AUTOMATIC SURJECTIVITY OF RING HOMOMORPHISMS ON H^*-ALGEBRAS AND ALGEBRAIC DIFFERENCES AMONG SOME GROUP ALGEBRAS OF COMPACT GROUPS

LAJOS MOLNÁR

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. In this paper we present two automatic surjectivity results concerning ring homomorphisms between p-classes of an H^*-algebra which, in some sense, improve the main theorem in a recent paper by the author (*Proc. Amer. Math. Soc.* **124** (1996), 169–175) quite significantly. Furthermore, we apply our results to show that for arbitrary infinite compact groups G, G', no quotient ring of $L^2(G)$ is isomorphic to $L^p(G')$ $(2 < p \leq \infty)$, a statement we conjecture to be true for every pair $L^p(G), L^q(G')$ of group rings corresponding to different exponents $1 \leq p, q \leq \infty$.

The study of ring homomorphisms between Banach algebras has a long history. Probably the first such result is due to Eidelheit [Eid] who proved that every ring isomorphism between the algebras of all bounded linear operators acting on real Banach spaces is implemented by an invertible operator and hence it is continuous and linear. For the case of complex spaces and a recent far-reaching generalization concerning not only ring but semigroup isomorphisms of standard operator algebras we refer to [Arn] and [Sem], respectively. Moreover, in the case of general semisimple Banach algebras, the most important result on the linearity of ring isomorphisms is due to Kaplansky [Kap].

Motivated by the theory of operator ranges, in our papers [Mol2], [Mol3] we studied homomorphism ranges and considered surjective and "almost" surjective ring homomoprhisms (continuity is never assumed) between some operator algebras and function algebras, respectively. As the main corollary in [Mol3], we obtained the existence of ring theoretical differences among some important function algebras. The result of [Mol2] is in a close relation to our present investigations. It states that there is no surjective ring homomorphism between different p-classes of an infinite-dimensional H^*-algebra. These classes are common generalizations of the p-classes of compact operators and l_p spaces. Our mentioned result shows that

the ring structure of these Banach algebras depends heavily on the parameter p, although there is deep similarity in their constructions.

In the present paper we revisit this problem and, in some sense, strengthen the result in [Mol2] quite significantly. Afterwards, emphasizing the complete analogy with the case of p-classes of an H^*-algebra, we verify the conjecture formulated in the abstract for the case of $L^2(G)$ and $L^p(G')$ ($2 < p \leq \infty$).

Let us begin with the notation and some basic results that we shall use. If H is a Hilbert space and $1 \leq p < \infty$, then let $\mathcal{B}(H)$, $\mathcal{C}(H)$ and $\mathcal{C}_p(H)$ denote the set of all bounded, compact and Schatten-von Neumann p-class operators on H, respectively.

The algebra $\mathcal{A}$ is called an H^*-algebra if it is a Banach *-algebra whose norm is a Hilbert space norm such that $\langle x, yz^* \rangle = \langle xz, y \rangle = \langle z, x^*y \rangle$ holds for every $x, y, z \in \mathcal{A}$. By the structure theorem of H^*-algebras [Amb], if $\mathcal{A}$ is semisimple, then it is the orthogonal direct sum of its minimal closed ideals $\mathcal{A}_\alpha$ ($\alpha \in \Gamma$). Moreover, every $\mathcal{A}_\alpha$ is isometric and *-isomorphic to the H^*-algebra $\mathcal{C}_2(H_\alpha)$ of Hilbert-Schmidt operators acting on a Hilbert space H_α, where the original Hilbert-Schmidt norm is multiplied by a constant $c_\alpha \geq 1$ which is the common value of the norms of the minimal nonzero projections in $\mathcal{A}_\alpha$.

In the case of H^*-algebras, centralizers play the same role as the bounded linear operators do in the case of Hilbert spaces. A (left) centralizer is a function $T : \mathcal{A} \to \mathcal{A}$ with the property that $T(xy) = (Tx)y$ holds for every $x, y \in \mathcal{A}$. In the paper [SG] it is shown that these mappings are automatically continuous linear operators and they form a C^*-subalgebra of the whole operator algebra on $\mathcal{A}$ (here $\mathcal{A}$ is considered as a Hilbert space). In our recent paper [Mol1] we brought the analogy between centralizer algebras and operator algebras closer by realizing algebra isomorphisms between some ideals of the centralizer algebra and those of certain direct sums of operator algebras acting on Hilbert spaces. To be more detailed, we first established a spectral theorem for the elements of $\mathcal{C}(\mathcal{A})$, the C^*-subalgebra of the centralizer algebra generated by the left multiplications. Roughly speaking, the elements of $\mathcal{C}(\mathcal{A})$ are the "compact operators" on $\mathcal{A}$. After this, we defined $\mathcal{C}_p(\mathcal{A})$, the p-class of $\mathcal{A}$, in a way very similar to that which is commonly done concerning the Schatten-von Neumann p-classes of compact operators (as a basic reference on these classes, see [GK]). To obtain the mentioned isomorphisms between these p-classes and certain operator ideals we need the following construction.

