NO SUBMAXIMAL TOPOLOGY ON A COUNTABLE SET IS $T_1$-COMPLEMENTARY

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Abstract. Two $T_1$-topologies $\tau$ and $\mu$ given on the same set $X$, are called transversal if their union generates the discrete topology on $X$. The topologies $\tau$ and $\mu$ are $T_1$-complementary if they are transversal and their intersection is the cofinite topology on $X$. We establish that for any connected Tychonoff topology there exists a connected Tychonoff transversal one. Another result is that no $T_1$-complementary topology exists for the maximal topology constructed by van Douwen on the rational numbers. This gives a negative answer to Problem 162 from Open Problems in Topology (1990).

0. Introduction

The lattice $\mathcal{L}_1(X)$ of all $T_1$-topologies on a given set $X$ has been under intensive study since 1966 when A.K. Steiner [St] showed that for any infinite set $X$ there exist $T_1$-topologies on $X$ which do not have a complement in the lattice $\mathcal{L}_1(X)$. Recall that a topology $\mu$ on $X$ is a complement of $\tau$ in $\mathcal{L}_1(X)$ if $\tau \cup \mu$ is a subbase of the discrete topology and $\tau \cap \mu$ coincides with the cofinite topology on $X$.

The papers [An], [AnSt], [StSt1] and [StSt2] contain positive results on the existence of complements in the lattice $\mathcal{L}_1(X)$ which are also called $T_1$-complements. It was proved, in particular, that every Hausdorff locally compact or Frechet-Urysohn topology has a $T_1$-complement [An]. In the same paper, Anderson constructed an example of an irresolvable ($\equiv$ not representable as a union of two disjoint dense subsets) dense in itself space which has a $T_1$-complement [An]. In the same paper, Anderson constructed an example of an irresolvable ($\equiv$ not representable as a union of two disjoint dense subsets) dense in itself space which has a $T_1$-complement and asked whether every MI-space has one [An, Question 1]. Recall that $(X, \tau)$ is a MI-space if it is dense in itself and every dense subset of $X$ is open. It is easy to see that every MI-space is irresolvable.

Later, S. Watson asked whether each Hausdorff space has a $T_1$-complement. This question is published as Problem 162 (Problem 94 in the internal enumeration of Watson’s paper) in Open Problems in Topology [Wa1]. The second part of Problem...
162 of [Wa1] is an inquiry whether every completely regular $T_1$-topology has a $T_1$-complement. The same question is repeated in [Wa2] (Problem 6.6).

In this paper we work only with $T_1$-spaces. We use the modern term “submaximal space” instead of “MI-space”. Our main result is Theorem 3.6 which implies that no submaximal Hausdorff topology on a countable set is $T_1$-complementary.

A narrower class than submaximal topologies is formed by the maximal ones. A topology $\tau$ is maximal if it is dense in itself but any strictly stronger one is not. As there exists in ZFC a Tychonoff countable maximal space [vD], Theorem 3.6 and Corollary 3.8 provide the negative answer to the respective questions from [An], [Wa1] and [Wa2].

Given a set $X$ and $\tau, \mu \in L_1(X)$ we say that $\tau$ and $\mu$ are transversal if $\tau \cup \mu$ is a subbase of the discrete topology on $X$. It is immediate that if $\tau$ and $\mu$ are $T_1$-complementary, then they are transversal.

We prove in particular, that every connected Tychonoff topology has a connected Tychonoff transversal topology. Examples of the non-existence of transversal connected topologies are given.

1. Notations and terminology

All spaces are assumed to be $T_1$. If $X$ is a space, then $\tau(X)$ is its topology and $\tau^*(X) = \tau(X) \setminus \{\emptyset\}$. Analogously, if $\tau$ is a topology, then $\tau^* = \tau \setminus \{\emptyset\}$. A topology $\tau$ on a set $X$ is called cofinite (and is denoted by $CF(X)$) if every non-empty element of $\tau$ is a complement of a finite set. If $\tau$ is a topology on a set $X$ and $x \in X$, then $\tau(x) = \{U \in \tau : x \in U\}$. We will write $\tau(x, X)$ instead of $\tau(X)(x)$. Given a space $(X, \tau)$ and a subset $A \subset X$ we denote by $cl_\tau(A)$ and $Int_\tau(A)$ the $\tau$-closure and $\tau$-interior of $A$ respectively. A space $X$ is called submaximal if it is dense in itself and every dense subset of $X$ is open. A topology $\tau$ on a set $X$ is maximal if $(X, \tau)$ has no isolated points, but $(X, \mu)$ has an isolated point if $\mu$ is a topology strictly stronger than $\tau$.

All other notions are standard and can be found in [En].

2. Transversal topologies

Let us start with the main definitions and some auxiliary results.

2.1. Definition. Two topologies $\tau$ and $\mu$ on the same set $X$ are called transversal if their join $\tau \vee \mu (\equiv$ the smallest topology that contains $\tau \cup \mu$) is discrete. The topology $\mu$ will be referred to as a $\tau$-transversal one.

2.2. Proposition. If $\tau$ and $\mu$ are topologies on the same set $X$, then the following conditions are equivalent:

1. $\tau$ and $\mu$ are transversal;
2. for each point $x \in X$ there exist $U \in \tau$ and $V \in \mu$ such that $U \cap V = \{x\}$;
3. there is a $\tau$-open cover $\gamma$ of $X$ such that $X$ is a union of $\mu$-isolated points of elements of $\gamma$.

