

## REFLECTION AND UNIQUENESS THEOREMS FOR HARMONIC FUNCTIONS

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(Communicated by J. Marshall Ash)

ABSTRACT. Suppose that  $h$  is harmonic on an open half-ball  $\beta$  in  $R^N$  such that the origin  $0$  is the centre of the flat part  $\tau$  of the boundary  $\partial\beta$ . If  $h$  has non-negative lower limit at each point of  $\tau$  and  $h$  tends to  $0$  sufficiently rapidly on the normal to  $\tau$  at  $0$ , then  $h$  has a harmonic continuation by reflection across  $\tau$ . Under somewhat stronger hypotheses, the conclusion is that  $h \equiv 0$ . These results strengthen recent theorems of Baouendi and Rothschild. While the flat boundary set  $\tau$  can be replaced by a spherical surface, it cannot in general be replaced by a smooth  $(N - 1)$ -dimensional manifold.

### 1. INTRODUCTION

Let  $x = (x_1, \dots, x_N)$  denote a typical point of  $R^N$ , where  $N \geq 2$ , and let  $\|\cdot\|$  be the Euclidean norm on  $R^N$ . For each positive number  $r$  let

$$\begin{aligned}\beta(r) &= \{x : \|x\| < r, x_N > 0\}, \\ \tau(r) &= \{x : \|x\| < r, x_N = 0\}, \\ \alpha(r) &= \{(0, \dots, 0, x_N) \in R^N : 0 < x_N < r\}.\end{aligned}$$

A modified form of a recent theorem of Baouendi and Rothschild [1, Theorem 3] may be stated as follows. (The theorem was originally proved with a relatively open subset of the unit sphere in place of  $\tau(r)$ , but see [1, §0, final paragraph].)

**Theorem BR.** *Let  $h$  be a continuous real-valued function on  $\overline{\beta(r)}$ , harmonic on  $\beta(r)$ . If  $h \geq 0$  on  $\tau(r)$  and*

$$(1) \quad \lim_{t \rightarrow 0^+} t^{-m} h(0, \dots, 0, t) = 0$$

*for each positive integer  $m$ , then  $h = 0$  on  $\alpha(r) \cup \tau(\rho)$  for some  $\rho \in (0, r]$ .*

The main result of this note is the following strengthened version of the above theorem.

**Theorem 1.** *Let  $h$  be harmonic on  $\beta(r)$ . If*

$$(2) \quad \liminf_{x \rightarrow y, x \in \beta(r)} h(x) \geq 0$$

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Received by the editors February 7, 1995 and, in revised form, March 4, 1998.

1991 *Mathematics Subject Classification.* Primary 31B05.

*Key words and phrases.* Harmonic function, reflection, uniqueness, continuation.

for each  $y \in \tau(r)$  and

$$(3) \quad \liminf_{t \rightarrow 0^+} t^{-m} |h(0, \dots, 0, t)| = 0$$

for each positive integer  $m$ , then  $h = 0$  on  $\alpha(r)$  and

$$(4) \quad \lim_{x \rightarrow y, x \in \beta(r)} h(x) = 0$$

for each  $y \in \tau(r)$ .

In Theorem 1 the hypotheses of Theorem BR that  $h$  is continuous on  $\overline{\beta(r)}$  and non-negative on  $\tau(r)$  are replaced by the milder hypothesis (2). Also, (3) is a relaxation of (1).

Recall that if (4) holds for each  $y \in \tau(r)$ , then  $h$  has a unique harmonic continuation  $\bar{h}$  to the ball

$$B(r) = \{x : \|x\| < r\}$$

and  $\bar{h}$  is obtained by reflection, that is,

$$(5) \quad \bar{h}(x_1, \dots, x_{N-1}, -x_N) = -h(x_1, \dots, x_N) \quad (x \in \beta(r)).$$

Thus Theorems BR and 1 may be regarded as reflection principles. Under more stringent hypotheses, these theorems become uniqueness results. For each number  $d \in [0, 1)$  let  $C(d)$  denote the spherical cap  $\{x : \|x\| = 1, x_N > d\}$ .