Let $I \neq \emptyset$ be a set and suppose that $\mathcal{A}_i$ is a Banach *-algebra and k_i is a positive real number for every $i \in I$. If $1 \leq p < \infty$, then let

$$l_p(\mathcal{A}_i, k_i, I) = \{(x_i)_{i \in I} : x_i \in \mathcal{A}_i \ (i \in I), \ \left(\sum_i k_i \|x_i\|^p\right)^{1/p} < \infty\}$$

and let

$$c_0(\mathcal{A}_i, k_i, I) = \{(x_i)_{i \in I} : x_i \in \mathcal{A}_i \ (i \in I), \ (k_i \|x_i\|) \in c_0(I)\}.$$

It is easy to see that these spaces (called the l_p- and c_0-direct sums of $\{\mathcal{A}_i\}_{i \in I}$) with the norms suggested in their definitions are Banach *-algebras.

Now, our result cited in the text above [Mol1, Theorem 3] states that if $\mathcal{A}$ is a semisimple H^*-algebra and $1 \leq p < \infty$, then $\mathcal{C}_p(\mathcal{A})$ is isometric and *-isomorphic to $l_p(\mathcal{C}_p(H_\alpha), c_\alpha, \Gamma)$, where H_α and c_α are as above. A similar assertion holds true for $\mathcal{C}(\mathcal{A})$ and $c_0(\mathcal{C}(H_\alpha), 1, \Gamma)$.

After this introduction, we are in a position to present and prove the following automatic surjectivity result, the main result of the paper, on which the partial answer to our conjecture mentioned in the abstract rests.

Main Theorem. *Let $\mathcal{A}, \mathcal{B}$ be semisimple H^*-algebras. Suppose that $\|e\| = \|f\|$ whenever e, f are minimal nonzero projections and $e \in \mathcal{A}'$, $f \in \mathcal{B}'$ for some minimal closed ideals $\mathcal{A}' \subset \mathcal{A}$, $\mathcal{B}' \subset \mathcal{B}$ with the same Hilbert dimension. Let $1 \leq p < \infty$ and let $\Phi : \mathcal{C}_p(\mathcal{A}) \to \mathcal{C}_p(\mathcal{B})$ be a ring homomorphism whose range contains a dense subalgebra. Then Φ is in fact surjective.*

Remark. Observe that the above condition on the equality of the norms of minimal projections in isomorphic minimal closed ideals cannot be removed as it is easily seen from the example of the natural injection of the H^*-algebra $l_2(\mathbb{C}, n, \mathbb{N})$ into $l_2(\mathbb{C}, 1, \mathbb{N})$.

Proof. We first show that the kernel J of Φ is closed. Let $x \in \overline{J}$ and let $a \in \mathcal{C}_p(\mathcal{A})$ be arbitrary. Since J is a ring ideal, we obtain $xa \in \overline{J}$. Let $m \in J$ be so that $\|xa - m\| < 1$. Then $xa - m$ is quasi-invertible, i.e. there exists a $y \in \mathcal{C}_p(\mathcal{A})$ such that

$$(xa - m)y = y(xa - m) = (xa - m) + y.$$

Taking images under Φ, we have

$$\Phi(x)\Phi(a)\Phi(y) = \Phi(y)\Phi(x)\Phi(a) = \Phi(x)\Phi(a) + \Phi(y).$$

Since $\mathcal{C}_p(\mathcal{B})$ is dense in $\mathcal{C}(B)$, hence $\Phi(x)z$ is quasi-invertible for every z from a dense subalgebra $\mathcal{D}$ of $\mathcal{C}(\mathcal{B})$. We infer that $\sigma(\Phi(x)z) = \{0\}$ and consequently $r(\Phi(x)z) = 0$ holds for every $z \in \mathcal{D}$. A classical theorem of Newburgh [New, Theorem 3] says that on a Banach algebra with the property that every element has totally disconnected spectrum, the spectrum is a continuous function with respect to the Hausdorff metric on the compact subsets of $\mathbb{C}$. The Banach algebra $\mathcal{C}(\mathcal{B})$ has this property. Indeed, $\mathcal{C}(\mathcal{B})$ is isometric and *-isomorphic to a C^*-subalgebra of compact operators acting on a Hilbert space [Mol1, Theorem 1]. Since the spectrum with respect to a C^*-algebra and to any of its C^*-subalgebras is the same, hence every element of $\mathcal{C}(\mathcal{B})$ has totally disconnected spectrum. Now, Newburgh's theorem applies and it follows that $r(\Phi(x)z) = 0$ $(z \in \mathcal{C}(\mathcal{B}))$. Since $\mathcal{C}(\mathcal{B})$ is a C^*-algebra, we have $\Phi(x) = 0$.

