ON THE EXTREMALITY OF QUASICONFORMAL MAPPINGS AND QUASICONFORMAL DEFORMATIONS

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Abstract. Given a family of quasiconformal deformations $F(w, t)$ such that $\partial F$ has a uniform bound $M$, the solution $f(z, t)(f(z, 0) = z)$ of the Löwner-type differential equation

$$\frac{dw}{dt} = F(w, t)$$

is an $e^{2Mt}$-quasiconformal mapping. An open question is to determine, for each fixed $t > 0$, whether the extremality of $f(z, t)$ is equivalent to that of $F(w, t)$. The note gives this a negative approach in both directions.

§1. Introduction

This paper deals with the quasiconformal solutions $w = f(z, t)(f(z, 0) = 0)$ of the following Löwner-type differential equation

$$\frac{dw}{dt} = F(w, t)$$

in the unit disk. The maximal dilatation $K[f]$ of $f$ can be estimated in terms of the sup norm of $\partial F$. It is of interest to find out whether minimizing the sup norm of $\partial F$ is equivalent to minimizing the maximal dilatation $K[f]$. To make this precise, we will need some definitions and notation.

In the terminology of Ahlfors [1], a complex-valued function $F(w)$ defined in a Jordan domain $\Omega$ is called a quasiconformal deformation if $F(w)$ is continuous in $\Omega \cup \partial \Omega$ and has locally $L^2$-generalized derivatives, $\partial F$, $\bar{\partial} F$, with $\partial F \in L^\infty(\Omega)$. The deformation provides a way of generating a family of quasiconformal mappings, and is suitable for obtaining some distortion theorems (see Reich [6]).

Now for a homeomorphism $h$ of the unit circle $\partial \Delta$ onto itself, denote by $QC(h)$ the class of all quasiconformal mappings of the unit disk $\Delta = \{ z : |z| < 1 \}$ with boundary values $h$. Then $QC(h)$ is non-empty if and only if $h$ is quasisymmetric in the sense of Beurling-Ahlfors [2]. A quasisymmetric function $h$ then determines the extremal maximal dilatation $K_h$, defined as

$$K_h = \inf\{ K[f] : f \in QC(h) \}.$$  

$f_0 \in QC(h)$ is extremal if $K[f_0] = K_h$. 

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Similarly, for a continuous function $H$ defined on $\partial \Delta$, denote by $QD(H)$ the class of all quasiconformal deformations $F$ on $\Delta$ with boundary values $H$. Gardiner-Sullivan [3] and Reich-Chen [7] respectively proved that $QD(H)$ is non-empty if and only if $H$ is a Zygmund function, providing that $F$ satisfies the following normalized conditions:

\[(2)\quad \text{Re}[\bar{w}F(w)] = 0 \quad (|w| = 1), \quad F(0) = F(1) = 0.\]

Recall that a continuous function $\phi(x)$ is called a Zygmund [9] function if it satisfies

\[|\phi(x + t) - 2\phi(x) + \phi(x - t)| = O(t)\]

for all real numbers $x$ and $t > 0$. Define

\[B_H = \inf\{\|\partial F\|_\infty : F \in QD(H)\}\]

for a Zygmund function $H$. Then $F_0 \in QD(H)$ is extremal if $\|\partial F_0\|_\infty = B_H$.

Now we can formulate our problem as follows. Given a family of quasiconformal deformations $F(w, t)$ being continuous on $\Delta \times \{t \geq 0\}$ such that $\partial F$ is uniformly bounded, that is, there exists some constant $M > 0$ such that $|\partial F(w, t)| \leq M$ for a.e. $w \in \Delta$ and all $t \geq 0$, it is known (see [6]) that the differential equation (1) has a unique solution $f(z, t)$ with the initial condition $f(z, 0) = z$, which is an $e^{2Mt}$-quasiconformal mapping on the unit disk $\Delta$. If, additionally, $F(w, t)$ satisfy the normalized conditions (2), then $f(z, t)$ map $\Delta$ onto itself with $f(0, t) = f(1, t) - 1 = 0$. An open question is to determine, for each fixed $t > 0$, whether the extremality of $f(z, t)$ is equivalent to that of $F(w, t)$. After giving some preliminary results, we will give a negative approach to the question in both directions. On the other hand, we will give a sufficient condition under which the answer to the question is affirmative.