Proof. It is clear that the family $B(\tau, \mu) = \{U \cap V : U \in \tau, V \in \mu\}$ is a base of $\tau \vee \mu$. As each singleton belongs to $\tau \vee \mu$, we have $\{x\} \in B(\tau, \mu)$ for every $x \in X$. This proves $(1) \implies (2)$.

For each $x \in X$ take $U_x \in \tau(x)$ and $V_x \in \mu(x)$ such that $\{x\} = U_x \cap V_x$. Let $\gamma = \{U_x : x \in X\}$. It is clear that $\gamma$ is a $\tau$-open cover of $X$ and $x$ is $\mu$-isolated in $U_x$ for each $x \in X$. Thus $(2) \implies (3)$. 

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To prove (3) ⇒ (1) consider any γ as in (3). For every \( x \in X \) there is a \( U_x \in \gamma \) such that \( x \) is \( \mu \)-isolated in \( U_x \). This means that \( U_x \cap V_x = \{x\} \) for some \( V_x \in \mu \) and therefore \( \{x\} \) is open in \( \tau \cap \mu \).

2.3. Theorem. Let \((X, \tau)\) be a space of weight \( \kappa \). If \( \mu \) is a \( \tau \)-transversal topology on \( X \), then \((X, \mu)\) is a union of \( \leq \kappa \) discrete subspaces.

Proof. Take a base \( B \) for \((X, \tau)\) of cardinality \( \kappa \). For every \( x \in X \) fix a \( U_x \in B \) and \( V_x \in \mu \) with \( U_x \cap V_x = \{x\} \). For every \( U \in B \) put \( A_U = \{x \in X : U = U_x\} \). Since for each \( x \in A_U \) we have \( V_x \cap A_U = \{x\} \), the set \( A_U \) is discrete in \((X, \mu)\). Now \( X = \bigcup \{A_U : U \in B\} \) is a union of \( \kappa \) many sets discrete in \((X, \mu)\).

2.4. Corollary. If \((X, \tau)\) is a second countable space and \( \mu \) is a \( \tau \)-transversal topology on \( X \), then \((X, \mu)\) is \( \sigma \)-discrete.

2.5. Corollary. Let \((X, \tau)\) be an infinite connected second countable space. Then there is no dense-in-itself compact (or even Baire) \( \tau \)-transversal topology \( \mu \) on \( X \).

Proof. Suppose that \( \mu \) is transversal for \( \tau \) and \((X, \mu)\) is a Baire space. According to Corollary 2.4 we have \( X = \bigcup \{X_n : n \in \omega\} \), where \( X_n \) is a \( \mu \)-discrete subset of \( X \). The Baire property of \((X, \mu)\) implies \( U = \text{Int} \{\overline{c}(\mu)(X_n)\} \neq \emptyset \) for some \( n \in \omega \). Then \( X_n \cap U \neq \emptyset \) and any \( x \in X_n \cap U \) is an isolated point of \((X, \mu)\) which contradicts the fact that \((X, \mu)\) is dense-in-itself.

2.6. Theorem. For every cardinal \( \kappa \geq \omega \) there exists a space \( D_\kappa \) with the following properties:

1. \( D_\kappa \) is homeomorphic to a dense subspace of \( I^\kappa \) and hence \( D_\kappa \) is a Tychonoff space without isolated points;
2. \( D_\kappa = \bigcup \{D_\kappa^n : n \in \omega\} \), where each \( D_\kappa^n \) is closed, discrete in \( D_\kappa \) and \( |D_\kappa^n| = \kappa \);
3. if \( A \subset D_\kappa \) and \( |A| < \kappa \), then \( A \) is closed and discrete in \( D_\kappa \);
4. if \( \kappa \geq \omega \), then the space \( D_\kappa \) is connected.

Proof. It is possible to represent the set \( \kappa \) in the following form: \( \kappa = \bigcup \{K_\alpha : \alpha < \kappa\} \), where \( |K_\alpha| = \kappa \) for all \( \alpha < \kappa \) and \( K_\alpha \cap K_\beta = \emptyset \) if \( \alpha \neq \beta \).

If \( \kappa < \omega \), then for every finite \( A \subset \kappa \) pick a countable dense subspace \( C_A \) of the space \( I^A \) and let \( F = \bigcup \{C_A : A \text{ is a finite subset of } \kappa\} \).

If \( \kappa \geq \omega \), then for every finite \( A \subset \kappa \) put \( C_A = I^A \) and let \( F = \bigcup \{C_A : A \text{ is a finite subset of } \kappa\} \).

It is clear that in both cases \( |F| = \kappa \), so it is possible to enumerate the elements of \( F \) as \( \{x_\alpha : \alpha < \kappa\} \) in such a way that for each \( x \in F \) the set \( \{\alpha : x_\alpha = x\} \) has cardinality \( \kappa \). Given an \( x_\alpha \in F \) we denote by \( S_\alpha \) the finite set of coordinates corresponding to the face to which the point \( x_\alpha \) belongs.

For each \( \alpha < \kappa \) let

\[
d_\alpha(t) = \begin{cases} 
0, & \text{if } t \notin K_\alpha \cup S_\alpha; \\
x_\alpha(t), & \text{if } t \in S_\alpha; \\
1, & \text{if } t \in K_\alpha \setminus S_\alpha,
\end{cases}
\]

and denote by \( D_\kappa \) the subspace \( \{d_\alpha : \alpha < \kappa\} \) of \( I^\kappa \).

Let us prove that the space \( D_\kappa \) has the properties we promised. Observe first that \( D_\kappa \) is dense in \( I^\kappa \) for all \( \kappa \geq \omega \).