**Corollary 1.** *Let  $h$  be harmonic on  $\beta(r)$ , and suppose that (2) holds for each  $y \in \tau(r)$ . If there exists a sequence  $(d_m)$  in  $[0, 1)$  such that*

$$(6) \quad \liminf_{t \rightarrow 0^+} t^{-m} |h(tz)| = 0 \quad (z \in C(d_m))$$

for each positive integer  $m$ , then  $h \equiv 0$ .

**Corollary 2.** *Let  $h$  be harmonic on  $\beta(r)$ . If*

$$(7) \quad \liminf_{x \rightarrow y, x \in \beta(r)} h(x)/x_N \geq 0$$

for each  $y \in \tau(r)$  and (3) holds for each positive integer  $m$ , then  $h \equiv 0$ .

Corollary 1 is an improvement of the half-space version of [1, Corollary 2.6], but Corollary 2 seems to have no counterpart in [1].

In §5 we give an example to show that Theorem BR (and *a fortiori* Theorem 1) may fail if  $\beta(r)$  is replaced by a domain whose boundary is analytic near 0 and tangential at 0 to  $\tau(1)$ . This partially answers the conjecture [1, p. 245] that results similar to those of [1] hold for more general domains.

Baouendi and Rothschild [2] have recently generalized the main results of [1] in a direction different from that considered here.

## 2. A REPRESENTATION LEMMA

We shall need the following lemma, which may be regarded as a local version of the Poisson integral representation of a positive harmonic function on a half-space (see, e.g., Helms [4, Theorem 2.25]).

**Lemma.** *Suppose that  $h$  is harmonic on  $\beta(r)$  and satisfies (2) at each  $y \in \tau(r)$ . Then there exists a measure  $\mu$  on  $\tau(r)$  with the following property: for each  $\rho \in (0, r)$ , there exists a harmonic function  $H$  on  $B(\rho)$  such that*

$$(8) \quad H(x_1, \dots, x_N) = -H(x_1, \dots, x_{N-1}, -x_N) \quad (x \in B(\rho))$$

and

$$(9) \quad h(x) = x_N \int_{\tau(\rho)} \|x - y\|^{-N} d\mu(y) + H(x) \quad (x \in \beta(\rho)).$$

This lemma is known, at least tacitly, but lacking on exact reference, I indicate a proof using a technique which involves passing to a space of higher dimension. This technique is due to Huber [5] and was rediscovered and more extensively exploited by Kuran [6]. The results that we shall need are half-ball versions of [6, Lemmas 1, 3, 6, Theorem 1], which were proved in [6] for a half-space rather than a half-ball, but which are indeed valid in the generality we require (see [6, p. 279]).

We verify first that if (2) holds for each  $y \in \tau(r)$ , then

$$(10) \quad \liminf_{x \rightarrow y, x \in \beta(r)} x_N^{-1} h(x) > -\infty \quad (y \in \tau(r)).$$

The function  $u$ , defined on  $\beta(r)$  by  $u(x) = \min\{h(x), x_N\}$ , is superharmonic on  $\beta(r)$  and tends to 0 at each point of  $\tau(r)$ . It follows from [6, Lemma 3] that (10) holds with  $u$  in place of  $h$ , and since  $h \geq u$  on  $\beta(r)$ , (10) itself is true.

