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NO SUBMAXIMAL TOPOLOGY ON A COUNTABLE SET IS T_1-COMPLEMENTARY

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ABSTRACT. Two T_1-topologies τ and μ given on the same set X, are called *transversal* if their union generates the discrete topology on X. The topologies τ and μ are T_1-complementary if they are transversal and their intersection is the cofinite topology on X. We establish that for any connected Tychonoff topology there exists a connected Tychonoff transversal one. Another result is that no T_1-complementary topology exists for the maximal topology constructed by van Douwen on the rational numbers. This gives a negative answer to Problem 162 from *Open Problems in Topology* (1990).

0. Introduction

The lattice $\mathcal{L}_1(X)$ of all T_1-topologies on a given set X has been under intensive study since 1966 when A.K. Steiner [St] showed that for any infinite set X there exist T_1-topologies on X which do not have a complement in the lattice $\mathcal{L}_1(X)$. Recall that a topology μ on X is a complement of τ in $\mathcal{L}_1(X)$ if $\tau \cup \mu$ is a subbase of the discrete topology and $\tau \cap \mu$ coincides with the cofinite topology on X.

The papers [An], [AnSt], [StSt1] and [StSt2] contain positive results on the existence of complements in the lattice $\mathcal{L}_1(X)$ which are also called T_1-complements. It was proved, in particular, that every Hausdorff locally compact or Frechet-Urysohn topology has a T_1-complement [An]. In the same paper, Anderson constructed an example of an irresolvable ($\equiv$ not representable as a union of two disjoint dense subsets) dense in itself space which has a T_1-complement and asked whether every MI-space has one [An, Question 1]. Recall that (X, τ) is a MI-space if it is dense in itself and every dense subset of X is open. It is easy to see that every MI-space is irresolvable.

Later, S. Watson asked whether each Hausdorff space has a T_1-complement. This question is published as Problem 162 (Problem 94 in the internal enumeration of Watson's paper) in Open Problems in Topology [Wa1]. The second part of Problem

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162 of [Wa1] is an inquiry whether every completely regular T_1-topology has a T_1-complement. The same question is repeated in [Wa2] (Problem 6.6).

In this paper we work only with T_1-spaces. We use the modern term "submaximal space" instead of "MI-space". Our main result is Theorem 3.6 which implies that no submaximal Hausdorff topology on a countable set is T_1-complementary. A narrower class than submaximal topologies is formed by the maximal ones. A topology τ is *maximal* if it is dense in itself but any strictly stonger one is not. As there exists in ZFC a Tychonoff countable maximal space [vD], Theorem 3.6 and Corollary 3.8 provide the negative answer to the respective questions from [An], [Wa1] and [Wa2].

Given a set X and $\tau, \mu \in \mathcal{L}_1(X)$ we say that τ and μ are *transversal* if $\tau \cup \mu$ is a subbase of the discrete topology on X. It is immediate that if τ and μ are T_1-complementary, then they are transversal.

We prove in particular, that every connected Tychonoff topology has a connected Tychonoff transversal topology. Examples of the non-existence of transversal connected topologies are given.

1. Notations and terminology

All spaces are assumed to be T_1. If X is a space, then $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\backslash\{\emptyset\}$. Analogously, if τ is a topology, then $\tau^* = \tau\backslash\{\emptyset\}$. A topology τ on a set X is called *cofinite* (and is denoted by $\mathcal{CF}(X)$) if every non-empty element of τ is a complement of a finite set. If τ is a topology on a set X and $x \in X$, then $\tau(x) = \{U \in \tau : x \in U\}$. We will write $\tau(x, X)$ instead of $\tau(X)(x)$. Given a space (X, τ) and a subset $A \subset X$ we denote by $\mathrm{cl}_\tau(A)$ and $\mathrm{Int}_\tau(A)$ the τ-closure and τ-interior of A respectively. A space X is called *submaximal* if it is dense in itself and every dense subset of X is open. A topology τ on a set X is *maximal* if (X, τ) has no isolated points, but (X, μ) has an isolated point if μ is a topology strictly stronger than τ.

All other notions are standard and can be found in [En].

2. Transversal topologies

Let us start with the main definitions and some auxiliary results.

2.1. Definition. Two topologies τ and μ on the same set X are called *transversal* if their join $\tau \vee \mu$ ($\equiv$ the smallest topology that contains $\tau \cup \mu$) is discrete. The topology μ will be referred to as a τ-transversal one.

2.2. Proposition. *If τ and μ are topologies on the same set X, then the following conditions are equivalent:*

(1) τ and μ are transversal;

(2) for each point $x \in X$ there exist $U \in \tau$ and $V \in \mu$ such that $U \cap V = \{x\}$;

(3) there is a τ-open cover γ of X such that X is a union of μ-isolated points of elements of γ.

Proof. It is clear that the family $\mathcal{B}(\tau, \mu) = \{U \cap V : U \in \tau, V \in \mu\}$ is a base of $\tau \vee \mu$. As each singleton belongs to $\tau \vee \mu$, we have $\{x\} \in \mathcal{B}(\tau, \mu)$ for every $x \in X$. This proves (1) $\Longrightarrow$ (2).