Indeed, for a finite \( S \subset \kappa \) every \( x \in C_S \) occurs \( \kappa \) times in the enumeration of \( F \). Thus, there is an \( \alpha(x) < \kappa \) such that \( x_{\alpha(x)} = x \) and hence \( S_{\alpha(x)} = S \). It is clear
that $\pi_S(d_\alpha(x)) = x$. Therefore, $\pi_S(D_\kappa) \supset C_S$ for every finite $S \subset \kappa$. Since the set $C_S$ is dense in $I^S$, we can conclude that $\pi_S(D_\kappa)$ is dense in $I^S$ for every finite $S \subset \kappa$. Thus $D_\kappa$ is dense in $I^\kappa$. This shows that (1) is true for $D_\kappa$.

Now let $D^n_\kappa = \{d_\alpha \in D_\kappa : |S_\alpha| = n\}$ for every $n \in \omega$. It is evident that $|D^n_\kappa| = \kappa$ for each $n \in \omega$ and $\bigcup\{D^n_\kappa : n \in \omega\} = D_\kappa$. Given an $\alpha < \kappa$ take any distinct $t_1, \ldots, t_{n+1} \in K_\alpha \setminus S_\alpha$. Then $d_\alpha(t_i) = 1$ for every $i \leq n + 1$. Thus, $W_\alpha = \{d \in D_\kappa : d(t_i) > \frac{1}{2} \text{ for all } i \leq n + 1\}$ is an open neighbourhood of $d_\alpha$. If $\beta \neq \alpha$ and $|S_\beta| = n$, then

$$\{t_1, \ldots, t_{n+1}\}\setminus (K_\beta \cup S_\beta) \neq \emptyset,$$

which implies $d_\beta(t_i) = 0$ for some $i \leq n + 1$. Therefore, $d_\beta \notin W_\alpha$. This proves that $D^n_\kappa$ is closed and discrete in $D_\kappa$. Hence we established (2) for $D_\kappa$.

Take a subset $A$ of $D_\kappa$ with $|A| < \kappa$. The sets $S = \{\alpha < \kappa : d_\alpha \in A\}$ and $H = \bigcup\{S_\alpha : \alpha \in B\}$ have cardinality less than $\kappa$. Take any $\alpha_0 < \kappa$. It is clear that $K_{\alpha_0} \setminus H \neq \emptyset$. Pick any $t \in K_{\alpha_0} \setminus H$ and let $V_{\alpha_0} = \{d \in D_\kappa : d(t) > \frac{1}{2}\}$. Then $V_{\alpha_0}$ is an open neighbourhood of $d_{\alpha_0}$ which does not contain any point of $A$, distinct from $d_{\alpha_0}$. Thus, $A$ is closed and discrete in $D_\kappa$. This proves (3).

Finally, suppose that $\kappa \geq \omega$. If $D_\kappa$ is disconnected, then there is a continuous surjective function $\varphi : D_\kappa \to \{0, 1\}$. As $D_\kappa$ is dense in $I^K$, there is a countable set $T \subset \kappa$ and a continuous function $\varphi_1 : \pi_T(D_\kappa) \to \{0, 1\}$ such that $\varphi_1 \circ \pi_T = \varphi \circ |\text{Ar}|$. In particular, $\varphi_1$ is surjective. Now, if $S$ is a finite subset of $T$ and $x \in I^S$, then the set $P(S) = \{\alpha < \kappa : x_\alpha = x\}$ has cardinality $\kappa > \omega$. Therefore, $K_{\alpha_0} \setminus T = \emptyset$ for some $\alpha_0 \in P(S)$ and hence $d_{\alpha_0}|S = x$ and $d_{\alpha_0}|T \setminus S = 0$. This proves that $\pi_T(D_\kappa)$ contains the set $\sigma = \{z \in I^T : |\{t \in T : z(t) \neq 0\}| < \omega\}$ which is connected and dense in $I^T$. Since $\pi_T(D_\kappa)$ contains a dense connected subspace, it is connected. This gives a contradiction with the continuity and surjectivity of $\varphi_1$. Consequently, $D_\kappa$ is connected.

2.7. Corollary. For every cardinal $\kappa \geq \omega$ there exists a Tychonoff space $E_\kappa$ with the following properties:

1. $E_\kappa = A_\kappa \cup B_\kappa$, where each one of the subspaces $A_\kappa, B_\kappa$ is closed in $E_\kappa$ and homeomorphic to the space $D_\kappa$ from Theorem 2.6. In particular, if $A \subset E_\kappa$ and $|A| < \kappa$, then $A$ is closed and discrete in $E_\kappa$;
2. there is a point $p \in E_\kappa$ such that $A_\kappa \cap B_\kappa = \{p\}$;
3. $A_\kappa \setminus \{p\} = \bigcup\{A^n_\kappa : n \in \omega\}$ and $B_\kappa \setminus \{p\} = \bigcup\{B^n_\kappa : n \in \omega\}$, where $A^n_\kappa$ and $B^n_\kappa$ are closed and discrete in $E_\kappa$ for all $n \in \omega$;
4. $|A^n_\kappa| = |B^n_\kappa| = \kappa$ for all $n \in \omega$;
5. $A^n_\kappa \cap A^m_\kappa = \emptyset$ and $B^n_\kappa \cap B^m_\kappa = \emptyset$ whenever $m \neq n$;
6. if $\kappa \geq \omega$, then the space $E_\kappa$ is connected.