§2. Preliminaries

2.1. Let $A$ denote the Banach space of all functions $\phi$ holomorphic in $\Delta$ with the usual $L^1$-norm. The natural pairing:

\[(\mu, \phi) = \iint_{\Delta} \mu \phi, \quad \mu \in L^\infty, \phi \in A,\]

induces a linear map $P$ from $L^\infty$ onto $A^*$, the dual of $A$, defined by $P\mu(\phi) = (\mu, \phi)$.

$\mu \in L^\infty$ is said to satisfy the Hamilton-Krushkal ([4], [5]) condition if $\int\int_{\Delta} \mu \phi \rightarrow \|\mu\|_\infty$ as $n \rightarrow \infty$.

We will use the following standard result.

**Proposition 1** ([4], [5], [7], [8]). (1) $f$ is an extremal quasiconformal mapping iff its complex dilatation $\mu$ satisfies the Hamilton-Krushkal condition.

(2) $F$ is an extremal quasiconformal deformation iff its $\overline{\partial}$-derivative $\overline{\partial}F$ satisfies the Hamilton-Krushkal condition.

Note that $\mu \in L^\infty$ satisfies the Hamilton-Krushkal condition iff there exists a sequence $(\phi_n)$ in $A$ with $\|\phi_n\| = 1$ such that $\int\int_{\Delta} \mu \phi_n \rightarrow \|\mu\|_\infty$ as $n \rightarrow \infty$. Such a sequence $(\phi_n)$ is called a Hamilton sequence for $\mu$. It is said to be degenerating if $\phi_n \rightarrow 0$ locally uniformly in $\Delta$.

We will also need the following

**Proposition 2** ([8]). $\mu$ satisfies the Hamilton-Krushkal condition iff $\mu/(1 - |\mu|^2)$ does.
2.2. Let \( f(z, t) \) be the solution of the system (1). As done in Reich [6], differentiating both sides of the equation
\[
\frac{df(z, t)}{dt} = F(f(z, t), t)
\]
partially with respect to \( z \) and \( \bar{z} \) yields the relation
\[
f_{\bar{z}}(z, t)^2 \partial_{\bar{z}} \mu(z, t) = (|f_{\bar{z}}(z, t)|^2 - |f_{\bar{z}}(z, t)|^2) \overline{\partial F}(f(z, t), t),
\]
or equivalently,
\[
(3) \quad \overline{\partial F}(f(z, t), t) = \frac{\partial_{\bar{z}} \mu(z, t)}{1 - |\mu(z, t)|^2} \frac{f_{\bar{z}}(z, t)}{f_z(z, t)},
\]
where \( \mu(z, t) \) is the complex dilatation of \( f(z, t) \). Denote by \( \nu(w, t) \) the complex dilatation of the inverse mapping \( f^{-1}(w, t) \); then (3) is equivalent to
\[
(4) \quad \overline{\partial F}(w, t) = -\frac{\partial_{\bar{z}} \mu(z, t)}{\mu(z, t)} \frac{\nu(w, t)}{1 - |\nu(w, t)|^2} \quad (z = f^{-1}(w, t))
\]
whenever \( \mu(z, t) \neq 0 \).

§3. Counterexample theorems

**Theorem 1.** There exists a family of quasiconformal deformations \( F(w, t) \) on \( \Delta \times [0, T] \) such that the solution \( f(z, t) \) of the system (1) and \( F(w, t) \) themselves satisfy the following conditions:

1. For \( t \in (0, t_1] \), both \( f(z, t) \) and \( F(w, t) \) are extremal.
2. For \( t \in (t_1, t_2) \), \( f(z, t) \) is extremal while \( F(w, t) \) is not.
3. For \( t \in (t_2, T] \), neither \( f(z, t) \) nor \( F(w, t) \) is extremal.

**The example.** Let \( \mu \) be an extremal Beltrami differential in \( \Delta \) which has a degenerating Hamilton sequence and satisfies \( |\mu(z)| = \|\mu\|_{\infty} = k^2 \) almost everywhere. Let \( G \subset \Delta \) be a compact positive-measure subset. Define
\[
\mu(z, t) = t^2 \chi_G(z) \mu(z) + t \chi_{\Delta - G}(z) \mu(z)
\]
for \( t \in [0, T] \), where \( \chi \) denotes the characteristic function of a set, while \( 1 < T < k^{-1} \) is a fixed number.

Let \( f(z, t) \) be the quasiconformal mapping of \( \Delta \) onto itself with complex dilatation \( \mu(z, t) \) and \( f(0, t) = f(1, t) - 1 = 0 \). Noting that \( \mu(z, t) \) satisfies the Hamilton-Krushkal condition if \( f \in [0, 1] \), by Proposition 1, \( f(z, t) \) is extremal iff \( t \in [0, 1] \).