For each $x \in X$ take $U_x \in \tau(x)$ and $V_x \in \mu(x)$ such that $\{x\} = U_x \cap V_x$. Let $\gamma = \{U_x : x \in X\}$. It is clear that γ is a τ-open cover of X and x is μ-isolated in U_x for each $x \in X$. Thus (2) $\Longrightarrow$ (3).

To prove (3) $\Longrightarrow$ (1) consider any γ as in (3). For every $x \in X$ there is a $U_x \in \gamma$ such that x is μ-isolated in U_x. This means that $U_x \cap V_x = \{x\}$ for some $V_x \in \mu$ and therefore $\{x\}$ is open in $\tau \vee \mu$. □

2.3. Theorem. *Let (X, τ) be a space of weight κ. If μ is a τ-transversal topology on X, then (X, μ) is a union of $\leq \kappa$ discrete subspaces.*

Proof. Take a base $\mathcal{B}$ for (X, τ) of cardinality κ. For every $x \in X$ fix a $U_x \in \mathcal{B}$ and $V_x \in \mu$ with $U_x \cap V_x = \{x\}$. For every $U \in \mathcal{B}$ put $A_U = \{x \in X : U = U_x\}$. Since for each $x \in A_U$ we have $V_x \cap A_U = \{x\}$, the set A_U is discrete in (X, μ). Now $X = \bigcup\{A_U : U \in \mathcal{B}\}$ is a union of κ many sets discrete in (X, μ). □

2.4. Corollary. *If (X, τ) is a second countable space and μ is a τ-transversal topology on X, then (X, μ) is σ-discrete.*

2.5. Corollary. *Let (X, τ) be an infinite connected second countable space. Then there is no dense-in-itself compact (or even Baire) τ-transversal topology μ on X.*

Proof. Suppose that μ is transversal for τ and (X, μ) is a Baire space. According to Corollary 2.4 we have $X = \bigcup\{X_n : n \in \omega\}$, where X_n is a μ-discrete subset of X. The Baire property of (X, μ) implies $U = \mathrm{Int}_\mu(\mathrm{cl}_\mu(X_n)) \neq \emptyset$ for some $n \in \omega$. Then $X_n \cap U \neq \emptyset$ and any $x \in X_n \cap U$ is an isolated point of (X, μ) which contradicts the fact that (X, μ) is dense-in-itself. □

2.6. Theorem. *For every cardinal $\kappa \geq \omega$ there exists a space D_κ with the following properties:*

(1) D_κ is homeomorphic to a dense subspace of I^κ and hence D_κ is a Tychonoff space without isolated points;

(2) $D_\kappa = \bigcup\{D^n_\kappa : n \in \omega\}$, where each D^n_κ is closed, discrete in D_κ and $|D^n_\kappa| = \kappa$;

(3) if $A \subset D_\kappa$ and $|A| < \kappa$, then A is closed and discrete in D_κ;

(4) if $\kappa \geq \mathfrak{c}$, then the space D_κ is connected.

Proof. It is possible to represent the set κ in the following form: $\kappa = \bigcup\{K_\alpha : \alpha < \kappa\}$, where $|K_\alpha| = \kappa$ for all $\alpha < \kappa$ and $K_\alpha \cap K_\beta = \emptyset$ if $\alpha \neq \beta$.

If $\kappa < \mathfrak{c}$, then for every finite $A \subset \kappa$ pick a countable dense subspace C_A of the space I^A and let $\mathcal{F} = \bigcup\{C_A : A \text{ is a finite subset of } \kappa\}$.

If $\kappa \geq \mathfrak{c}$, then for every finite $A \subset \kappa$ put $C_A = I^A$ and let $\mathcal{F} = \bigcup\{C_A : A \text{ is a finite subset of } \kappa\}$.

It is clear that in both cases $|\mathcal{F}| = \kappa$, so it is possible to enumerate the elements of $\mathcal{F}$ as $\{x_\alpha : \alpha < \kappa\}$ in such a way that for each $x \in \mathcal{F}$ the set $\{\alpha : x_\alpha = x\}$ has cardinality κ. Given an $x_\alpha \in \mathcal{F}$ we denote by S_α the finite set of coordinates corresponding to the face to which the point x_α belongs.

For each $\alpha < \kappa$ let

$$d_\alpha(t) = \begin{cases} 0, & \text{if } t \notin K_\alpha \cup S_\alpha; \\ x_\alpha(t), & \text{if } t \in S_\alpha; \\ 1, & \text{if } t \in K_\alpha \backslash S_\alpha, \end{cases}$$

and denote by D_κ the subspace $\{d_\alpha : \alpha < \kappa\}$ of I^κ.

Let us prove that the space D_κ has the properties we promised. Observe first that D_κ is dense in I^κ for all $\kappa \geq \omega$.

Indeed, for a finite $S \subset \kappa$ every $x \in C_S$ occurs κ times in the enumeration of $\mathcal{F}$. Thus, there is an $\alpha(x) < \kappa$ such that $x_{\alpha(x)} = x$ and hence $S_{\alpha(x)} = S$. It is clear that $\pi_S(d_{\alpha(x)}) = x$. Therefore, $\pi_S(D_\kappa) \supset C_S$ for every finite $S \subset \kappa$. Since the set C_S is dense in I^S, we can conclude that $\pi_S(D_\kappa)$ is dense in I^S for every finite $S \subset \kappa$. Thus D_κ is dense in I^κ. This shows that (1) is true for D_κ.