We may and do suppose that

$$\mathcal{C}_p(\mathcal{A}) = l_p(\mathcal{C}_p(H_\alpha), c_\alpha, \Gamma) \quad \text{and} \quad \mathcal{C}_p(\mathcal{B}) = l_p(\mathcal{C}_p(K_\beta), d_\beta, \Delta).$$

In what follows, for any fixed β, let Φ_β denote the βth coordinate function of Φ. Let M denote the kernel of Φ_β and for every α let M_α be the image of M under the natural projection onto the αth coordinate space of $l_p(\mathcal{C}_p(H_\alpha), c_\alpha, \Gamma)$. Since Φ_β has a dense subalgebra in its range, hence M is closed and we claim that for every α either $M_\alpha = \{0\}$ or $M_\alpha = \mathcal{C}_p(H_\alpha)$. To see this, for an arbitrary $x_\alpha \in \mathcal{C}_p(H_\alpha)$ let $\overline{x_\alpha}$ denote the element of $l_p(\mathcal{C}_p(H_\alpha), c_\alpha, \Gamma)$ whose coordinates are 0 except for the αth one which is x_α. Suppose $0 \neq x_\alpha \in M_\alpha$. Since M is an ideal, we obtain for any $a_\alpha, b_\alpha \in \mathcal{C}_p(H_\alpha)$ that $\overline{a_\alpha x_\alpha b_\alpha} \in M$. Consequently, for every finite rank operator f_α on H_α we have $\overline{f_\alpha} \in M$. But we know that M is closed and the finite rank operators form a dense subspace in $\mathcal{C}_p(H_\alpha)$. Thus $\overline{x_\alpha} \in M$ for any $x_\alpha \in \mathcal{C}_p(H_\alpha)$ and we have $M_\alpha = \mathcal{C}_p(H_\alpha)$.

We next assert that there is at most one index α for which $M_\alpha = \{0\}$ holds. Assume that there are two such indexes, say α_1 and α_2. Let

$$N = \{\Phi_\beta(x) \,:\, x \in \mathcal{C}_p(\mathcal{A}),\, x_{\alpha_1} = 0\}.$$

If $0 \neq x_{\alpha_2} \in \mathcal{C}_p(H_{\alpha_2})$, then $0 \neq \Phi_\beta(\overline{x_{\alpha_2}}) \in N$. One can easily verify that the closure C of N in the operator norm is an ideal. We then have $C = \mathcal{C}(K_\beta)$. Let $0 \neq x_{\alpha_1} \in \mathcal{C}_p(H_{\alpha_1})$. For the element $\Phi_\beta(\overline{x_{\alpha_1}}) \neq 0$ we obtain that $\Phi_\beta(\overline{x_{\alpha_1}})N = \{0\}$ and hence that $\Phi_\beta(\overline{x_{\alpha_1}})C = \{0\}$. Since this is a contradiction, we have the assertion.

We now prove that there exists an index α such that $M_\alpha = \{0\}$. If we suppose, on the contrary, that there is no such α, then we have $M_\alpha = \mathcal{C}_p(H_\alpha)$ for every $\alpha \in \Gamma$. Since M is closed, this implies $M = \mathcal{C}_p(\mathcal{A})$ which is untenable.

So, let α denote the uniquely determined index corresponding to β for which $M_\alpha = \{0\}$. Then we have

$$M = \{x \in \mathcal{C}_p(\mathcal{A}) \,:\, x_\alpha = 0\}$$

and this implies that there exists a ring homomorphism $\phi_\beta : \mathcal{C}_p(H_\alpha) \to \mathcal{C}(K_\beta)$ such that

$$\Phi_\beta(x) = \phi_\beta(x_\alpha) \qquad (x \in \mathcal{C}_p(\mathcal{A})).$$

Now we need the following

Lemma. *Suppose that H, K are complex Hilbert spaces, H is infinite dimensional and $1 \leq p < \infty$. Let $\Phi : \mathcal{C}_p(H) \to \mathcal{C}(K)$ be a ring homomorphism whose range contains a dense subalgebra. Then there exists a bounded invertible linear or conjugate-linear operator $T : H \to K$ such that*

$$\Phi(A) = TAT^{-1} \qquad (A \in \mathcal{C}_p(H)).$$

In particular, $rng\, \Phi = \mathcal{C}_p(K)$ and Φ is a continuous linear or conjugate-linear algebra isomorphism of $\mathcal{C}_p(H)$ onto $\mathcal{C}_p(K)$.