A typical point of  $R^{N+2}$  is denoted by  $\xi = (\xi_1, \dots, \xi_{N+2})$ , and with such a point we associate the number

$$\delta_\xi = \sqrt{(\xi_N^2 + \xi_{N+1}^2 + \xi_{N+2}^2)}.$$

Let  $\mathcal{B}(r)$  denote the open ball of radius  $r$  centred at the origin of  $R^{N+2}$  and let  $E = \{\xi : \delta_\xi = 0\}$ . Define  $h^*$  on  $\mathcal{B}(r) \setminus E$  by

$$h^*(\xi) = \delta_\xi^{-1} h(\xi_1, \dots, \xi_{N-1}, \delta_\xi).$$

By [6, Lemma 1],  $h^*$  is harmonic on  $\mathcal{B}(r) \setminus E$ , and since (10) holds, it follows from [6, Theorem 1] that  $h^*$  has a superharmonic extension  $U$  to  $\mathcal{B}(r)$ . The support of the Riesz measure  $\mu^*$  associated to  $U$  is contained in  $E$ . Hence, by the local form of the Riesz decomposition theorem, if  $0 < \rho < r$ , then there exists a harmonic function  $H^*$  on  $\mathcal{B}(\rho)$  such that

$$U(\xi) = \int_{E \cap \mathcal{B}(\rho)} \|\xi - \eta\|^{-N} d\mu^*(\eta) + H^*(\xi) \quad (x \in \mathcal{B}(\rho)).$$

If  $x \in \beta(\rho)$  and  $\xi_x$  is defined to be the point  $(x_1, \dots, x_N, 0, 0)$ , then  $\xi_x \in \mathcal{B}(\rho) \setminus E$  and  $\delta_{\xi_x} = x_N$ , so that

$$\begin{aligned} h(x) &= x_N h^*(\xi_x) \\ &= x_N U(\xi_x) \\ &= x_N \int_{E \cap \mathcal{B}(\rho)} \|\xi_x - \eta\|^{-N} d\mu^*(\eta) + x_N H^*(\xi_x) \\ &= x_N \int_{\tau(\rho)} \|x - y\|^{-N} d\mu(y) + \tilde{H}(x), \end{aligned}$$

where  $\mu$  is the measure defined on the Borel subsets of  $\tau(r)$  by

$$\mu(F) = \mu^*(\{\xi \in E : (\xi_1, \dots, \xi_{N-1}, 0) \in F\})$$

and  $\tilde{H}$  is defined on  $\beta(\rho)$  by  $\tilde{H}(x) = x_N H^*(\xi_x)$ . By [6, Lemma 6],  $\tilde{H}$  is harmonic on  $\beta(\rho)$ , and since  $H^*$  is locally bounded on  $\mathcal{B}(\rho)$  it follows that  $\tilde{H}$  tends to 0 at each point of  $\tau(\rho)$ . Therefore, by the reflection principle,  $\tilde{H}$  has a harmonic continuation  $H$  to  $B(\rho)$  satisfying (8).

### 3. PROOF OF THEOREM 1

Suppose that  $h$  satisfies the hypotheses of Theorem 1. It is enough to fix  $\rho \in (0, r)$  and to show that  $h = 0$  on  $\alpha(\rho)$  and (4) holds for each  $y \in \tau(\rho)$ .

According to the lemma,  $h$  has the representation (9) on  $\beta(\rho)$ . Since the function  $H$  in (9) is harmonic on  $B(\rho)$ , there exists a series  $\sum_{j=0}^{\infty} H_j$ , where  $H_j$  is a homogeneous harmonic polynomial of degree  $j$  on  $R^N$ , which converges to  $H$  on  $B(\rho)$  (see, e.g., Brelot [3, Appendix]). In particular, the function  $t \mapsto H(0, \dots, 0, t)$  is given on the interval  $(-\rho, \rho)$  by its Taylor series about 0. Also, since by (8) this function is odd, the Taylor series contains only odd powers of  $t$ . Thus we have a representation of the form

$$(11) \quad H(0, \dots, 0, t) = \sum_{j=0}^{\infty} a_{2j+1} t^{2j+1} \quad (-\rho < t < \rho).$$

We next aim to prove inductively that

$$(12) \quad a_{2j+1} = (-1)^{j+1} \left( j + \frac{N}{2} - 1 \right) \int_{\tau(\rho)} \|y\|^{-N-2j} d\mu(y)$$

for each non-negative integer  $j$ . An argument by contradiction will then show that  $\mu \equiv 0$  on  $\tau(\rho)$ , and the conclusions of Theorem 1 will then follow easily.