Proof. Let $P_\kappa$ and $Q_\kappa$ be two disjoint copies of the space $D_\kappa$ with $P^n_\kappa \subset P_\kappa$ and $Q^n_\kappa \subset Q_\kappa$ as respective copies of $D^n_\kappa$. Pick points $a \in P_\kappa$, $b \in Q_\kappa$, and identify $a$ and $b$ in the space $P_\kappa \cup Q_\kappa$. Denote by $E_\kappa$ the resulting quotient space and by $\varphi_\kappa : P_\kappa \oplus Q_\kappa \to E_\kappa$ the relevant quotient map. Define the point $p$ by the equality $\{p\} = \varphi_\kappa(\{a, b\})$ and let $A_\kappa = \varphi_\kappa(P_\kappa)$, $A^n_\kappa = \varphi_\kappa(P^n_\kappa \setminus \{a\})$ and $B_\kappa = \varphi_\kappa(Q_\kappa)$, $B^n_\kappa = \varphi_\kappa(Q^n_\kappa \setminus \{b\})$. It is straightforward that the space $E_\kappa$ satisfies (1)-(6).

The following example shows that not every Tychonoff space has a transversal connected or even dense-in-itself topology.
2.8. Example. The one-point compactification of any infinite discrete space has no transversal connected (or even dense-in-itself) topology.

Proof. Let $X$ be the one-point compactification of an infinite discrete space. If $a$ is the only non-isolated point of $X$, then for every $U \in \tau(a, X)$ the set $X \setminus U$ is finite. Suppose that $\mu$ is a dense-in-itself topology, transversal to $\tau(X)$. Then there is a $V \in \mu$ such that $U \cap V = \{a\}$ for some $U \in \tau(a, X)$. Hence $V \subset (X \setminus U) \cup \{a\}$ is a non-empty finite set. Since $(X, \mu)$ is a $T_1$-space, any point of $V$ is isolated in $(X, \mu)$, which is a contradiction. \hfill \Box

2.9. Example. There exists a dense-in-itself Tychonoff space $X$ of cardinality $\kappa$ such that $\tau(X)$ has no transversal connected Tychonoff topology.

Proof. Consider the discrete union $Y = \bigoplus\{Q_\alpha : \alpha < \kappa\}$ of $\kappa$ copies of rationals. Take any point $z \notin Y$ and let $X = \{z\} \cup Y$. Declare the family $\tau(y, Y)$ to be the local base in $X$ at any $y \in Y$. Let $B = \{U_A = \{z\} \cup \{Q_\beta : \beta \in \kappa \setminus A\}$ where $A$ is a countable subset of $\kappa$. Let $\tau$ be the neighbourhood base of $z$ in $X$.

Suppose that $X$ has a connected transversal Tychonoff topology $\mu$. By Proposition 2.2 there exist $U \in \tau(X)$ and $V \in \mu$ such that $\{z\} = U \cap V$. But the complement of $U$ is countable, which implies that $V \subset (X \setminus U) \cup \{z\}$ is countable. Since in a connected Tychonoff space no non-empty proper open set can be countable, we have a contradiction. \hfill \Box

2.10. Lemma. Let $(X, \tau)$ be a space of cardinality $\kappa \geq \omega$. Suppose that there exist families $\{U_n : n \in \omega\}$ and $\{V_n : n \in \omega\}$ of open subsets of $X$ with the following properties:

1. $\bigcup_{n+1} U_n$ and $\bigcap_{n+1} V_n$ for any $n \in \omega$;
2. $\bigcap_0 \bigcap_0 = 0$;
3. $|P| = \kappa$, where $P = X \setminus (U_0 \cup V_0)$;
4. $|U_n \setminus U_{n+1}| = |V_n \setminus V_{n+1}| = \kappa$ for all $n \in \omega$.

Then there exists a $\tau$-transversal topology $\mu$ on the set $X$ such that $(X, \mu)$ is homeomorphic to $E_\kappa$ from Corollary 2.7. In particular, $\tau$ has a transversal dense-in-itself Tychonoff topology and if $\kappa \geq \omega$, then $\tau$ has a transversal connected Tychonoff topology.

Proof. Let $F = \bigcap\{U_n : n \in \omega\}$ and $G = \bigcap\{V_n : n \in \omega\}$ (the sets $F$ and $G$ can be empty). Take a point $p' \in P$. Using evident decompositions of $E_\kappa$ and $X$ into countably many pieces of cardinality $\kappa$, we conclude that there exists a bijection $\varphi : E_\kappa \to X$ such that

- (i) $\varphi(p) = p'$;
- (ii) $\varphi(A^0_\kappa) = F \cup (P \setminus \{p'\})$;
- (iii) $\varphi(B^0_\kappa) = G \cup (U_0 \setminus U_1)$;
- (iv) $\varphi(B^\kappa_i) = U_i \setminus U_{i+1}$ for all $i \geq 1$;
- (v) $\varphi(A^{i+1}_\kappa) = V_i \setminus V_{i+1}$ for every $i \in \omega$.

Let $\mu = \{\varphi(W) \setminus W \text{ open in } E_\kappa\}$. It is clear that $(X, \mu)$ is homeomorphic to $E_\kappa$. We are going to prove that the topology $\mu$ is $\tau$-transversal.

Given an $x \in X \setminus (F \cup G)$, there exists an $n \in \omega$ such that $x \notin U_n \cup V_n$. Let $U = X \setminus (U_n \cup V_n)$. Then $x \in U$ and $\varphi^{-1}(U)$ is contained in the closed and discrete subset

$$A = \{p \cup A^0_\kappa \cup \ldots \cup A^\kappa_\kappa \cup B^0_\kappa \cup \ldots \cup B^\kappa_\kappa$$
of the space $E_\kappa$. Therefore, there is a $W \in \tau(E_\kappa)$ such that $\{\varphi^{-1}(x)\} = W \cap \varphi^{-1}(U)$. Then $V = \varphi(W) \in \mu$ and $U \cap V = \{x\}$.