Now let $D_\kappa^n = \{d_\alpha \in D_\kappa : |S_\alpha| = n\}$ for every $n \in \omega$. It is evident that $|D_\kappa^n| = \kappa$ for each $n \in \omega$ and $\bigcup\{D_\kappa^n : n \in \omega\} = D_\kappa$. Given an $\alpha < \kappa$ take any distinct $t_1, \ldots, t_{n+1} \in K_\alpha \backslash S_\alpha$. Then $d_\alpha(t_i) = 1$ for every $i \le n+1$. Thus, $W_\alpha = \{d \in D_\kappa : d(t_i) > \frac{1}{2}$ for all $i \le n+1\}$ is an open neighbourhood of d_α. If $\beta \ne \alpha$ and $|S_\beta| = n$, then

$$\{t_1, \ldots, t_{n+1}\} \backslash (K_\beta \cup S_\beta) \ne \emptyset,$$

which implies $d_\beta(t_i) = 0$ for some $i \le n+1$. Therefore, $d_\beta \notin W_\alpha$. This proves that D_κ^n is closed and discrete in D_κ. Hence we established (2) for D_κ.

Take a subset A of D_κ with $|A| < \kappa$. The sets $B = \{\alpha < \kappa : d_\alpha \in A\}$ and $H = \bigcup\{S_\alpha : \alpha \in B\}$ have cardinality less than κ. Take any $\alpha_0 < \kappa$. It is clear that $K_{\alpha_0} \backslash H \ne \emptyset$. Pick any $t \in K_{\alpha_0} \backslash H$ and let $V_{\alpha_0} = \{d \in D_\kappa : d(t) > \frac{1}{2}\}$. Then V_{α_0} is an open neighbourhood of d_{α_0} which does not contain any point of A, distinct from d_{α_0}. Thus, A is closed and discrete in D_κ. This proves (3).

Finally, suppose that $\kappa \ge \mathfrak{c}$. If D_κ is disconnected, then there is a continuous surjective function $\varphi : D_\kappa \to \{0, 1\}$. As D_κ is dense in I^K, there is a countable set $T \subset \kappa$ and a continuous function $\varphi_1 : \pi_T(D_\kappa) \to \{0, 1\}$ such that $\varphi_1 \circ \pi_T = \varphi$ [Ar]. In particular, φ_1 is surjective. Now, if S is a finite subset of T and $x \in I^S$, then the set $P(S) = \{\alpha < \kappa : x_\alpha = x\}$ has cardinality $\kappa > \omega$. Therefore, $K_\alpha \cap T = \emptyset$ for some $\alpha \in P(S)$ and hence $d_\alpha|_S = x$ and $d_\alpha|_{T\backslash S} \equiv 0$. This proves that $\pi_T(D_\kappa)$ contains the set $\sigma_T = \{z \in I^T : |\{t \in T : z(t) \ne 0\}| < \omega\}$ which is connected and dense in I^T. Since $\pi_T(D_\kappa)$ contains a dense connected subspace, it is connected. This gives a contradiction with the continuity and surjectivity of φ_1. Consequently, D_κ is connected. ☐

2.7. Corollary. *For every cardinal $\kappa \ge \omega$ there exists a Tychonoff space E_κ with the following properties:*

(1) $E_\kappa = A_\kappa \cup B_\kappa$, where each one of the subspaces A_κ, B_κ is closed in E_κ and homeomorphic to the space D_κ from Theorem 2.6. In particular, if $A \subset E_\kappa$ and $|A| < \kappa$, then A is closed and discrete in E_κ;

(2) there is a point $p \in E_\kappa$ such that $A_\kappa \cap B_\kappa = \{p\}$;

(3) $A_\kappa \backslash \{p\} = \bigcup\{A_\kappa^n : n \in \omega\}$ and $B_\kappa \backslash \{p\} = \bigcup\{B_\kappa^n : n \in \omega\}$, where A_κ^n and B_κ^n are closed and discrete in E_κ for all $n \in \omega$;

(4) $|A_\kappa^n| = |B_\kappa^n| = \kappa$ for all $n \in \omega$;

(5) $A_\kappa^n \cap A_\kappa^m = \emptyset$ and $B_\kappa^n \cap B_\kappa^m = \emptyset$ whenever $m \ne n$;

(6) if $\kappa \ge \mathfrak{c}$, then the space E_κ is connected.

Proof. Let P_κ and Q_κ be two disjoint copies of the space D_κ with $P_\kappa^n \subset P_\kappa$ and $Q_\kappa^n \subset Q_\kappa$ as respective copies of D_κ^n. Pick points $a \in P_\kappa$, $b \in Q_\kappa$ and identify a and b in the space $P_\kappa \oplus Q_\kappa$. Denote by E_κ the resulting quotient space and by $\varphi_\kappa : P_\kappa \oplus Q_\kappa \to E_\kappa$ the relevant quotient map. Define the point p by the equality $\{p\} = \varphi_\kappa(\{a, b\})$ and let $A_\kappa = \varphi_\kappa(P_\kappa)$, $A_\kappa^n = \varphi_\kappa(P_\kappa^n \backslash \{a\})$ and $B_\kappa = \varphi_\kappa(Q_\kappa)$, $B_\kappa^n = \varphi_\kappa(Q_\kappa^n \backslash \{b\})$. It is straightforward that the space E_κ satisfies (1)-(6). ☐

The following example shows that not every Tychonoff space has a transversal connected or even dense-in-itself topology.