Throughout this paragraph  $x$  denotes a point of  $\alpha(\rho)$  with coordinates  $(0, \dots, 0, t)$ . By (9) and (11),

$$\begin{aligned} t^{-1}h(x) &= \int_{\tau(\rho)} (t^2 + \|y\|^2)^{-N/2} d\mu(y) + a_1 + O(t^2) \\ &\rightarrow \int_{\tau(\rho)} \|y\|^{-N} d\mu(y) + a_1 \quad (t \rightarrow 0+), \end{aligned}$$

by monotone convergence. Hypothesis (3) with  $m = 1$  now implies that (12) holds with  $j = 0$ . To proceed with the inductive proof of (12), we introduce the function  $\phi$ , defined by

$$\phi(\theta) = (1 + \theta)^{-N/2} \quad (\theta > -1),$$

and note that

$$(13) \quad \frac{\phi^{(j)}(0)}{j!} = (-1)^j \left( j + \frac{N}{2} - 1 \right) \quad (j = 0, 1, 2, \dots)$$

and that by Taylor's theorem,

$$(14) \quad \phi(\theta) - \sum_{j=0}^k \frac{\phi^{(j)}(0)}{j!} \theta^j = \frac{\theta^{k+1}}{k!} \int_0^1 (1 - \zeta)^k \phi^{(k+1)}(\theta\zeta) d\zeta \quad (\theta > -1, k = 0, 1, 2, \dots).$$

It is easy to see that  $(-1)^{k+1}\phi^{(k+1)}$  is positive and decreasing on  $(-1, +\infty)$ . Hence it follows from (14) that if  $\Phi_k$  is defined by

$$\Phi_k(\theta) = \frac{(-1)^{k+1}}{\theta^{k+1}} \left( \phi(\theta) - \sum_{j=0}^k \frac{\phi^{(j)}(0)}{j!} \theta^j \right),$$

then  $\Phi_k$  is positive and decreasing on  $(0, +\infty)$  and

$$(15) \quad \lim_{\theta \rightarrow 0^+} \Phi_k(\theta) = (-1)^{k+1} \frac{\phi^{(k+1)}(0)}{(k+1)!}.$$

Suppose now that (12) holds for  $j = 0, \dots, k$ . Then by (9) and (11),

$$\begin{aligned} h(x) &= t \int_{\tau(\rho)} (t^2 + \|y\|^2)^{-N/2} d\mu(y) \\ &\quad - \sum_{j=0}^k (-1)^j \left( j + \frac{N}{2} - 1 \right) t^{2j+1} \int_{\tau(\rho)} \|y\|^{-N-2j} d\mu(y) \\ &\quad + \sum_{j=k+1}^{\infty} a_{2j+1} t^{2j+1}. \end{aligned}$$

Using (13), we can write this equation in the form

$$\begin{aligned} \frac{h(x)}{t^{2k+3}} &= \frac{1}{t^{2k+2}} \left\{ \int_{\tau(\rho)} \|y\|^{-N} \left( \phi \left( \frac{t^2}{\|y\|^2} \right) - \sum_{j=0}^k \frac{\phi^{(j)}(0)}{j!} \left( \frac{t^2}{\|y\|^2} \right)^j \right) d\mu(y) \right\} \\ &\quad + a_{2k+3} + O(t^2) \\ &= (-1)^{k+1} \int_{\tau(\rho)} \|y\|^{-N-2k-2} \Phi_k \left( \frac{t^2}{\|y\|^2} \right) d\mu(y) + a_{2k+3} + O(t^2). \end{aligned}$$

Since  $\Phi_k$  is positive and decreasing on  $(0, +\infty)$ , it now follows from (15) that

$$\lim_{t \rightarrow 0^+} t^{-2k-3} h(x) = \frac{\phi^{(k+1)}(0)}{(k+1)!} \int_{\tau(\rho)} \|y\|^{-N-2k-2} d\mu(y) + a_{2k+3}.$$

This, together with (13) and hypothesis (3), implies that (12) holds for  $j = k+1$ , and the inductive proof of (12) is complete.