Now if $x \in F$, then let $U = U_1$. Observe that $\varphi^{-1}(U) = (B_\kappa \setminus B_\kappa^0) \cup \varphi^{-1}(F) \subset B_\kappa \cup A_\kappa^0$ and $\varphi^{-1}(x) \in \varphi^{-1}(F) \subset A_\kappa^0$. Thus $A_\kappa \setminus \{p\}$ is an open neighbourhood of $\varphi^{-1}(x)$, whose intersection with $\varphi^{-1}(U)$ is contained in $A_\kappa^0$. Therefore, $\varphi^{-1}(x)$ is isolated in $\varphi^{-1}(U)$. Take any $W \in \tau(E_\kappa)$ such that $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$. Then $V = \varphi(W) \in \mu$ and $V \cap U = \{x\}$. We have established that for any $x \in X$ property (2) of Proposition 2.2 is fulfilled and hence $\mu$ is $\tau$-transversal.

2.11. Lemma. Suppose that in a space $(X, \tau)$ of cardinality $\kappa$ there exist $a, b \in X$ and $U_0, U_1, V_0, V_1 \in \tau$ such that

1. $a \in U_1 \subset \overline{U_1} \subset U_0$;
2. $b \in V_1 \subset \overline{V_1} \subset V_0$;
3. $|U_1| = |V_1| = \kappa$;
4. $U_0 \cap V_0 = \emptyset$ and $X \setminus (U_0 \cup V_0) \neq \emptyset$;
5. for any $x \in (U_0 \setminus \{a\}) \cup (V_0 \setminus \{b\})$ there exists $a U \in \tau(x, X)$ such that $|U| < \kappa$.

Then there exists a $\tau$-transversal topology $\mu$ on the set $X$ such that $(X, \mu)$ is homeomorphic to the space $E_\kappa$ from Corollary 2.7. In particular, $\tau$ has a transversal dense-in-itself Tychonoff topology and if $\kappa \geq \mathfrak{c}$, then $\tau$ has a transversal connected Tychonoff topology.

Proof. Pick any $p' \in X \setminus (U_0 \cup V_0)$ and denote by $P$ the set $X \setminus (U_0 \cup V_1 \cup \{p'\})$. Using evident decompositions of $X$ and $E_\kappa$ into finitely many pieces we can construct a bijection $\varphi : E_\kappa \to X$ such that

1. $\varphi(p) = p'$;
2. $\varphi(A_\kappa^0) \supset P \cup \{b\}$ and $\varphi(A_\kappa \setminus \{p\}) = (U_1 \setminus \{a\}) \cup P \cup \{b\}$;
3. $\varphi(B_\kappa^0) \supset U_0 \setminus U_1$ and $\varphi(B_\kappa \setminus \{p\}) = (V_1 \setminus \{b\}) \cup (U_0 \setminus U_1) \cup \{a\}$.

Let $\mu = \{\varphi(W) : W \text{ is open in } E_\kappa\}$. It is clear that $(X, \mu)$ is homeomorphic to $E_\kappa$. We are going to prove that the topology $\mu$ is $\tau$-transversal.

Given an $x \in X \setminus (U_0 \cup V_0)$ let $U = X \setminus (\overline{U}_1 \cup \overline{V}_1)$. The set $U$ is a $\tau$-open neighbourhood of the point $x$ and $\varphi^{-1}(U) \subset \varphi^{-1}(P \cup \{p'\} \cup (U_0 \setminus U_1)) \subset A_\kappa^0 \cup \{p\} \cup B_\kappa^0$. Since the set $A_\kappa^0 \cup \{p\} \cup B_\kappa^0$ is closed and discrete in $E_\kappa$, there is a $W \in \tau(E_\kappa)$ such that $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$. Then $V = \varphi(W) \in \mu$ and $U \cap V = \{x\}$.

Suppose that $x \in (U_0 \setminus \{a\}) \cup (V_0 \setminus \{b\})$. Apply (5) to find a $U \in \tau$ such that $x \in U$ and $|U| < \kappa$. Then $|\varphi^{-1}(U)| < \kappa$ and therefore $\varphi^{-1}(U)$ is closed and discrete in $E_\kappa$ by condition (1) of Proposition 2.7. Pick a $W \in \tau(E_\kappa)$ such that $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$. Then $V = \varphi(W) \in \mu$ and $U \cap V = \{x\}$.

If $x = a$, let $U = U_1$. Then $\varphi^{-1}(U) \cap (B_\kappa \setminus \{p\}) = \{\varphi^{-1}(a)\}$. Thus for $W = B_\kappa \setminus \{p\}$ we have $V = \varphi(W) \in \mu$ and $U \cap V = \{a\}$.

If $x = b$, let $U = V_1$. Then $\varphi^{-1}(U) \cap (A_\kappa \setminus \{p\}) = \{\varphi^{-1}(b)\}$. Thus for $W = A_\kappa \setminus \{p\}$ we have $V = \varphi(W) \in \mu$ and $U \cap V = \{b\}$.

Condition (2) of Proposition 2.2 having been checked for every $x \in X$ we conclude that $\tau$ and $\mu$ are transversal.

\[ \square \]
2.12. Theorem. Given a cardinal number $\lambda \geq \omega$ suppose that $(X, \tau)$ is a regular space such that $|U| \geq \lambda$ for any $U \in \tau^*$. Then $\tau$ has a dense-in-itself transversal Tychonoff topology $\mu$. Moreover, if $\lambda \geq \mathfrak{c}$, then $\mu$ can be chosen to be Tychonoff and connected.

