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REFLECTION AND UNIQUENESS THEOREMS FOR HARMONIC FUNCTIONS

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ABSTRACT. Suppose that h is harmonic on an open half-ball β in R^N such that the origin 0 is the centre of the flat part τ of the boundary $\partial\beta$. If h has non-negative lower limit at each point of τ and h tends to 0 sufficiently rapidly on the normal to τ at 0, then h has a harmonic continuation by reflection across τ. Under somewhat stronger hypotheses, the conclusion is that $h \equiv 0$. These results strengthen recent theorems of Baouendi and Rothschild. While the flat boundary set τ can be replaced by a spherical surface, it cannot in general be replaced by a smooth $(N-1)$-dimensional manifold.

1. INTRODUCTION

Let $x = (x_1, \dots, x_N)$ denote a typical point of R^N, where $N \geq 2$, and let $||\cdot||$ be the Euclidean norm on R^N. For each positive number r let

$$\beta(r) = \{x : ||x|| < r, x_N > 0\},$$
$$\tau(r) = \{x : ||x|| < r, x_N = 0\},$$
$$\alpha(r) = \{(0, \dots, 0, x_N) \in R^N : 0 < x_N < r\}.$$

A modified form of a recent theorem of Baouendi and Rothschild [1, Theorem 3] may be stated as follows. (The theorem was originally proved with a relatively open subset of the unit sphere in place of $\tau(r)$, but see [1, §0, final paragraph].)

Theorem BR. *Let h be a continuous real-valued function on $\overline{\beta(r)}$, harmonic on $\beta(r)$. If $h \geq 0$ on $\tau(r)$ and*

$$\lim_{t \to 0+} t^{-m} h(0, \dots, 0, t) = 0 \tag{1}$$

for each positive integer m, then $h = 0$ on $\alpha(r) \cup \tau(\rho)$ for some $\rho \in (0, r]$.

The main result of this note is the following strengthened version of the above theorem.

Theorem 1. *Let h be harmonic on $\beta(r)$. If*

$$\liminf_{x \to y, x \in \beta(r)} h(x) \geq 0 \tag{2}$$

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for each $y \in \tau(r)$ *and*

$$\liminf_{t \to 0+} t^{-m} |h(0, \ldots, 0, t)| = 0 \tag{3}$$

for each positive integer m, *then* $h = 0$ *on* $\alpha(r)$ *and*

$$\lim_{x \to y, x \in \beta(r)} h(x) = 0 \tag{4}$$

for each $y \in \tau(r)$.

In Theorem 1 the hypotheses of Theorem BR that h is continuous on $\overline{\beta(r)}$ and non-negative on $\tau(r)$ are replaced by the milder hypothesis (2). Also, (3) is a relaxation of (1).

Recall that if (4) holds for each $y \in \tau(r)$, then h has a unique harmonic continuation $\overline{h}$ to the ball

$$B(r) = \{x : ||x|| < r\}$$

and $\overline{h}$ is obtained by reflection, that is,

$$\overline{h}(x_1, \ldots, x_{N-1}, -x_N) = -h(x_1, \ldots, x_N) \quad (x \in \beta(r)). \tag{5}$$

Thus Theorems BR and 1 may be regarded as reflection principles. Under more stringent hypotheses, these theorems become uniqueness results. For each number $d \in [0, 1)$ let $C(d)$ denote the spherical cap $\{x : ||x|| = 1, x_N > d\}$.

Corollary 1. *Let* h *be harmonic on* $\beta(r)$, *and suppose that (2) holds for each* $y \in \tau(r)$. *If there exists a sequence* (d_m) *in* [0,1) *such that*

$$\liminf_{t \to 0+} t^{-m} |h(tz)| = 0 \qquad (z \in C(d_m)) \tag{6}$$

for each positive integer m, *then* $h \equiv 0$.

Corollary 2. *Let* h *be harmonic on* $\beta(r)$. *If*

$$\liminf_{x \to y, x \in \beta(r)} h(x)/x_N \geq 0 \tag{7}$$

for each $y \in \tau(r)$ *and (3) holds for each positive integer* m, *then* $h \equiv 0$.

Corollary 1 is an improvement of the half-space version of [1, Corollary 2.6], but Corollary 2 seems to have no counterpart in [1].

In §5 we give an example to show that Theorem BR (and *a fortiori* Theorem 1) may fail if $\beta(r)$ is replaced by a domain whose boundary is analytic near 0 and tangential at 0 to $\tau(1)$. This partially answers the conjecture [1, p. 245] that results similar to those of [1] hold for more general domains.

Baouendi and Rothschild [2] have recently generalized the main results of [1] in a direction different from that considered here.

2. A representation lemma

We shall need the following lemma, which may be regarded as a local version of the Poisson integral representation of a positive harmonic function on a half-space (see, e.g., Helms [4, Theorem 2.25]).