2.8. Example. The one-point compactification of any infinite discrete space has no transversal connected (or even dense-in-itself) topology.

Proof. Let X be the one-point compactification of an infinite discrete space. If a is the only non-isolated point of X, then for every $U \in \tau(a, X)$ the set $X \backslash U$ is finite. Suppose that μ is a dense-in-itself topology, transversal to $\tau(X)$. Then there is a $V \in \mu$ such that $U \cap V = \{a\}$ for some $U \in \tau(a, X)$. Hence $V \subset (X \backslash U) \cup \{a\}$ is a non-empty finite set. Since (X, μ) is a T_1-space, any point of V is isolated in (X, μ), which is a contradiction. $\square$

2.9. Example. There exists a dense-in-itself Tychonoff space X of cardinality $\mathfrak{c}$ such that $\tau(X)$ has no transversal connected Tychonoff topology.

Proof. Consider the discrete union $Y = \bigoplus\{Q_\alpha : \alpha < \mathfrak{c}\}$ of $\mathfrak{c}$ copies of rationals. Take any point $z \notin Y$ and let $X = \{z\} \cup Y$. Declare the family $\tau(y, Y)$ to be the local base in X at any $y \in Y$. Let $\mathcal{B} = \{U_A = \{z\} \cup \bigcup\{Q_\beta : \beta \in \mathfrak{c} \backslash A\}$ where A is a countable subset of $\mathfrak{c}\}$ be the neighbourhood base of z in X.

Suppose that X has a connected transversal Tychonoff topology μ. By Proposition 2.2 there exist $U \in \tau(X)$ and $V \in \mu$ such that $\{z\} = U \cap V$. But the complement of U is countable, which implies that $V \subset (X \backslash U) \cup \{z\}$ is countable. Since in a connected Tychonoff space no non-empty proper open set can be countable, we have a contradiction. $\square$

2.10. Lemma. *Let (X, τ) be a space of cardinality $\kappa \geq \omega$. Suppose that there exist families $\{U_n : n \in \omega\}$ and $\{V_n : n \in \omega\}$ of open subsets of X with the following properties:*

(1) $\overline{U}_{n+1} \subset U_n$ and $\overline{V}_{n+1} \subset V_n$ for any $n \in \omega$;
(2) $\overline{U}_0 \cap \overline{V}_0 = \emptyset$;
(3) $|P| = \kappa$, where $P = X \backslash (U_0 \cup V_0)$;
(4) $|U_n \backslash U_{n+1}| = |V_n \backslash V_{n+1}| = \kappa$ for all $n \in \omega$.

Then there exists a τ-transversal topology μ on the set X such that (X, μ) is homeomorphic to E_κ from Corollary 2.7. In particular, τ has a transversal dense-in-itself Tychonoff topology and if $\kappa \geq \mathfrak{c}$, then τ has a transversal connected Tychonoff topology.

Proof. Let $F = \bigcap\{U_n : n \in \omega\}$ and $G = \bigcap\{V_n : n \in \omega\}$ (the sets F and G can be empty). Take a point $p' \in P$. Using evident decompositions of E_κ and X into countably many pieces of cardinality κ, we conclude that there exists a bijection $\varphi : E_\kappa \to X$ such that

(i) $\varphi(p) = p'$;
(ii) $\varphi(A^0_\kappa) = F \cup (P \backslash \{p'\})$;
(iii) $\varphi(B^0_\kappa) = G \cup (U_0 \backslash U_1)$;
(iv) $\varphi(B^i_\kappa) = U_i \backslash U_{i+1}$ for all $i \geq 1$;
(v) $\varphi(A^{i+1}_\kappa) = V_i \backslash V_{i+1}$ for every $i \in \omega$.

Let $\mu = \{\varphi(W) : W$ is open in $E_\kappa\}$. It is clear that (X, μ) is homeomorphic to E_κ. We are going to prove that the topology μ is τ-transversal.

Given an $x \in X \backslash (F \cup G)$, there exists an $n \in \omega$ such that $x \notin \overline{U}_n \cup \overline{V}_n$. Let $U = X \backslash (\overline{U}_n \cup \overline{V}_n)$. Then $x \in U$ and $\varphi^{-1}(U)$ is contained in the closed and discrete subset

$$A = \{p\} \cup A^0_\kappa \cup \ldots \cup A^n_\kappa \cup B^0_\kappa \cup \ldots \cup B^n_\kappa$$

of the space E_κ. Therefore, there is a $W \in \tau(E_\kappa)$ such that $\{\varphi^{-1}(x)\} = W \cap \varphi^{-1}(U)$. Then $V = \varphi(W) \in \mu$ and $U \cap V = \{x\}$.