Define
\[
F(w, t) = \partial_t f \circ f^{-1}(w, t).
\]
Then \( f(z, t) \) automatically satisfy the equation (1). Now we investigate the extremality of \( F(w, t) \).

By the relation (3), we get
\[
|\overline{\partial F}(f(z, t), t)| = \frac{|\partial_{\bar{z}} \mu(z, t)|}{1 - |\mu(z, t)|^2} = \begin{cases} 
\frac{k^2}{1 - t^2 k^4}, & z \in \Delta - G, \\
\frac{2t k^2}{1 - t^4 k^4}, & z \in G.
\end{cases}
\]
Thus, if
\[ \frac{2tk^2}{1-t^4k^4} > \frac{k^2}{1-t^2k^4}, \]
\( \overline{\partial}F(w,t) \) cannot satisfy the Hamilton-Krushkal condition, and consequently \( F(w,t) \) cannot be extremal.

A direct computation will show that there exists a unique number \( t_0 \in (0,1) \) such that
\[ \frac{2tk^2}{1-t^4k^4} \leq \frac{k^2}{1-t^2k^4} \quad \text{as } t \in (0,t_0] \]
while
\[ \frac{2tk^2}{1-t^4k^4} > \frac{k^2}{1-t^2k^4} \quad \text{as } t \in (t_0,T]. \]

On the other hand, by the relation (4), we get
\[ \overline{\partial}F(w,t) = \begin{cases} 
\frac{2}{t} \cdot \frac{\nu(w,t)}{1-t^4k^4}, & w \in f(\cdot,t)(G), \\
1 - \frac{\nu(w,t)^2}{1-t^2k^4}, & w \in \Delta - f(\cdot,t)(G).
\end{cases} \]

Let \( t \in (0,t_0] \). Then \( f(z,t) \) is extremal and so is \( f^{-1}(w,t) \). By Proposition 1, \( \nu(w,t) \) satisfies the Hamilton-Krushkal condition. Now the relations (5) and (7) imply that \( \overline{\partial}F(w,t) \) also satisfies the Hamilton-Krushkal condition. Consequently, \( F(w,t) \) is extremal by Proposition 1 again.

Setting \( t_1 = t_0, t_2 = 1 \), we conclude that \( F(w,t) \) satisfy all the conditions in Theorem 1. This completes the proof of Theorem 1.

By considering
\[ \mu(z,t) = t^2 \chi_{\Delta-G}(z)\mu(z) + t \chi_G(z)\mu(z) \]
in the above example, we get

**Theorem 2.** There exists a family of quasiconformal deformations \( F(w,t) \) on \( \overline{\Delta} \times [0,T] \) such that the solution \( f(z,t) \) of the system (1) and \( F(w,t) \) themselves satisfy the following conditions:

1. For \( t \in (0,t_1) \), neither \( f(z,t) \) nor \( F(w,t) \) is extremal.
2. For \( t \in [t_1,t_2) \), \( f(z,t) \) is not extremal while \( F(w,t) \) is.
3. For \( t \in [t_2,T] \), both \( f(z,t) \) and \( F(w,t) \) are extremal.

§4. A SUFFICIENT CONDITION

**Theorem 3.** Let \( f(z,t) \) be the solution of the system (1). If \( \mu(z,t) \), the complex dilatation of \( f(z,t) \), has the form \( \mu(z,t) = k(t)\mu(z) \) for some differentiable function \( k(t) \) with \( k(0) = 0, k'(t) > 0 \). Then, for each fixed \( t > 0 \), \( f(z,t) \) is extremal iff \( F(w,t) \) is extremal.

**Proof.** By the relation (4),
\[ \overline{\partial}F(w,t) = -\frac{k'(t)}{k(t)} \cdot \frac{\nu(w,t)}{1-[\nu(w,t)]^2}. \]

Therefore, \( f(z,t) \) is extremal \( \iff f^{-1}(w,t) \) is extremal \( \iff \nu(w,t) \) satisfies the Hamilton-Krushkal condition (by Proposition 1) \( \iff \nu(w,t)/(1-[\nu(w,t)]^2) \) satisfies the Hamilton-Krushkal condition (by Proposition 2) \( \iff \overline{\partial}F(w,t) \) satisfies the
Hamilton-Krushkal condition (by relation (8)) ⇔ $F(w,t)$ is extremal (by Proposition 1).

Remark. From the proof of Theorem 1, we find that the condition in Theorem 3 is not necessary.

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