Proof. By the first part of the proof of Main Theorem, the kernel of Φ is closed and hence Φ is injective. It is well-known that any additive function between Banach spaces is continuous if and only if it is bounded on the unit ball. Now, let (P_n) be a sequence of pairwise orthogonal rank-one projections. We assert that for some $n \in \mathbb{N}$, the mapping

$$A \longmapsto \Phi(AP_n)$$

is continuous on $\mathcal{C}_p(H)$ (cf. the proof of [Dra, 14.3. Teorema]). Suppose, on the contrary, that for every $n \in \mathbb{N}$ there exists an operator $A_n \in \mathcal{C}_p(H)$ such that $\|A_n\|_p \leq 1$ and $\|\Phi(A_nP_n)\| \geq r_n n 2^n$, where $r_n = \|\Phi(P_n)\|$. Let

$$C = \sum_n \frac{1}{2^n} A_n P_n,$$

where the sum is unconditionally convergent in $\mathcal{C}_p(H)$. We then infer

$$\|\Phi(C)\| r_n \geq \|\Phi(C)\Phi(P_n)\| = \frac{1}{2^n} \|\Phi(A_nP_n)\| \geq r_n n.$$

Since this holds for every $n \in \mathbb{N}$, we arrive at a contradiction. Now, let $P = y_0 \otimes y_0 \in \mathcal{C}_p(H)$ be a projection such that the mapping $A \mapsto \Phi(AP)$ is continuous. Let $z_0 \in K$ be with $\Phi(P)z_0 \neq 0$. Define the function $T : H \to K$ by

$$Tx = \Phi(x \otimes y_0)z_0 \qquad (x \in H).$$

Since $Tx = \Phi(x \otimes y_0 \cdot P)z_0$ $(x \in H)$, hence T is continuous. Moreover, T is additive and

$$\Phi(A)Tx = \Phi(A)\Phi(x \otimes y_0)z_0 = \Phi(Ax \otimes y_0)z_0 = T(Ax) \quad (A \in \mathcal{C}_p(H),\ x \in H),$$

that is,

$$\Phi(A)T = TA \qquad (A \in \mathcal{C}_p(H)).$$

It is apparent that T is injective. Indeed, if $0 \neq x \in H$ and $Tx = 0$, then $TAx = \Phi(A)Tx = 0$ for every $A \in \mathcal{C}_p(H)$. As a consequence, we have $T = 0$ which contradicts $Ty_0 \neq 0$.

Our next claim is that the range of T is dense. Let $z \in K$ be arbitrary and let $R \in \mathcal{C}(K)$ be a rank-one operator such that $RTy_0 = z$. By the density of the range of Φ, there exists a sequence (A_n) in $\mathcal{C}_p(H)$ such that $\Phi(A_n) \to R$. This implies

$$TA_n y_0 = \Phi(A_n)Ty_0 \longrightarrow RTy_0 = z$$

and the claim is reached.

We now prove that T is in fact surjective. Let $y \in H$ and let (x_n) be a sequence in H such that $Tx_n \to y$. Then $TAx_n = \Phi(A)Tx_n \to \Phi(A)y$ and this implies that (TAx_n) is convergent for every $A \in \mathcal{C}_p(H)$. If $0 \neq u \in H$ is fixed and $v \in H$ is arbitrary, then by the real-linearity of T (recall that T is continuous and additive) we obtain that

$$\Re\langle x_n, v \rangle Tu + \Im\langle x_n, v \rangle T(iu) = T(u \otimes v)x_n.$$

Since Tu and $T(iu)$ are $\mathbb{R}$-independent, we infer that $(\Re\langle x_n, v \rangle)$ and $(\Im\langle x_n, v \rangle)$ are convergent. This gives that $(\langle x_n, v \rangle)$ is convergent for every $v \in H$. By the principle of uniform boundedness and Riesz representation theorem, it follows that (x_n) is weakly convergent. So, let $x_n \xrightarrow{w} x$ for some $x \in H$. Up to this point we have considered H as a complex Banach space. Now, let us consider it as a real one. Clearly, the real dual of a complex Banach space coincides with the real part of its complex dual. Since T is real weak-weak continuous which follows from the norm continuity of T, hence we have

$$\Re\langle Tx_n, z \rangle \longrightarrow \Re\langle Tx, z \rangle \qquad (z \in K).$$

But z runs through K and thus we get $\langle Tx_n, z \rangle \to \langle Tx, z \rangle$ $(z \in K)$. Taking the convergence $Tx_n \to y$ into consideration, we obtain $y = Tx$. This gives the surjectivity of T.