Proof. Let

$$M = \{x \in X : |U| = |X| = \kappa \text{ for any } U \in \tau(x)\}.$$

There are three cases to consider.

Case 1. The set $M$ has at least two cluster points, say $a$ and $b$. It is clear that $M$ has to be infinite and $a, b \in M$.

Pick two distinct points $x_0, y_0 \in M\setminus\{a, b\}$. There exist $U_0 \in \tau(a)$, $V_0 \in \tau(b)$ such that $U_0 \cap V_0 = \emptyset$ and $\{x_0, y_0\} \subset X\setminus(U_0 \cup V_0)$. In particular $|P| = \kappa$, where $P = X\setminus(U_0 \cup V_0)$.

Suppose that we have constructed open sets $U_i \in \tau(a)$, $V_i \in \tau(b)$ and points $x_i, y_i \in M$ for all $i \leq n$ in such a way that

(i) $U_{i+1} \subset U_i$ and $V_{i+1} \subset V_i$ for all $i < n$;

(ii) $x_{i+1} \in U_i \setminus V_{i+1}$ and $y_{i+1} \in V_i \setminus U_{i+1}$ for all $i < n$.

Since $a$ and $b$ are cluster points of $M$, there exist $x_{n+1} \in (\mathcal{U}_n\{a\}) \cap M$ and $y_{n+1} \in (\mathcal{V}_n\{b\}) \cap M$. Since the space $X$ is regular, we can find $U_{n+1} \in \tau(a)$ and $V_{n+1} \in \tau(b)$ such that $U_{n+1} \subset (\mathcal{U}_n\{x_{n+1}\})$ and $V_{n+1} \subset (\mathcal{V}_n\{y_{n+1}\})$. It is evident that the properties (i) and (ii) are fulfilled for all $i \leq n$.

Observe that the families $\mathcal{U} = \{U_i : i \in \omega\}$ and $\mathcal{V} = \{V_i : i \in \omega\}$ satisfy the conditions (1)-(3) of Lemma 2.10. The condition (4) is also fulfilled because each of the sets $U_n \setminus U_{n+1}$ and $V_n \setminus V_{n+1}$ is open and meets $M$ for any $n \in \omega$. Therefore, we can apply Lemma 2.10 and conclude that $(X, \tau)$ has a dense-in-itself transversal topology $\mu$ which will be connected if $\kappa \geq \mathfrak{c}$.

Case 2. The set $M$ has at least two isolated points, say $a$ and $b$.

Pick any $p' \in X\setminus\{a, b\}$. Since $X$ is Hausdorff, there exist $U_0 \in \tau(a)$, $V_0 \in \tau(b)$ such that $U_0 \cap V_0 = \emptyset$ and $U_0 \cap M = \{a\}$, $V_0 \cap M = \{b\}$. Applying the regularity of $X$ find $U_1 \in \tau(a)$ and $V_1 \in \tau(b)$ such that $U_1 \subset U_0$ and $V_1 \subset V_0$. It is clear that the conditions (1), (2) and (4) of Lemma 2.11 are satisfied for $a, b, U_0, U_1, V_0, V_1$. The condition (3) is fulfilled because $a \in M$ and $b \in M$. The condition (5) holds due to the fact that there are no points of $M$ in $U_0 \cup V_0$ distinct from $a$ and $b$.

Thus we can apply Lemma 2.11 and conclude that $(X, \tau)$ has a dense-in-itself transversal Tychonoff topology $\mu$ which will be connected if $\kappa \geq \mathfrak{c}$.

Case 3. The set $M$ has at most one point.

If $\mathcal{M} = \emptyset$, then let $\varphi : E_\kappa \to X$ be any bijection. The topology $\mu = \{\varphi(W) : W \in \tau(E_\kappa)\}$ is as promised and to prove it we must only establish $\tau$-transversality of $\mu$. Let $x \in X$. As $\mathcal{M} = \emptyset$, there is a $U \in \tau(x)$ with $|U| < \kappa$. Therefore, the set $\varphi^{-1}(U)$ is closed and discrete in $E_\kappa$ by (1) of Corollary 2.7. Take any $W \in \tau(E_\kappa)$ such that $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$. Then $V = \varphi(W) \in \mu$ and $U \cap V = \{x\}$ so that $\mu$ is $\tau$-transversal.

Assume that $\mathcal{M} = \{a\}$. Let $\nu = \min\{|Z| : Z \in \tau^*\}$. Since the space $X$ is regular, there exists an $H \in \tau^*$ such that $|H| = \nu$ and $a \notin \overline{H}$. Since $|Z| = \nu \geq \lambda$ for any $Z \in \tau^*(H)$, the conclusion of Case 1 is applicable to the space $H$. This makes it possible to find a bijection $\xi : E_\nu \to H$ such that the topology $\{\xi(W) : W \in \tau(E_\nu)\}$ is $\tau(H)$-transversal.
Since \( \nu < \kappa \) we have \( |X \setminus \{a\} \cup H| = \kappa \). Let \( \psi : E_\kappa \to X \setminus \{H \cup \{a\}\} \) be any bijection. The spaces \( E_\nu \) and \( E_\kappa \) are not compact by Corollary 2.7, so it is possible to choose points \( z \in \beta E_\nu \setminus E_\nu \) and \( t \in \beta E_\kappa \setminus E_\kappa \). Take any \( q \in H \) and in the space \( E^* = (E_\nu \cup \{t\}) \oplus (E_\nu \cup \{z\}) \) identify the points \( q' = \xi^{-1}(q) \) and \( t \). Denote the resulting quotient space by \( E \), the point \( \{q, t\} \) by \( w \), and let \( f : E^* \to E \) be the relevant quotient map. It is clear that \( E \) is connected if \( \lambda \geq \mathfrak{c} \). Identifying \( E_\kappa \cup (E_\nu \setminus \{q'\}) \) with \( f(E_\kappa \cup (E_\nu \setminus \{q'\})) \) we have

\[
E = E_\kappa \cup (E_\nu \setminus \{q'\}) \cup \{w\} \cup \{z\}.
\]