Now if $x \in F$, then let $U = U_1$. Observe that $\varphi^{-1}(U) = (B_\kappa \backslash B_\kappa^0) \cup \varphi^{-1}(F) \subset B_\kappa \cup A_\kappa^0$ and $\varphi^{-1}(x) \in \varphi^{-1}(F) \subset A_\kappa^0$. Thus $A_\kappa \backslash \{p\}$ is an open neighbourhood of $\varphi^{-1}(x)$, whose intersection with $\varphi^{-1}(U)$ is contained in A_κ^0. Therefore, $\varphi^{-1}(x)$ is isolated in $\varphi^{-1}(U)$. Take any $W \in \tau(E_\kappa)$ such that $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$. Then $V = \varphi(W) \in \mu$ and $V \cap U = \{x\}$.

Finally, assume that $x \in G$. For $U = V_0$ we have $\varphi^{-1}(U) = (A_\kappa \backslash A_\kappa^0) \cup \varphi^{-1}(G) \subset A_\kappa \cup B_\kappa^0$. Thus $B_\kappa \backslash \{p\}$ is an open neighbourhood of $\varphi^{-1}(x)$, whose intersection with $\varphi^{-1}(U)$ is contained in B_κ^0. Therefore, $\varphi^{-1}(x)$ is isolated in $\varphi^{-1}(U)$. Take any $W \in \tau(E_\kappa)$ such that $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$. Then $V = \varphi(W) \in \mu$ and $V \cap U = \{x\}$. We have established that for any $x \in X$ property (2) of Proposition 2.2 is fulfilled and hence μ is τ-transversal. $\square$

2.11. Lemma. *Suppose that in a space (X, τ) of cardinality κ there exist $a, b \in X$ and $U_0, U_1, V_0, V_1 \in \tau$ such that*

(1) $a \in U_1 \subset \overline{U}_1 \subset U_0$;
(2) $b \in V_1 \subset \overline{V}_1 \subset V_0$;
(3) $|U_1| = |V_1| = \kappa$;
(4) $U_0 \cap V_0 = \emptyset$ and $X \backslash (U_0 \cup V_0) \neq \emptyset$;
(5) for any $x \in (U_0 \backslash \{a\}) \cup (V_0 \backslash \{b\})$ there exists a $U \in \tau(x, X)$ such that $|U| < \kappa$.

Then there exists a τ-transversal topology μ on the set X such that (X, μ) is homeomorphic to the space E_κ from Corollary 2.7. In particular, τ has a transversal dense-in-itself Tychonoff topology and if $\kappa \geq \mathfrak{c}$, then τ has a transversal connected Tychonoff topology.

Proof. Pick any $p' \in X \backslash (U_0 \cup V_0)$ and denote by P the set $X \backslash (U_0 \cup V_1 \cup \{p'\})$. Using evident decompositions of X and E_κ into finitely many pieces we can construct a bijection $\varphi : E_\kappa \to X$ such that

(i) $\varphi(p) = p'$;
(ii) $\varphi(A_\kappa^0) \supset P \cup \{b\}$ and $\varphi(A_\kappa \backslash \{p\}) = (U_1 \backslash \{a\}) \cup P \cup \{b\}$;
(iii) $\varphi(B_\kappa^0) \supset U_0 \backslash U_1$ and $\varphi(B_\kappa \backslash \{p\}) = (V_1 \backslash \{b\}) \cup (U_0 \backslash U_1) \cup \{a\}$.

Let $\mu = \{\varphi(W) : W$ is open in $E_\kappa\}$. It is clear that (X, μ) is homeomorphic to E_κ. We are going to prove that the topology μ is τ-transversal.

Given an $x \in X \backslash (U_0 \cup V_0)$ let $U = X \backslash (\overline{U}_1 \cup \overline{V}_1)$. The set U is a τ-open neighbourhood of the point x and $\varphi^{-1}(U) \subset \varphi^{-1}(P \cup \{p'\} \cup (U_0 \backslash U_1)) \subset A_\kappa^0 \cup \{p\} \cup B_\kappa^0$. Since the set $A_\kappa^0 \cup \{p\} \cup B_\kappa^0$ is closed and discrete in E_κ, there is a $W \in \tau(E_\kappa)$ such that $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$. Then $V = \varphi(W) \in \mu$ and $U \cap V = \{x\}$.

Suppose that $x \in (U_0 \backslash \{a\}) \cup (V_0 \backslash \{b\})$. Apply (5) to find a $U \in \tau$ such that $x \in U$ and $|U| < \kappa$. Then $|\varphi^{-1}(U)| < \kappa$ and therefore $\varphi^{-1}(U)$ is closed and discrete in E_κ by condition (1) of Proposition 2.7. Pick a $W \in \tau(E_\kappa)$ such that $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$. Then $V = \varphi(W) \in \mu$ and $U \cap V = \{x\}$.

If $x = a$, let $U = U_1$. Then $\varphi^{-1}(U) \cap (B_\kappa \backslash \{p\}) = \{\varphi^{-1}(a)\}$. Thus for $W = B_\kappa \backslash \{p\}$ we have $V = \varphi(W) \in \mu$ and $U \cap V = \{a\}$.

If $x = b$, let $U = V_1$. Then $\varphi^{-1}(U) \cap (A_\kappa \backslash \{p\}) = \{\varphi^{-1}(b)\}$. Thus for $W = A_\kappa \backslash \{p\}$ we have $V = \varphi(W) \in \mu$ and $U \cap V = \{b\}$.

