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ON THE EXTREMALITY OF QUASICONFORMAL MAPPINGS AND QUASICONFORMAL DEFORMATIONS

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ABSTRACT. Given a family of quasiconformal deformations $F(w,t)$ such that $\overline{\partial}F$ has a uniform bound M, the solution $f(z,t)(f(z,0)=z)$ of the Löwner-type differential equation

$$\frac{dw}{dt} = F(w,t)$$

is an e^{2Mt}-quasiconformal mapping. An open question is to determine, for each fixed $t>0$, whether the extremality of $f(z,t)$ is equivalent to that of $F(w,t)$. The note gives this a negative approach in both directions.

§1. INTRODUCTION

This paper deals with the quasiconformal solutions $w = f(z,t)(f(z,0)=0)$ of the following Löwner-type differential equation

$$\frac{dw}{dt} = F(w,t) \tag{1}$$

in the unit disk. The maximal dilatation $K[f]$ of f can be estimated in terms of the sup norm of $\overline{\partial}F$. It is of interest to find out whether minimizing the sup norm of $\overline{\partial}F$ is equivalent to minimizing the maximal dilatation $K[f]$. To make this precise, we will need some definitions and notation.

In the terminology of Ahlfors [1], a complex-valued function $F(w)$ defined in a Jordan domain Ω is called a quasiconformal deformation if $F(w)$ is continuous in $\Omega\cup\partial\Omega$ and has locally L^2-generalized derivatives, ∂F, $\overline{\partial}F$, with $\overline{\partial}F \in L^\infty(\Omega)$. The deformation provides a way of generating a family of quasiconformal mappings, and is suitable for obtaining some distortion theorems (see Reich [6]).

Now for a homemorphism h of the unit circle $\partial\Delta$ onto itself, denote by $QC(h)$ the class of all quasiconformal mappings of the unit disk $\Delta = \{z : |z| < 1\}$ with boundary values h. Then $QC(h)$ is non-empty if and only if h is quasisymmetric in the sense of Beurling-Ahlfors [2]. A quasisymmetric function h then determines the extremal maximal dilatation K_h, defined as

$$K_h = \inf\{K[f] : f \in QC(h)\}.$$

$f_0 \in QC(h)$ is extremal if $K[f_0] = K_h$.

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Similarly, for a continuous function H defined on $\partial\Delta$, denote by $QD(H)$ the class of all quasiconformal deformations F on Δ with boundary values H. Gardiner-Sullivan [3] and Reich-Chen [7] respectively proved that $QD(H)$ is non-empty if and only if H is a Zygmund function, providing that F satisfies the following normalized conditions:

$$Re[\bar{w}F(w)] = 0 \quad (|w| = 1), \quad F(0) = F(1) = 0. \tag{2}$$

Recall that a continuous function $\phi(x)$ is called a Zygmund [9] function if it satisfies

$$|\phi(x+t) - 2\phi(x) + \phi(x-t)| = O(t)$$

for all real numbers x and $t > 0$. Define

$$B_H = \inf\{\|\overline{\partial}F\|_\infty : F \in QD(H)\}$$

for a Zygmund function H. Then $F_0 \in QD(H)$ is extremal if $\|\overline{\partial}F_0\|_\infty = B_H$.

Now we can formulate our problem as follows. Given a family of quasiconformal deformations $F(w,t)$ being continuous on $\overline{\Delta} \times \{t \geq 0\}$ such that $\overline{\partial}F$ is uniformly bounded, that is, there exists some constant $M > 0$ such that $|\overline{\partial}F(w,t)| \leq M$ for a.e. $w \in \Delta$ and all $t \geq 0$, it is known (see [6]) that the differential equation (1) has a unique solution $f(z,t)$ with the initial condition $f(z,0) = z$, which is an e^{2Mt}-quasiconformal mapping on the unit disk Δ. If, additionally, $F(w,t)$ satisfy the normalized conditions (2), then $f(z,t)$ map Δ onto itself with $f(0,t) = f(1,t) - 1 = 0$. An open question is to determine, for each fixed $t > 0$, whether the extremality of $f(z,t)$ is equivalent to that of $F(w,t)$. After giving some preliminary results, we will give a negative approach to the question in both directions. On the other hand, we will give a sufficient condition under which the answer to the question is affirmative.