Now let

\[
\varphi(y) = \begin{cases} 
\xi(y), & \text{if } y \in E_\nu \setminus \{q'\}; \\
\psi(y), & \text{if } y \in E_\kappa; \\
a, & \text{if } y = z, \\
q, & \text{if } y = w.
\end{cases}
\]

Then \( \varphi : E \to X \) is a bijection. To conclude our proof it suffices to establish that \( \mu = \{\varphi(W) : W \in \tau(E)\} \) is a \( \tau \)-transversal topology.

Take any \( x \in X \setminus \{a\} \). If \( x \in H \), then there is a \( U \in \tau(H) \) and \( W' \in \tau(E_\nu) \) such that \( W' \cap \xi^{-1}(U) = \{\xi^{-1}(x)\} \). If \( x \neq q \), then \( \varphi^{-1}(x) = \xi^{-1}(x) \). Otherwise \( \varphi^{-1}(x) = w \). But in both cases \( \varphi^{-1}(U) \subset \xi^{-1}(U) \cup \{w\} \) and the set \( W = f(W') \cup E_\kappa \) is open in \( E \). It is immediate that \( U \cap V = \{x\} \), where \( V = \varphi(W) \in \mu \). This shows that the condition (2) of Proposition 2.2 holds for \( x \).

Assume that \( x \in X \setminus \{H \cup \{a\}\} \). There exists a \( U \in \tau(x) \) with \( |U| < \kappa \). Therefore, \( \psi^{-1}(U \setminus H) \) is closed and discrete in \( E_\kappa \). Take any \( W \in \tau(E_\kappa) \) such that \( W \cap \psi^{-1}(U \setminus H) = \{\psi^{-1}(x)\} \). Then \( W \in \tau(E) \) and \( W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\} \). Therefore, \( V = \varphi(W) \in \mu \) and \( U \cap V = \{x\} \) which proves the property (2) of Proposition 2.2 for \( x \).

Finally, if \( x = a \), then \( W = \{z\} \cup (E_\nu \setminus \{q'\}) \) is an open neighbourhood of \( z \) in \( E \). If \( U = X \setminus \overline{\mathcal{H}} \), then \( \varphi^{-1}(U) \cap W = \{z\} \). Therefore, \( U \in \tau \), \( V = \varphi(W) \in \mu \) and \( U \cap V = \{a\} \) which shows that the condition (2) of Proposition 2.2 is fulfilled for \( x = a \). Since this condition holds for every \( x \in X \) we conclude that \( \mu \) is a \( \tau \)-transversal topology.

**2.13. Corollary.** Let \( X \) be a regular space without isolated points. Then \( X \) has a transversal dense-in-itself Tychonoff topology.

**2.14. Corollary.** Let \( X \) be a connected Tychonoff space. Then \( X \) has a connected Tychonoff transversal topology.

**Proof.** Any open subset of a Tychonoff connected space has cardinality \( \geq \mathfrak{c} \). Now apply Theorem 2.12. \( \square \)

### 3. Complementary topologies

Recall that topologies \( \tau \) and \( \mu \) on the same set \( X \) are called \( T_1 \)-complementary [Wa2], if their join is discrete and \( \tau \cap \mu = \mathcal{CF}(X) \).

**3.1. Definition.** A closed subset \( F \) of a space \( X \) is called well-placed if \( F \) and \( X \setminus F \) are infinite.
3.2. Proposition. A space \((X, \tau)\) has a well-placed subset if and only if
\[
\tau|_{X\setminus A} = \{U \cap (X\setminus A) : U \in \tau\} \neq \mathcal{CF}(X\setminus A)
\]
for each finite \(A \subset X\).

Proof. If there is a finite \(A \subset X\) such that the topology of \(X\setminus A\) is cofinite, then
for any infinite closed \(F \subset X\) the set \(F \cap (X\setminus A)\) is infinite and closed in \(X\setminus A\).
Therefore \((X\setminus A)\setminus F\) is finite whence \(X\setminus F\) is finite.

Suppose that there is no finite \(A \subset X\) with \(\tau|_{X\setminus A} = \mathcal{CF}(X\setminus A)\). If \(D\) is an infinite
discrete subspace of \(X\), then splitting it into two disjoint infinite parts \(D_0, D_1\) we see that \(D_0\) is closed, infinite and
does not intersect \(D_1\) which implies \(F = D_0\) is well-placed.

Thus, if the set \(A\) of isolated points of \(X\) is infinite, our proof is complete. If not, then \(\tau|_{X\setminus A}\) is not cofinite and hence \(X\setminus A\) has a proper infinite closed subset \(F\). If \(B = (X\setminus A)\setminus F\) is finite, then every point of \(B\) is isolated in \(X\setminus A\) and hence
in \(X\) which is a contradiction because \(B \cap A = \emptyset\). Thus \(F\) is well-placed. \(\square\)

3.3. Lemma. Let \(\tau\) and \(\mu\) be transversal topologies on a set \(X\). Suppose that
\(\mu|_A = \mathcal{CF}(A)\) for some \(A \subset X\). Then \(A\) is a discrete subspace of \((X, \tau)\).