We know that $\Phi(A) = TAT^{-1}$ $(A \in \mathcal{C}_p(H))$. In particular, for every $x, y \in H$, the function $T(x \otimes y)T^{-1}$ is a complex-linear operator. We then have

$$\Re\langle T^{-1}(iz), y \rangle Tx + \Im\langle T^{-1}(iz), y \rangle T(ix) = T(\langle T^{-1}(iz), y \rangle x) = (T(x \otimes y)T^{-1})(iz)$$

$$= i(T(x \otimes y)T^{-1})(z) = iT(\langle T^{-1}z, y \rangle x) = i\Re\langle T^{-1}z, y \rangle Tx + i\Im\langle T^{-1}z, y \rangle T(ix)$$

for any $z \in H$. Let x be arbitrary but fixed, $0 \neq y \in H$ and $z = T(iy)$. Then $i\Im\langle T^{-1}z, y \rangle \neq \Im\langle T^{-1}(iz), y \rangle$ and by

$$(\Im\langle T^{-1}(iz), y \rangle - i\Im\langle T^{-1}z, y \rangle)T(ix) = (i\Re\langle T^{-1}z, y \rangle - \Re\langle T^{-1}(iz), y \rangle)Tx$$

we infer that there exists a complex number λ_x such that $T(ix) = \lambda_x Tx$. Consequently, for every $x \in H$ there is a function $\tau_x : \mathbb{C} \to \mathbb{C}$ with the property that

$$T(\lambda x) = \tau_x(\lambda)Tx \quad (\lambda \in \mathbb{C}).$$

We show that τ_x does not depend on x. If $y \in H$ and $\{Tx, Ty\}$ is linearly independent, then for every $\lambda \in \mathbb{C}$ we have

$$\tau_{x+y}(\lambda)Tx + \tau_{x+y}(\lambda)Ty = \tau_{x+y}(\lambda)T(x+y) = T(\lambda(x+y))$$
$$= T(\lambda x) + T(\lambda y) = \tau_x(\lambda)Tx + \tau_y(\lambda)Ty.$$

This implies

$$\tau_x(\lambda) = \tau_{x+y}(\lambda) = \tau_y(\lambda).$$

If $x, y \neq 0$ and $\{Tx, Ty\}$ is dependent, then take a nonzero vector $z \in H$ such that $\{Tx, Tz\}$ and $\{Tz, Ty\}$ are independent. Hence, for every $\lambda \in \mathbb{C}$ we obtain

$$\tau_x(\lambda) = \tau_z(\lambda) = \tau_y(\lambda).$$

Now, let $\tau : \mathbb{C} \to \mathbb{C}$ be such that $T(\lambda x) = \tau(\lambda)Tx$ ($x \in H$, $\lambda \in \mathbb{C}$). One can easily verify that τ is a nonzero ring endomorphism of $\mathbb{C}$. By the continuity of T we have the continuity of τ. It is well-known (see [AD, pp. 52–57] or [Kuc, Lemma 1, p. 356]) that in this case τ is either the identity or the conjugation. Hence, T is either a linear or conjugate-linear continuous bijection from H onto K. Since $\Phi(A) = TAT^{-1}$ ($A \in \mathcal{C}_p(H)$), we immediately have the second statement of the lemma. $\square$

We shall also need the finite dimensional version of the previous lemma. Observe that if H is finite dimensional, then so is K. One can quickly obtain this statement by considering the identity in $\mathcal{C}_p(H)$ and its image which must be an identity too. Consequently, in this case we have a ring isomorphism Ψ between matrix algebras $M_{n \times n}$ and $M_{m \times m}$. Now, similarly to [Mol2, Lemma 3 and its corollary] we conclude that there is a ring automorphism $h : \mathbb{C} \to \mathbb{C}$ and an invertible matrix T such that Φ is of the form

$$\Phi(A) = T[h(A)]T^{-1},$$

where $h(A)$ denotes the matrix obtained from $A \in M_{n \times n}$ by applying h for every entry of it.

