1}x_2^{2m_2}, \text{ and } k_3, \ldots, k_n \in \{0,1\} \}
\]

vanish simultaneously, so that \( M(\mathbf{m}) \leq 3 \cdot 2^{n-2} \) for \( n \geq 2 \). Therefore we can replace the inequality \( M(\mathbf{m}) \leq 2^{r(\mathbf{m})} \) in (9) by \( M(\mathbf{m}) \leq 3 \cdot 2^{r(\mathbf{m})-2} \) whenever \( r(\mathbf{m}) \geq 2 \).

Let \( l \) be an integer greater than 3. If it were true that for each \( m = 1, 2, \ldots \) there are nonnegative integers \( k_1, \ldots, k_l \) such that the zero sets of \( \psi_m^{(k_1)}(z), \ldots, \psi_m^{(k_l)}(z) \) are mutually disjoint, then a similar argument as above would imply that \( M(\mathbf{m}) = r(\mathbf{m}) + 1 \) if \( r(\mathbf{m}) \leq l - 1 \), and \( M(\mathbf{m}) \leq l \cdot 2^{r(\mathbf{m})-l+1} \) if \( r(\mathbf{m}) > l - 1 \). We do not know whether or not such nonnegative integers exist, so we conclude this paper with the following question.

**Question.** Let \( l \geq 4 \) and \( m \geq 1 \) be integers. Do there exist nonnegative integers \( k_1, \ldots, k_l \) such that the zero sets of \( \psi_m^{(k_1)}(z), \ldots, \psi_m^{(k_l)}(z) \) are mutually disjoint?
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REFERENCES


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