Now suppose that  $\mu \not\equiv 0$  on  $\tau(\rho)$ . Choose  $\sigma \in (0, \rho)$  such that  $\mu(\tau(\sigma)) > 0$ . Defining

$$\lambda = \int_{\tau(\sigma)} \|y\|^{-N} d\mu(y),$$

we have  $\lambda > 0$  and

$$\int_{\tau(\sigma)} \|y\|^{-N-2j} d\mu(y) \geq \lambda \sigma^{-2j} \quad (j = 0, 1, \dots).$$

Hence, by (12),

$$|a_{2j+1}| \geq \lambda \left( j + \frac{N}{2} - 1 \right) \sigma^{-2j} \geq \lambda \sigma^{-2j},$$

so that  $\sum a_{2j+1} t^{2j+1}$  diverges when  $t \geq \sigma$ . As this series converges to  $H(0, \dots, 0, t)$  when  $-\rho < t < \rho$ , we have arrived at a contradiction. This shows that  $\mu \equiv 0$  on  $\tau(\rho)$ .

The representation (9) now reduces to  $h = H$  on  $\beta(\rho)$ , and since  $H = 0$  on  $\tau(\rho)$ , (4) holds for each  $y \in \tau(\rho)$ . Also, (12) now gives  $a_{2j+1} = 0$  for each  $j$ , so that by (11),  $h = H = 0$  on  $\alpha(\rho)$ . Since  $\rho$  is an arbitrary number in  $(0, r)$ , this completes the proof.

#### 4. PROOFS OF THE COROLLARIES

Note first that  $(0, \dots, 0, 1)$  lies in the spherical cap  $C(d)$  for each  $d \in (0, 1)$ . Hence if the hypotheses of Corollary 1 hold, then so do the hypotheses of Theorem 1, and therefore  $h$  has a harmonic continuation  $\bar{h}$  to  $B(r)$  such that  $\bar{h} = 0$  on  $\tau(r)$ . The function  $\bar{h}$  is given on  $B(r)$  by a series  $\sum_{j=0}^{\infty} H_j$ , where  $H_j$  is a homogeneous harmonic polynomial of degree  $j$  on  $R^N$ . We show by induction that  $H_j \equiv 0$  for each  $j$ . First we have  $H_0 \equiv H_0(0) = \bar{h}(0) = 0$ . Now suppose that  $H_j \equiv 0$  for each  $j = 0, \dots, k$ . If  $z \in C(d_{k+1})$  and  $0 < t < r$ , then

$$t^{-k-1}h(tz) = H_{k+1}(z) + O(t),$$

and hypothesis (6) implies that  $H_{k+1}(z) = 0$ . Hence, by homogeneity,  $H_{k+1} = 0$  on the truncated cone  $\{tz : z \in C(d_{k+1}), 0 < t < r\}$ , and therefore  $H_{k+1} \equiv 0$ . This completes the induction.

We note in passing that the hypotheses of Corollary 1 can be relaxed a little: instead of assuming (6) for each  $z \in C(d_m)$ , we need only suppose that (6) holds for each  $z \in S_m$ , where  $S_m$  is a subset of the hemisphere  $C(0)$  such that  $(0, \dots, 0, 1) \in S_m$  and the closure of  $S_m$  has non-empty interior in the topology of  $C(0)$ .