Condition (2) of Proposition 2.2 having been checked for every $x \in X$ we conclude that τ and μ are transversal. $\square$

2.12. Theorem. *Given a cardinal number $\lambda \geq \omega$ suppose that (X,τ) is a regular space such that $|U| \geq \lambda$ for any $U \in \tau^*$. Then τ has a dense-in-itself transversal Tychonoff topology μ. Moreover, if $\lambda \geq \mathfrak{c}$, then μ can be chosen to be Tychonoff and connected.*

Proof. Let

$$M = \{x \in X : |U| = |X| = \kappa \text{ for any } U \in \tau(x)\}.$$

There are three cases to consider.
Case 1. The set M has at least two cluster points, say a and b. It is clear that M has to be infinite and $a, b \in M$.

Pick two distinct points $x_0, y_0 \in M\backslash\{a,b\}$. There exist $U_0 \in \tau(a)$, $V_0 \in \tau(b)$ such that $\overline{U}_0 \cap \overline{V}_0 = \emptyset$ and $\{x_0, y_0\} \subset X\backslash(\overline{U}_0 \cup \overline{V}_0)$. In particular $|P| = \kappa$, where $P = X\backslash(U_0 \cup V_0)$.

Suppose that we have constructed open sets $U_i \in \tau(a)$, $V_i \in \tau(b)$ and points $x_i, y_i \in M$ for all $i \leq n$ in such a way that

(i) $\overline{U}_{i+1} \subset U_i$ and $\overline{V}_{i+1} \subset V_i$ for all $i < n$;

(ii) $x_{i+1} \in U_i\backslash\overline{U}_{i+1}$ and $y_{i+1} \in V_i\backslash\overline{V}_{i+1}$ for all $i < n$.

Since a and b are cluster points of M, there exist $x_{n+1} \in (U_n\backslash\{a\}) \cap M$ and $y_{n+1} \in (V_n\backslash\{b\}) \cap M$. Since the space X is regular, we can find $U_{n+1} \in \tau(a)$ and $V_{n+1} \in \tau(b)$ such that $\overline{U}_{n+1} \subset (U_n\backslash\{x_{n+1}\})$ and $\overline{V}_{n+1} \subset (V_n\backslash\{y_{n+1}\})$. It is evident that the properties (i) and (ii) are fulfilled for all $i \leq n$.

Observe that the families $\mathcal{U} = \{U_i : i \in \omega\}$ and $\mathcal{V} = \{V_i : i \in \omega\}$ satisfy the conditions (1)-(3) of Lemma 2.10. The condition (4) is also fulfilled because each of the sets $U_n\backslash\overline{U}_{n+1}$ and $V_n\backslash\overline{V}_{n+1}$ is open and meets M for any $n \in \omega$. Therefore, we can apply Lemma 2.10 and conclude that (X,τ) has a dense-in-itself transversal topology μ which will be connected if $\kappa \geq \mathfrak{c}$.
Case 2. The set M has at least two isolated points, say a and b.

Pick any $p' \in X\backslash\{a,b\}$. Since X is Hausdorff, there exist $U_0 \in \tau(a)$, $V_0 \in \tau(b)$ such that $U_0 \cap V_0 = \emptyset$, $p' \in X\backslash(U_0 \cup V_0)$ and $U_0 \cap M = \{a\}$, $V_0 \cap M = \{b\}$. Applying the regularity of X find $U_1 \in \tau(a)$ and $V_1 \in \tau(b)$ such that $\overline{U}_1 \subset U_0$ and $\overline{V}_1 \subset V_0$. It is clear that the conditions (1), (2) and (4) of Lemma 2.11 are satisfied for a, b, U_0, U_1, V_0, V_1. The condition (3) is fulfilled because $a \in M$ and $b \in M$. The condition (5) holds due to the fact that there are no points of M in $U_0 \cup V_0$ distinct from a and b.

Thus we can apply Lemma 2.11 and conclude that (X,τ) has a dense-in-itself transversal Tychonoff topology μ which will be connected if $\kappa \geq \mathfrak{c}$.
Case 3. The set M has at most one point.

If $M = \emptyset$, then let $\varphi : E_\kappa \to X$ be any bijection. The topology $\mu = \{\varphi(W) : W \in \tau(E_\kappa)\}$ is as promised and to prove it we must only establish τ-transversality of μ. Let $x \in X$. As $M = \emptyset$, there is a $U \in \tau(x)$ with $|U| < \kappa$. Therefore, the set $\varphi^{-1}(U)$ is closed and discrete in E_κ by (1) of Corollary 2.7. Take any $W \in \tau(E_\kappa)$ such that $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$. Then $V = \varphi(W) \in \mu$ and $U \cap V = \{x\}$ so that μ is τ-transversal.

Assume that $M = \{a\}$. Let $\nu = \min\{|Z| : Z \in \tau^*\}$. Since the space X is regular, there exists an $H \in \tau^*$ such that $|H| = \nu$ and $a \notin \overline{H}$. Since $|Z| = \nu \geq \lambda$ for any $Z \in \tau^*(H)$, the conclusion of Case 1 is applicable to the space H. This makes it possible to find a bijection $\xi : E_\nu \to H$ such that the topology $\{\xi(W) : W \in \tau(E_\nu)\}$ is $\tau(H)$-transversal.