Let us turn back to the proof of our theorem. We assert that to different β's there correspond different α's. Suppose, on the contrary, that to $\beta \neq \beta'$ there corresponds the same α. Suppose that H_α is infinite dimensional. Let $a \in \mathcal{C}(K_\beta)$ be a nonzero operator. There are bounded invertible linear or conjugate-linear operators $t_\alpha, t_{\alpha'}$ such that, by the condition on density, there exists a sequence (x_n) in $\mathcal{C}_p(H_\alpha)$ with

$$t_\alpha x_n t_\alpha^{-1} \longrightarrow a, \quad t_{\alpha'} x_n t_{\alpha'}^{-1} \longrightarrow 0 \quad (n \longrightarrow \infty).$$

Since this is an obvious contradiction, we obtain the assertion. Assume now that H_α is finite dimensional. It is easy to see that in this case we have ring automorphisms h, h' of $\mathbb{C}$ such that the set of all pairs $(h(x), h'(x))$ contains a nontrivial algebra $\mathcal{D}$. If $h(x) = a, h'(x) = b$, then we have $h'h^{-1}(a) = b$. Hence, the ring automorphism $h'' = h'h^{-1}$ of $\mathbb{C}$ has the property that $h''(a) = b$ for every $(a, b) \in \mathcal{D}$. Since $\mathcal{D}$ is an algebra, we readily obtain that h'' is linear and hence it is the identity on $\mathbb{C}$. Therefore, we have $h = h'$ and now the assertion follows as in the infinite dimensional case.

We next claim that there is only a finite number of the involved h's which are discontinuous. Indeed, let us suppose that we have infinitely many such ring automorphisms (h_n). Clearly, if an additive function is not continuous on $\mathbb{C}$, then it is unbounded on every disc. Hence, we have a sequence (λ_n) of complex numbers and a subsequence (c_{α_n}) such that $\sum_n c_{\alpha_n} \dim(H_{\alpha_n})|\lambda_n|^p < \infty$ but $h_n(\lambda_n) \nrightarrow 0$. Considering an appropriate element x of $l_p(\mathcal{C}_p(H_\alpha), c_\alpha, \Gamma)$ whose coordinates are either $\lambda_n i_n$ (i_n being the identity on H_{α_n}) or 0, we have $\Phi(x) \notin c_0(\mathcal{C}(K_\beta), 1, \Delta)$. Since this is untenable, we get the claim.

Observe that until this point we have used only the property of Φ that $rng\,\Phi \subset \mathcal{C}(\mathcal{B})$ contains a dense subalgebra. Now, for every β, let $f(\beta)$ denote the uniquely determined α corresponding to β. From what is proved above, we conclude that, eliminating a finite number of finite dimensional minimal closed ideals in $\mathcal{A}$ and $\mathcal{B}$, we may suppose that there are invertible bounded linear or conjugate-linear operators $t_\beta : H_{f(\beta)} \to K_\beta$ such that

$$\Phi : (x_\alpha) \longmapsto (t_\beta x_{f(\beta)} t_\beta^{-1})$$

and it is sufficient to prove that the range of this mapping is equal to the l_p-direct sum of the corresponding $\mathcal{C}_p(K_\beta)$ spaces. We know that $\sum_\beta c_{f(\beta)}\|x_{f(\beta)}\|_p^p < \infty$ implies $\sum_\beta d_\beta\|t_\beta x_{f(\beta)} t_\beta^{-1}\|_p^p < \infty$. Observe that the norm of a minimal idempotent $e \in \mathcal{A}$ is $\sqrt{c_\alpha}$ if e belongs to the minimal closed ideal $\mathcal{C}_2(H_\alpha)$. Since $H_{f(\beta)}$ and K_β are isomorphic as topological vector spaces, it is easy to see (by the polar decomposition, for example) that they are isomorphic as Hilbert spaces. This results in that $\mathcal{C}_2(H_{f(\beta)})$ and $\mathcal{C}_2(K_\beta)$ are also isomorphic as Hilbert spaces. It now follows from the condition of the theorem that $d_\beta = c_{f(\beta)}$ holds true for every β. Thus we have the property that

$$\sum_\beta d_\beta\|x_{f(\beta)}\|_p^p < \infty \Longrightarrow \sum_\beta d_\beta\|t_\beta x_{f(\beta)} t_\beta^{-1}\|_p^p < \infty.$$