Proof. By Proposition 2.2 there exist \(U \in \tau\) and \(V \in \mu\) such that \(U \cap V = \{a\}\) for
every \(a \in A\). But \(F = A \setminus V\) is finite, so that \(U \cap A \subset \{a\} \cup F\) is also finite. Hence
\(a\) is a \(\tau\)-isolated point of \(A\). \(\square\)

3.4. Corollary. Let \((X, \tau)\) be a space in which every discrete subset is closed. If
\(\mu\) is a \(T_1\)-complementary topology for \(\tau\), then for every well-placed set \(F\) of \((X, \mu)\)
there exists a well-placed set \(G\) of \((X, \mu)\) such that \(G \subset F\) and \(G \neq F\).

Proof. Indeed, if \(\mu|_F = \mathcal{CF}(F)\), then by Lemma 3.3 the set \(F\) is \(\tau\)-discrete and
hence closed in \((X, \tau)\), which is a contradiction with \(\tau \cap \mu = \mathcal{CF}(X)\). Thus there
exists an infinite closed proper subset \(G\) of \(F\). It is clear that \(G\) is as required. \(\square\)

3.5. Lemma. Let \((X, \tau)\) be a Hausdorff space in which every discrete subset is closed. Suppose that \(\mu\) is a \(T_1\)-complementary topology for \(\tau\) and \(F = \{F_n : n \in \omega\}\)
is a family of well-placed subsets of \((X, \mu)\) with \(F_{n+1} \subset F_n\) for each \(n \in \omega\). Then
\(F = \bigcap F\) is well-placed in \((X, \mu)\).

Proof. It suffices to prove that \(F\) is infinite. Suppose not. Since every \(F_n\) is infinite,
we can assume that \(F_n \setminus F_{n+1} \neq \emptyset\) for each \(n \in \omega\). Let \(x_n \in F_n \setminus F_{n+1}\). It is
straightforward that \(F \cup A\) is \(\mu\)-closed for every \(A \subset Y = \{x_n : n \in \omega\}\). Any
infinite subspace of a Hausdorff space has an infinite discrete subspace, so there is
an infinite \(A \subset Y\) such that \(A\) is \(\tau\)-discrete and hence closed in \((X, \tau)\). Then
\(F \cup A\) is also \(\tau\)-closed and hence it is well-placed in \((X, \mu)\) and \((X, \tau)\), which is a
contradiction with \(\tau \cap \mu = \mathcal{CF}(X)\). \(\square\)

3.6. Theorem. Let \((X, \tau)\) be a dense-in-itself Hausdorff countable space in which
every discrete subset is closed. Then \(\tau\) does not have a \(T_1\)-complement.

Proof. Assume that \(\mu\) is a \(T_1\)-complementary topology for \(\tau\). If \(A\) is a finite subset
of \(X\) and \(\mu|_{X\setminus A} = \mathcal{CF}(A)\), then by Lemma 3.3 the set \(X\setminus A\) is closed and discrete
in \((X, \tau)\). Then \(X = (X\setminus A) \cup A\) is discrete, which is a contradiction. Consequently,
we can apply Proposition 3.2 to conclude that \((X, \mu)\) has a well-placed subset \(F\).
Let \(F_0 = F\).
Suppose that for some $\alpha < \omega_1$ we have constructed $\mu$-well-placed subsets $\{F_\beta : \beta < \alpha\}$ such that $F_\delta$ is a proper subset of $F_\beta$ if $\beta < \delta < \alpha$. If $\alpha$ is a limit ordinal, let $F_\alpha = \bigcap\{F_\beta : \beta < \alpha\}$. Lemma 3.5 makes it possible to assert that $F_\alpha$ is $\mu$-well-placed. If $\alpha = \beta + 1$ use Corollary 3.4 to find a $\mu$-well-placed $G$ which is a proper subset of $F_\beta$. Putting $F_\alpha = G$ we finish our transfinite construction.

As a result we get a family $F = \{F_\alpha : \alpha < \omega_1\}$ of subsets of a countable set $X$ such that $F_\beta \subset F_\alpha$ and $F_\beta \neq F_\alpha$ if $\alpha < \beta$. It is evident that such an $F$ cannot exist so the theorem is proved.

3.7. **Corollary.** If $(X, \tau)$ is a submaximal Hausdorff countable space, then $\tau$ has no $T_1$-complement.

*Proof.* It is well-known (see e.g. [ArCo]) that in a submaximal Hausdorff space any discrete subspace is closed.

Corollary 3.7 gives a negative answer to Question 1 from [An].

3.8. **Corollary.** There exists a Tychonoff countable dense-in-itself space $(X, \tau)$ which has no $T_1$-complement.

*Proof.* Van Douwen constructed in [vD] an example of a Tychonoff maximal (and hence submaximal) countable space $(X, \tau)$. Now apply 3.6 to see that $(X, \tau)$ is as promised.

Corollary 3.8 gives a negative answer to Problem 162 (Problem 94 in its internal enumeration) of [Wa1] as well as to Problem 6.6 of [Wa2].

3.9. **Question.** Let $X$ be a Hausdorff dense-in-itself space. Does $\tau(X)$ have a transversal dense-in-itself Hausdorff (or Tychonoff) topology?

3.10. **Question.** Let $X$ be a Hausdorff connected space. Does $\tau(X)$ have a transversal connected Hausdorff (or Tychonoff) topology? What is the answer if $X$ is regular?

**References**


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