Since $\nu < \kappa$ we have $|X\backslash(\{a\} \cup H)| = \kappa$. Let $\psi : E_\kappa \to X\backslash(H \cup \{a\})$ be any bijection. The spaces E_ν and E_κ are not compact by Corollary 2.7, so it is possible to choose points $z \in \beta E_\nu \backslash E_\nu$ and $t \in \beta E_\kappa \backslash E_\kappa$. Take any $q \in H$ and in the space $E^* = (E_\kappa \cup \{t\}) \oplus (E_\nu \cup \{z\})$ identify the points $q' = \xi^{-1}(q)$ and t. Denote the resulting quotient space by E, the point $\{q', t\}$ by w, and let $f : E^* \to E$ be the relevant quotient map. It is clear that E is a dense-in-itself Tychonoff space which is connected if $\lambda \geq \mathfrak{c}$. Identifying $E_\kappa \cup (E_\nu \backslash \{q'\})$ with $f(E_\kappa \cup (E_\nu \backslash \{q'\}))$ we have

$$E = E_\kappa \cup (E_\nu \backslash \{q'\}) \cup \{w\} \cup \{z\}.$$

Now let

$$\varphi(y) = \begin{cases} \xi(y), & \text{if } y \in E_\nu \backslash \{q'\}; \\ \psi(y), & \text{if } y \in E_\kappa; \\ a, & \text{if } y = z, \\ q, & \text{if } y = w. \end{cases}$$

Then $\varphi : E \to X$ is a bijection. To conclude our proof it suffices to establish that $\mu = \{\varphi(W) : W \in \tau(E)\}$ is a τ-transversal topology.

Take any $x \in X\backslash\{a\}$. If $x \in H$, then there is a $U \in \tau(H)$ and $W' \in \tau(E_\nu)$ such that $W' \cap \xi^{-1}(U) = \{\xi^{-1}(x)\}$. If $x \neq q$, then $\varphi^{-1}(x) = \xi^{-1}(x)$. Otherwise $\varphi^{-1}(x) = w$. But in both cases $\varphi^{-1}(U) \subset \xi^{-1}(U) \cup \{w\}$ and the set $W = f(W') \cup E_\kappa$ is open in E. It is immediate that $U \cap V = \{x\}$, where $V = \varphi(W) \in \mu$. This shows that the condition (2) of Proposition 2.2 holds for x.

Assume that $x \in X\backslash(H \cup \{a\})$. There exists a $U \in \tau(x)$ with $|U| < \kappa$. Therefore, $\psi^{-1}(U\backslash H)$ is closed and discrete in E_κ. Take any $W \in \tau(E_\kappa)$ such that $W \cap \psi^{-1}(U\backslash H) = \{\psi^{-1}(x)\}$. Then $W \in \tau(E)$ and $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$. Therefore, $V = \varphi(W) \in \mu$ and $U \cap V = \{x\}$ which proves the property (2) of Proposition 2.2 for x.

Finally, if $x = a$, then $W = \{z\} \cup (E_\nu \backslash \{q'\})$ is an open neighbourhood of z in E. If $U = X\backslash\overline{H}$, then $\varphi^{-1}(U) \cap W = \{z\} = \{\varphi^{-1}(a)\}$. Therefore, $U \in \tau,\ V = \varphi(W) \in \mu$ and $U \cap V = \{a\}$ which shows that the condition (2) of Proposition 2.2 is fulfilled for $x = a$. Since this condition holds for every $x \in X$ we conclude that μ is a τ-transversal topology. □

2.13. Corollary. *Let X be a regular space without isolated points. Then X has a transversal dense-in-itself Tychonoff topology.*

2.14. Corollary. *Let X be a connected Tychonoff space. Then X has a connected Tychonoff transversal topology.*

Proof. Any open subset of a Tychonoff connected space has cardinality $\geq \mathfrak{c}$. Now apply Theorem 2.12. □

3. Complementary topologies

Recall that topologies τ and μ on the same set X are called T_1-complementary [Wa2], if their join is discrete and $\tau \cap \mu = \mathcal{CF}(X)$.

3.1. Definition. A closed subset F of a space X is called well-placed if F and $X\backslash F$ are infinite.

3.2. Proposition. *A space (X,τ) has a well-placed subset if and only if*

$$\tau|_{X\setminus A} = \{U \cap (X\setminus A) : U \in \tau\} \neq \mathcal{CF}(X\setminus A)$$

for each finite $A \subset X$.

Proof. If there is a finite $A \subset X$ such that the topology of $X\setminus A$ is cofinite, then for any infinite closed $F \subset X$ the set $F \cap (X\setminus A)$ is infinite and closed in $X\setminus A$. Therefore $(X\setminus A)\setminus F$ is finite whence $X\setminus F$ is finite.

Suppose that there is no finite $A \subset X$ with $\tau|_{X\setminus A} = \mathcal{CF}(X\setminus A)$. If D is an infinite discrete subspace of X, then splitting it into two disjoint infinite parts D_0, D_1 we see that $\overline{D}_0$ is closed, infinite and does not intersect D_1 which implies $F = \overline{D}_0$ is well-placed.