We prove that this implies that $\sup_\beta \|t_\beta\|\|t_\beta^{-1}\| < \infty$. Indeed, if this set is not bounded, then we can choose a subsequence (β_n) with pairwise different terms such that

$$\|t_{\beta_n}\|^p \|t_{\beta_n}^{-1\,*}\|^p > n^2$$

holds for every n. Let $x_{f(\beta_n)}$ be the tensor product of two vectors constructed in the following way. Let u_n, v_n be unit vectors in $H_{f(\beta_n)}$ such that

$$\|t_{\beta_n}(u_n)\|^p \|t_{\beta_n}^{-1\,*}(v_n)\|^p > n^2.$$

Let

$$x_{f(\beta_n)} = \frac{1}{\sqrt[p]{d_{\beta_n} n^2}} u_n \otimes v_n.$$

It is easy to check that $\sum_n d_{\beta_n}\|x_{f(\beta_n)}\|_p^p < \infty$ but $\sum_n d_{\beta_n}\|t_{\beta_n} x_{f(\beta_n)} t_{\beta_n}^{-1}\|_p^p = \infty$. Since this is a contradiction, thus we have $\sup_\beta \|t_\beta\|\|t_\beta^{-1}\| < \infty$. The boundedness of this set immediately gives the property that

$$\sum_\beta d_\beta\|y_\beta\|_p^p < \infty \Longrightarrow \sum_\beta d_\beta\|t_\beta^{-1} y_\beta t_\beta\|_p^p < \infty.$$

The surjectivity of Φ is now easy to see. $\qquad\square$

Remark. Observe that an argument very similar to that discussed above yields the validity of the statement for the algebras $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{B})$ instead of $\mathcal{C}_p(\mathcal{A})$ and $\mathcal{C}_p(\mathcal{B})$.

By a slight modification in the last part of the proof of our main theorem, one can verify the following statement.

Theorem 1. *Let $\mathcal{A}, \mathcal{B}$ be semisimple H^*-algebras. Suppose that the set of all nonzero minimal projections in $\mathcal{B}$ is norm bounded. Let $\|e\| = \|f\|$ hold true for every pair of nonzero minimal projections $e \in \mathcal{A}$, $f \in \mathcal{B}$ from minimal closed ideals with the same Hilbert dimension. Let $1 \le p < \infty$ and let $\Phi : \mathcal{C}_p(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$ be a ring homomorphism whose range contains a dense subalgebra. Then we have $rng\,\Phi = \mathcal{C}_p(\mathcal{B})$.*

Example. To see that the condition on boundedness cannot be omitted, let us consider the H^*-algebras $\mathcal{A} = \mathcal{B} = l_2(\mathcal{C}_2(\mathbb{C}^2), n^4, \mathbb{N})$. For every $n \in \mathbb{N}$ let

$$t_n = \begin{pmatrix} n & 0 \\ 0 & 1/n \end{pmatrix}.$$

It is easy to see that the homomorphism $\Phi : \mathcal{C}_2(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$ defined by

$$\Phi((x_n)) = (t_n x_n t_n^{-1})$$

has the property that its range contains a dense subalgebra (for example, the algebra of all cofinite sequences) of $\mathcal{C}(\mathcal{B})$. But for

$$x_n = \begin{pmatrix} 1/n^3 & 1/n^3 \\ 1/n^3 & 1/n^3 \end{pmatrix} \qquad (n \in \mathbb{N})$$

we have $(x_n) \in \mathcal{C}_2(\mathcal{A})$, $\Phi((x_n)) \notin \mathcal{C}_2(\mathcal{B})$.

Let us turn now to our problem concerning group rings which we formulated in the abstract and in the introduction of the paper. We sum up a few basic facts from harmonic analysis that we shall use. Our general reference is [HR, Chapters VII, VIII]. Let G be a compact group with dual object Σ. For any $1 \le p < \infty$ let

$$\mathcal{C}_p(\Sigma) = l_p(\mathcal{C}_p(H_\sigma), d_\sigma, \Sigma),$$
$$\mathcal{C}_0(\Sigma) = c_0(\mathcal{C}(H_\sigma), 1, \Sigma),$$

where for every $\sigma \in \Sigma$ we select a fixed member $U^{(\sigma)}$ of σ with representation space H_σ of dimension $d_\sigma < \infty$. We note that $\mathcal{C}_0(\Sigma)$ and $\mathcal{C}_p(\Sigma)$ are *-isomorphic and isometric to $\mathcal{C}(L^2(G))$ and $\mathcal{C}_p(L^2(G))$, respectively. Let $\mathcal{C}_{00}(\Sigma)$ stand for the subspace of $\mathcal{C}_0(\Sigma)$ consisting of all cofinite systems. The Fourier transform $\hat{}: f \mapsto \hat{f}$ is an algebra *-isomorphism of $L^1(G)$ onto a dense subalgebra of $\mathcal{C}_0(\Sigma)$. Moreover, the mapping $\hat{}: f \mapsto \hat{f}$ is an inner product preserving linear bijection between the Hilbert spaces $L^2(G)$ and $\mathcal{C}_2(\Sigma)$.