To prove Corollary 2, note first that (7) implies (2), so that by Theorem 1 a function  $h$  satisfying the hypotheses of Corollary 2 has a harmonic continuation  $\bar{h}$  to  $B(r)$  satisfying (5). The function  $\partial\bar{h}/\partial x_N$  is also harmonic on  $B(r)$  and if  $y \in \tau(r)$ , then

$$\begin{aligned} \frac{\partial\bar{h}}{\partial x_N}(y) &= \lim_{t \rightarrow 0^+} h(y_1, \dots, y_{N-1}, t)/t \\ &\geq \liminf_{x \rightarrow y, x \in \beta(r)} h(x)/x_N \geq 0. \end{aligned}$$

It also follows from Theorem 1 that  $h = 0$  on  $\alpha(r)$  and hence  $\partial h/\partial x_N = 0$  on  $\alpha(r)$ . We have now shown that the hypotheses of Theorem 1 are satisfied with  $\partial h/\partial x_N$  in place of  $h$ . Hence  $\partial h/\partial x_N$  has a harmonic continuation  $H$  to  $B(r)$  satisfying

$$(16) \quad H(x_1, \dots, x_{N-1}, -x_N) = -\frac{\partial h}{\partial x_N}(x_1, \dots, x_N) \quad (x \in \beta(r)).$$

The harmonic functions  $H$  and  $\partial\bar{h}/\partial x_N$  are both equal to  $\partial h/\partial x_N$  on  $\beta(r)$ , and hence  $H = \partial\bar{h}/\partial x_N$  on  $B(r)$ . Therefore, differentiating (5), we obtain

$$(17) \quad H(x_1, \dots, x_{N-1}, -x_N) = \frac{\partial h}{\partial x_N}(x_1, \dots, x_N) \quad (x \in \beta(r)).$$

Equations (16) and (17) imply that  $H \equiv 0$ , so that  $\bar{h}(x)$  is independent of  $x_N$ . Since  $\bar{h} = 0$  on  $\tau(r)$ , it follows that  $h(=\bar{h}) = 0$  on  $\beta(r)$ .

#### 5. OTHER DOMAINS

We briefly indicate how results analogous to Theorem 1 and its corollaries can be proved with a spherical cap in place of the  $(N-1)$ -dimensional ball  $\tau(r)$ . In this section, the open ball of centre  $x$  and radius  $r$  in  $R^N$  is denoted by  $B(x, r)$ , and

its boundary is denoted by  $S(x, r)$ . Let  $\rho, s$  be numbers such that  $0 < s < \rho$ , let  $x_0 = (0, \dots, 0, -\rho)$ , and let  $\Omega = B(x_0, \rho) \cap B(0, s)$ . Suppose that  $h$  is harmonic on  $\Omega$  and let  $h^*$  be the image of  $h$  under the Kelvin transform relative to  $S(2x_0, 2\rho)$ , that is,

$$h^*(x) = \left( \frac{2\rho}{\|x - 2x_0\|} \right)^{N-2} h(x^*),$$

where

$$x^* = \frac{4\rho^2(x - 2x_0)}{\|x - 2x_0\|^2} + 2x_0.$$

Then  $h^*$  is harmonic on the domain  $\Omega^* = \{x^* : x \in \Omega\}$  (see, e.g., [4, p. 36]). It is easy to check that  $\beta(r) \subseteq \Omega^* \subseteq \beta(r')$  for some  $r, r' > 0$ . Suppose now that

$$\liminf_{x \rightarrow y, x \in \Omega} h(x) \geq 0 \quad (y \in S(x_0, \rho) \cap B(0, s))$$

and

$$\liminf_{t \rightarrow 0^-} |t|^{-m} |h(0, \dots, 0, t)| = 0 \quad (m = 1, 2, \dots).$$

Then  $h^*$  satisfies the hypotheses of Theorem 1 and hence  $h^* = 0$  on  $\alpha(r)$  and  $h^*$  has limit 0 at each point of  $\tau(r)$ . These conclusions imply that  $h = 0$  on the line segment  $\{(0, \dots, 0, t) : -s < t < 0\}$  and that  $h$  has limit 0 at each point of the spherical cap  $\{x^* : x \in \tau(r)\} \subset S(x_0, \rho)$ .