§2. Preliminaries

2.1. Let A denote the Banach space of all functions ϕ holomorphic in Δ with the usual L^1-norm. The natural pairing:

$$(\mu, \phi) = \iint_\Delta \mu\phi, \quad \mu \in L^\infty, \phi \in A,$$

induces a linear map P from L^∞ onto A^*, the dual of A, defined by $P\mu(\phi) = (\mu, \phi)$. $\mu \in L^\infty$ is said to satisfy the Hamilton-Krushkal ([4], [5]) condition if $\|P\mu\| = \|\mu\|_\infty$.

We will use the following standard result.

Proposition 1 ([4], [5], [7], [8])**.** (1) *f is an extremal quasiconformal mapping iff its complex dilatation μ satisfies the Hamilton-Krushkal condition.*

(2) *F is an extremal quasiconformal deformation iff its $\overline{\partial}$-derivative $\overline{\partial}F$ satisfies the Hamilton-Krushkal condition.*

Note that $\mu \in L^\infty$ satisfies the Hamilton-Krushkal condition iff there exists a sequence (ϕ_n) in A with $\|\phi_n\| = 1$ such that $\iint_\Delta \mu\phi_n \to \|\mu\|_\infty$ as $n \to \infty$. Such a sequence (ϕ_n) is called a Hamilton sequence for μ. It is said to be degenerating if $\phi_n \to 0$ locally uniformly in Δ.

We will also need the following

Proposition 2 ([8])**.** *μ satisfies the Hamilton-Krushkal condition iff $\mu/(1 - |\mu|^2)$ does.*

2.2. Let $f(z,t)$ be the solution of the system (1). As done in Reich [6], differentiating both sides of the equation

$$\frac{df(z,t)}{dt} = F(f(z,t),t)$$

partially with respect to z and $\bar{z}$ yields the relation

$$f_z(z,t)^2 \partial_t \mu(z,t) = (|f_z(z,t)|^2 - |f_{\bar{z}}(z,t)|^2)\overline{\partial} F(f(z,t),t),$$

or equivalently,

$$\overline{\partial} F(f(z,t),t) = \frac{\partial_t \mu(z,t)}{1 - |\mu(z,t)|^2} \cdot \frac{f_z(z,t)}{\overline{f_z(z,t)}}, \tag{3}$$

where $\mu(z,t)$ is the complex dilatation of $f(z,t)$. Denote by $\nu(w,t)$ the complex dilatation of the inverse mapping $f^{-1}(w,t)$; then (3) is equivalent to

$$\overline{\partial} F(w,t) = -\frac{\partial_t \mu(z,t)}{\mu(z,t)} \cdot \frac{\nu(w,t)}{1 - |\nu(w,t)|^2} \quad (z = f^{-1}(w,t)) \tag{4}$$

whenever $\mu(z,t) \neq 0$.

§3. Counterexample theorems

Theorem 1. *There exists a family of quasiconformal deformations $F(w,t)$ on $\overline{\Delta} \times [0,T]$ such that the solution $f(z,t)$ of the system (1) and $F(w,t)$ themselves satisfy the following conditions:*

(1) For $t \in (0, t_1]$, both $f(z,t)$ and $F(w,t)$ are extremal.
(2) For $t \in (t_1, t_2]$, $f(z,t)$ is extremal while $F(w,t)$ is not.
(3) For $t \in (t_2, T]$, neither $f(z,t)$ nor $F(w,t)$ is extremal.

The example. Let μ be an extremal Beltrami differential in Δ which has a degenerating Hamilton sequence and satisfies $|\mu(z)| = \|\mu\|_\infty = k^2$ almost everywhere. Let $G \subset \Delta$ be a compact positive-measure subset. Define

$$\mu(z,t) = t^2 \chi_G(z) \mu(z) + t \chi_{\Delta - G}(z) \mu(z)$$

for $t \in [0,T]$, where χ denotes the characteristic function of a set, while $1 < T < k^{-1}$ is a fixed number.

Let $f(z,t)$ be the quasiconformal mapping of Δ onto itself with complex dilatation $\mu(z,t)$ and $f(0,t) = f(1,t) - 1 = 0$. Noting that $\mu(z,t)$ satisfies the Hamilton-Krushkal condition iff $t \in [0,1]$, by Proposition 1, $f(z,t)$ is extremal iff $t \in [0,1]$.

Define

$$F(w,t) = \partial_t f \circ f^{-1}(w,t).$$

Then $f(z,t)$ automatically satisfy the equation (1). Now we investigate the extremality of $F(w,t)$.