Theorem 2. *Let G, G' be compact groups, let G' be infinite and let $2 < p \le \infty$. Then there does not exist a surjective ring homomorphism from $L^2(G)$ onto $L^p(G')$. In particular, these Banach algebras are not isomorphic even as rings.*

Proof. Suppose, on the contrary, that there is an epimorphism from $L^2(G)$ onto $L^p(G')$. Since $L^p(G') \subset L^2(G')$, we have a ring homomorphism Φ from $\mathcal{C}_2(\Sigma)$ into $\mathcal{C}_2(\Sigma')$ whose range contains $L^p(G')\hat{\ }$. It is known that the set $T(G')$ of all trigonometric polynomials on G' is contained in $C(G') \subset L^p(G')$ and its image under the Fourier transform is equal to $\mathcal{C}_{00}(\Sigma')$. It is easy to see that $\mathcal{C}_{00}(\Sigma') \subset$

$\mathcal{C}_2(\Sigma')$ is a dense subalgebra and hence by our main result we obtain that Φ is surjective. As a consequence, using the inverse Fourier transform, we have $L^p(G') = L^2(G')$. Since G' is infinite, the measure of every singleton is 0. By the regularity of the Haar measure we can choose an infinite sequence of pairwise disjoint Borel sets with positive measure. Obviously, this contradicts $L^p(G') = L^2(G')$, completing the proof. $\qquad\square$

Since the minimal closed ideals of the L^2-algebra of any compact Abelian group G are one-dimensional, hence the set of all minimal projections in $L^2(G)$ is bounded. Using Theorem 1 and the proof of Theorem 2 we obtain our final result which follows.

Theorem 3. *Suppose that G, G' are compact groups, G' is infinite and Abelian. Let $1 \leq p \leq \infty$, $p \neq 2$. Then there does not exist a surjective ring homomorphism from $L^2(G)$ onto $L^p(G')$.*

We conclude the paper by leaving our general conjecture as an open problem.

References

[AD]   J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Encyclopedia Math. Appl. 31, Cambridge University Press, 1989. MR **90h:**39001

[Amb]   W. Ambrose, *Structure theorems for a special class of Banach algebras*, Trans. Amer. Math. Soc. **57** (1945), 364–386. MR **7:**126c

[Arn]   B.H. Arnold, *Rings of operators on vector spaces*, Ann. of Math. **45** (1944), 24–49. MR **5:**147c

[Dra]   D.D. Draghia, *Continuitate in Algebre Banach*, Editure didactica si pedagogica, Bucuresti, 1995.

[Eid]   M. Eidelheit, *On isomorphisms of rings of linear operators*, Studia Math. **9** (1940), 97–105. MR **3:**51e

[GK]   I.C. Gohberg and M.G. Krein, *Introduction to The Theory of Linear Nonselfadjoint Operators*, Translations of Mathematical Monographs Vol. 18, American Mathematical Society, 1969. MR **39:**7447

[HR]   E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis II.*, Springer-Verlag, 1970. MR **41:**7378

[Kap]   I. Kaplansky, *Ring isomorphisms of Banach algebras*, Canad. Math. J. **6** (1954), 374–381. MR **16:**49e

[Kuc]   M. Kuczma, *An Introduction to The Theory of Functional Equations and Inequalities*, Państwowe Wydawnictwo Naukowe, Warszawa, 1985. MR **86i:**39008

[Mol1] L. Molnár, *p-classes of an H*-algebra and their representations*, Acta Sci. Math. (Szeged) **58** (1993), 411–423. MR **95c:**46081

[Mol2] L. Molnár, *Algebraic difference between p-classes of an H*-algebra*, Proc. Amer. Math. Soc. **124** (1996), 169-175. MR **96d:**46072

[Mol3] L. Molnár, *The range of a ring homomorphism from a commutative C^*-algebra*, Proc. Amer. Math. Soc. **124** (1996), 1789–1794. MR **96h:**46090

[New]   J.D. Newburgh, *The variation of spectra*, Duke Math. J. **18** (1951), 165–176. MR **14:**481b

[SF]   P.P. Saworotnow and J.C. Friedell, *Trace-class for an arbitrary H*-algebra*, Proc. Amer. Math. Soc. **26** (1970), 95–100. MR **42:**2304

[SG]   P.P. Saworotnow and G.R. Giellis, *Continuity and linearity of centralizers on a comple-mented algebra*, Proc. Amer. Math. Soc. **31** (1972), 142–146. MR **44:**5781

[Sem]   P. Šemrl, *Isomorphisms of standard operator algebras*, Proc. Amer. Math. Soc. **123** (1995), 1851–1855. MR **95g:**47066

Institute of Mathematics, Lajos Kossuth University, 4010 Debrecen, P. O. Box 12, Hungary

E-mail address: molnarl@math.klte.hu