Thus, if the set A of isolated points of X is infinite, our proof is complete. If not, then $\tau|_{X\setminus A}$ is not cofinite and hence $X\setminus A$ has a proper infinite closed subset F. If $B = (X\setminus A)\setminus F$ is finite, then every point of B is isolated in $X\setminus A$ and hence in X which is a contradiction because $B \cap A = \emptyset$. Thus F is well-placed. □

3.3. Lemma. *Let τ and μ be transversal topologies on a set X. Suppose that $\mu|_A = \mathcal{CF}(A)$ for some $A \subset X$. Then A is a discrete subspace of (X,τ).*

Proof. By Proposition 2.2 there exist $U \in \tau$ and $V \in \mu$ such that $U \cap V = \{a\}$ for every $a \in A$. But $F = A\setminus V$ is finite, so that $U \cap A \subset \{a\} \cup F$ is also finite. Hence a is a τ-isolated point of A. □

3.4. Corollary. *Let (X,τ) be a space in which every discrete subset is closed. If μ is a T_1-complementary topology for τ, then for every well-placed set F of (X,μ) there exists a well-placed set G of (X,μ) such that $G \subset F$ and $G \neq F$.*

Proof. Indeed, if $\mu|_F = \mathcal{CF}(F)$, then by Lemma 3.3 the set F is τ-discrete and hence closed in (X,τ), which is a contradiction with $\tau \cap \mu = \mathcal{CF}(X)$. Thus there exists an infinite closed proper subset G of F. It is clear that G is as required. □

3.5. Lemma. *Let (X,τ) be a Hausdorff space in which every discrete subset is closed. Suppose that μ is a T_1-complementary topology for τ and $\mathcal{F} = \{F_n : n \in \omega\}$ is a family of well-placed subsets of (X,μ) with $F_{n+1} \subset F_n$ for each $n \in \omega$. Then $F = \bigcap \mathcal{F}$ is well-placed in (X,μ).*

Proof. It suffices to prove that F is infinite. Suppose not. Since every F_n is infinite, we can assume that $F_n\setminus F_{n+1} \neq \emptyset$ for each $n \in \omega$. Let $x_n \in F_n\setminus F_{n+1}$. It is straightforward that $F \cup A$ is μ-closed for every $A \subset Y = \{x_n : n \in \omega\}$. Any infinite subspace of a Hausdorff space has an infinite discrete subspace, so there is an infinite $A \subset Y$ such that A is τ-discrete and hence closed in (X,τ). Then $F \cup A$ is also τ-closed and hence it is well-placed in (X,μ) and (X,τ), which is a contradiction with $\tau \cap \mu = \mathcal{CF}(X)$. □

3.6. Theorem. *Let (X,τ) be a dense-in-itself Hausdorff countable space in which every discrete subset is closed. Then τ does not have a T_1-complement.*

Proof. Assume that μ is a T_1-complementary topology for τ. If A is a finite subset of X and $\mu|_{X\setminus A} = \mathcal{CF}(A)$, then by Lemma 3.3 the set $X\setminus A$ is closed and discrete in (X,τ). Then $X = (X\setminus A) \cup A$ is discrete, which is a contradiction. Consequently, we can apply Proposition 3.2 to conclude that (X,μ) has a well-placed subset F. Let $F_0 = F$.

Suppose that for some $\alpha < \omega_1$ we have constructed μ-well-placed subsets $\{F_\beta : \beta < \alpha\}$ such that F_δ is a proper subset of F_β if $\beta < \delta < \alpha$. If α is a limit ordinal, let $F_\alpha = \bigcap\{F_\beta : \beta < \alpha\}$. Lemma 3.5 makes it possible to assert that F_α is μ-well-placed. If $\alpha = \beta + 1$ use Corollary 3.4 to find a μ-well-placed G which is a proper subset of F_β. Putting $F_\alpha = G$ we finish our transfinite construction.

As a result we get a family $\mathcal{F} = \{F_\alpha : \alpha < \omega_1\}$ of subsets of a countable set X such that $F_\beta \subset F_\alpha$ and $F_\beta \neq F_\alpha$ if $\alpha < \beta$. It is evident that such an $\mathcal{F}$ cannot exist so the theorem is proved. □

3.7. Corollary. *If (X, τ) is a submaximal Hausdorff countable space, then τ has no T_1-complement.*

Proof. It is well-known (see e.g. [ArCo]) that in a submaximal Hausdorff space any discrete subspace is closed. □

Corollary 3.7 gives a negative answer to Question 1 from [An].

3.8. Corollary. *There exists a Tychonoff countable dense-in-itself space (X, τ) which has no T_1-complement.*

Proof. Van Douwen constructed in [vD] an example of a Tychonoff maximal (and hence submaximal) countable space (X, τ). Now apply 3.6 to see that (X, τ) is as promised. □

Corollary 3.8 gives a negative answer to Problem 162 (Problem 94 in its internal enumeration) of [Wa1] as well as to Problem 6.6 of [Wa2].

3.9. Question. Let X be a Hausdorff dense-in-itself space. Does $\tau(X)$ have a transversal dense-in-itself Hausdorff (or Tychonoff) topology?

3.10. Question. Let X be a Hausdorff connected space. Does $\tau(X)$ have a transversal connected Hausdorff (or Tychonoff) topology? What is the answer if X is regular?

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