Finally we give an example to show that Theorems BR and 1 may fail in a smooth domain.

**Example.** Let  $\Omega = \{x : x_N > x_1^3, \|x\| < 1\}$ . There exists a continuous, real-valued function  $h$  on  $\bar{\Omega}$ , harmonic on  $\Omega$ , such that

- (i)  $h \geq 0$  on  $\{x \in \partial\Omega : \|x\| < 1\}$ ,
- (ii)  $\lim_{t \rightarrow 0^+} t^{-m} h(0, \dots, 0, t) = 0 \quad (m = 1, 2, \dots)$ , but
- (iii)  $h > 0$  on  $\alpha(1) \cup \{x \in \partial\Omega : \|x\| < 1, x_N \neq 0\}$ ,
- (iv)  $h$  has no harmonic continuation to any neighbourhood of 0.

It is enough to work in the plane, for if we produce an example  $h$  with  $N = 2$ , then the function  $(x_1, \dots, x_N) \mapsto h(x_1, x_N)$  will be a corresponding example in  $R^N$ .

We identify  $R^2$  with the complex plane  $\mathbb{C}$  in the usual way and first define a function  $h_1$  on the cut plane  $\mathbb{C} \setminus \{ix_2 : x_2 \leq 0\}$  by

$$h_1(re^{i\theta}) = \operatorname{Re}(\exp(r^{-2/3} e^{-2(\theta+\pi)i/3})) \quad (r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}).$$

Then  $h_1$ , being the real part of a regular function, is harmonic on its domain of definition. Also,

$$\begin{aligned} |h_1(re^{i\theta})| &\leq \exp(r^{-2/3} \cos \frac{2(\theta + \pi)}{3}) \\ &< \exp(-ar^{-2/3}) \quad (r > 0, -\frac{\pi}{6} < \theta < \frac{7\pi}{6}), \end{aligned}$$

where  $a = -\cos \frac{5\pi}{9} > 0$ . This implies that

- (a) if we define  $h_1(0) = 0$ , then the restriction of  $h_1$  to  $\bar{\Omega}$  is continuous on  $\bar{\Omega}$ ,
- (b) there exists a positive constant  $A$  such that

$$|h_1(z)| \leq A|z|^4 \quad (z \in \bar{\Omega}),$$

(c)  $\lim_{t \rightarrow 0^+} t^{-m} h_1(it) = 0 \quad (m = 1, 2, \dots)$ .

Also,

(d)  $h_1(it) = \exp(-t^{-2/3}) > 0 \quad (t > 0)$ .

Now define  $h_2$  on  $\mathbb{C}$  by  $h_2(x_1 + ix_2) = x_1 x_2$ . Then  $h_2$  is harmonic on  $\mathbb{C}$  and if  $z = x_1 + ix_2 \in \{z \in \partial\Omega : |z| < 1\}$ , then

$$h_2(z) = x_1^4 \geq \frac{1}{4}(x_1^2 + x_1^6)^2 = \frac{1}{4}(x_1^2 + x_2^2)^2 = \frac{1}{4}|z|^4.$$

Finally define  $h = h_1 + 5Ah_2$  on  $\bar{\Omega}$ . Then  $h$  is continuous on  $\bar{\Omega}$  and harmonic on  $\Omega$ , and using the above properties of  $h_1$  and  $h_2$  we find that  $h \geq 0$  on  $\{z \in \partial\Omega : |z| < 1\} \cup \{it : 0 < t < 1\}$  with equality only at 0 and

$$\lim_{t \rightarrow 0^+} t^{-m} h(it) = 0 \quad (m = 1, 2, \dots).$$

Also it is clear that  $h_1$  has no harmonic continuation to any neighbourhood of 0, and therefore  $h$  has no such continuation.

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