By the relation (3), we get

$$|\overline{\partial} F(f(z,t),t)| = \frac{|\partial_t \mu(z,t)|}{1 - |\mu(z,t)|^2} = \begin{cases} \dfrac{k^2}{1 - t^2 k^4}, & z \in \Delta - G, \\ \dfrac{2tk^2}{1 - t^4 k^4}, & z \in G. \end{cases}$$

Thus, if

$$\frac{2tk^2}{1-t^4k^4} > \frac{k^2}{1-t^2k^4},$$

$\overline{\partial}F(w,t)$ cannot satisfy the Hamilton-Krushkal condition, and consequently $F(w,t)$ cannot be extremal.

A direct computation will show that there exists a unique number $t_0 \in (0,1)$ such that

$$\frac{2tk^2}{1-t^4k^4} \le \frac{k^2}{1-t^2k^4} \quad \text{as } t \in (0,t_0] \tag{5}$$

while

$$\frac{2tk^2}{1-t^4k^4} > \frac{k^2}{1-t^2k^4} \quad \text{as } t \in (t_0,T]. \tag{6}$$

On the other hand, by the relation (4), we get

$$\overline{\partial}F(w,t) = \begin{cases} -\dfrac{2}{t}\cdot\dfrac{\nu(w,t)}{1-t^4k^4}, & w \in f(\cdot,t)(G), \\ -\dfrac{1}{t}\cdot\dfrac{\nu(w,t)}{1-t^2k^4}, & w \in \Delta - f(\cdot,t)(G). \end{cases} \tag{7}$$

Ler $t \in (0,t_0]$. Then $f(z,t)$ is extremal and so is $f^{-1}(w,t)$. By Proposition 1, $\nu(w,t)$ satisfies the Hamilton-Krushkal condition. Now the relations (5) and (7) imply that $\overline{\partial}F(w,t)$ also satisfies the Hamilton-Krushkal condition. Consequently, $F(w,t)$ is extremal by Proposition 1 again.

Setting $t_1 = t_0, t_2 = 1$, we conclude that $F(w,t)$ satisfy all the conditions in Theorem 1. This completes the proof of Theorem 1.

By considering

$$\mu(z,t) = t^2\chi_{\Delta-G}(z)\mu(z) + t\chi_G(z)\mu(z)$$

in the above example, we get

Theorem 2. *There exists a family of quasiconformal deformations $F(w,t)$ on $\overline{\Delta}\times[0,T]$ such that the solution $f(z,t)$ of the system (1) and $F(w,t)$ themselves satisfy the following conditions:*

(1) For $t \in (0,t_1)$, neither $f(z,t)$ nor $F(w,t)$ is extremal.

(2) For $t \in [t_1,t_2)$, $f(z,t)$ is not extremal while $F(w,t)$ is.

(3) For $t \in [t_2,T]$, both $f(z,t)$ and $F(w,t)$ are extremal.

§4. A sufficient condition

Theorem 3. *Let $f(z,t)$ be the solution of the system (1). If $\mu(z,t)$, the complex dilatation of $f(z,t)$, has the form $\mu(z,t) = k(t)\mu(z)$ for some differentiable function $k(t)$ with $k(0) = 0, k'(t) > 0$. Then, for each fixed $t > 0$, $f(z,t)$ is extremal iff $F(w,t)$ is extremal.*

Proof. By the relation (4),

$$\overline{\partial}F(w,t) = -\frac{k'(t)}{k(t)}\cdot\frac{\nu(w,t)}{1-|\nu(w,t)|^2}. \tag{8}$$

Therefore, $f(z,t)$ is extremal $\Leftrightarrow$ $f^{-1}(w,t)$ is extremal $\Leftrightarrow$ $\nu(w,t)$ satisfies the Hamilton-Krushkal condition (by Proposition 1) $\Leftrightarrow$ $\nu(w,t)/(1-|\nu(w,t)|^2)$ satisfies the Hamilton-Krushkal condition (by Proposition 2) $\Leftrightarrow$ $\overline{\partial}F(w,t)$ satisfies the

Hamilton-Krushkal condition (by relation (8)) $\Leftrightarrow F(w,t)$ is extremal (by Proposition 1). $\square$

Remark. From the proof of Theorem 1, we find that the condition in Theorem 3 is not necessary.

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