Lemma. *Suppose that h is harmonic on $\beta(r)$ and satisfies (2) at each $y \in \tau(r)$. Then there exists a measure μ on $\tau(r)$ with the following property: for each $\rho \in (0, r)$, there exists a harmonic function H on $B(\rho)$ such that*

$$H(x_1, \dots, x_N) = -H(x_1, \dots, x_{N-1}, -x_N) \qquad (x \in B(\rho)) \tag{8}$$

and

$$h(x) = x_N \int_{\tau(\rho)} ||x - y||^{-N} d\mu(y) + H(x) \qquad (x \in \beta(\rho)). \tag{9}$$

This lemma is known, at least tacitly, but lacking on exact reference, I indicate a proof using a technique which involves passing to a space of higher dimension. This technique is due to Huber [5] and was rediscovered and more extensively exploited by Kuran [6]. The results that we shall need are half-ball versions of [6, Lemmas 1, 3, 6, Theorem 1], which were proved in [6] for a half-space rather than a half-ball, but which are indeed valid in the generality we require (see [6, p. 279]).

We verify first that if (2) holds for each $y \in \tau(r)$, then

$$\liminf_{x \to y, x \in \beta(r)} x_N^{-1} h(x) > -\infty \qquad (y \in \tau(r)). \tag{10}$$

The function u, defined on $\beta(r)$ by $u(x) = \min\{h(x), x_N\}$, is superharmonic on $\beta(r)$ and tends to 0 at each point of $\tau(r)$. It follows from [6, Lemma 3] that (10) holds with u in place of h, and since $h \geq u$ on $\beta(r)$, (10) itself is true.

A typical point of R^{N+2} is denoted by $\xi = (\xi_1, \dots, \xi_{N+2})$, and with such a point we associate the number

$$\delta_\xi = \sqrt{(\xi_N^2 + \xi_{N+1}^2 + \xi_{N+2}^2)}.$$

Let $\mathcal{B}(r)$ denote the open ball of radius r centred at the origin of R^{N+2} and let $E = \{\xi : \delta_\xi = 0\}$. Define h^* on $\mathcal{B}(r) \backslash E$ by

$$h^*(\xi) = \delta_\xi^{-1} h(\xi_1, \dots, \xi_{N-1}, \delta_\xi).$$

By [6, Lemma 1], h^* is harmonic on $\mathcal{B}(r) \backslash E$, and since (10) holds, it follows from [6, Theorem 1] that h^* has a superharmonic extension U to $\mathcal{B}(r)$. The support of the Riesz measure μ^* associated to U is contained in E. Hence, by the local form of the Riesz decomposition theorem, if $0 < \rho < r$, then there exists a harmonic function H^* on $\mathcal{B}(\rho)$ such that

$$U(\xi) = \int_{E \cap \mathcal{B}(\rho)} ||\xi - \eta||^{-N} d\mu^*(\eta) + H^*(\xi) \qquad (x \in \mathcal{B}(\rho)).$$

If $x \in \beta(\rho)$ and ξ_x is defined to be the point $(x_1, \dots, x_N, 0, 0)$, then $\xi_x \in \mathcal{B}(\rho) \backslash E$ and $\delta_{\xi_x} = x_N$, so that

$$\begin{aligned} h(x) &= x_N h^*(\xi_x) \\ &= x_N U(\xi_x) \\ &= x_N \int_{E \cap \mathcal{B}(\rho)} ||\xi_x - \eta||^{-N} d\mu^*(\eta) + x_N H^*(\xi_x) \\ &= x_N \int_{\tau(\rho)} ||x - y||^{-N} d\mu(y) + \tilde{H}(x), \end{aligned}$$

where μ is the measure defined on the Borel subsets of $\tau(r)$ by

$$\mu(F) = \mu^*(\{\xi \in E : (\xi_1, \dots, \xi_{N-1}, 0) \in F\})$$

and $\tilde{H}$ is defined on $\beta(\rho)$ by $\tilde{H}(x) = x_N H^*(\xi_x)$. By [6, Lemma 6], $\tilde{H}$ is harmonic on $\beta(\rho)$, and since H^* is locally bounded on $\mathcal{B}(\rho)$ it follows that $\tilde{H}$ tends to 0 at each point of $\tau(\rho)$. Therefore, by the reflection principle, $\tilde{H}$ has a harmonic continuation H to $B(\rho)$ satisfying (8).

3. Proof of Theorem 1

Suppose that h satisfies the hypotheses of Theorem 1. It is enough to fix $\rho \in (0, r)$ and to show that $h = 0$ on $\alpha(\rho)$ and (4) holds for each $y \in \tau(\rho)$.

According to the lemma, h has the representation (9) on $\beta(\rho)$. Since the function H in (9) is harmonic on $B(\rho)$, there exists a series $\sum_{j=0}^{\infty} H_j$, where H_j is a homogeneous harmonic polynomial of degree j on R^N, which converges to H on $B(\rho)$ (see, e.g., Brelot [3, Appendix]). In particular, the function $t \mapsto H(0, \dots, 0, t)$ is given on the interval $(-\rho, \rho)$ by its Taylor series about 0. Also, since by (8) this function is odd, the Taylor series contains only odd powers of t. Thus we have a representation of the form

$$H(0, \dots, 0, t) = \sum_{j=0}^{\infty} a_{2j+1} t^{2j+1} \qquad (-\rho < t < \rho). \tag{11}$$

We next aim to prove inductively that

$$a_{2j+1} = (-1)^{j+1} \binom{j + \frac{N}{2} - 1}{j} \int_{\tau(\rho)} ||y||^{-N-2j} d\mu(y) \tag{12}$$

for each non-negative integer j. An argument by contradiction will then show that $\mu \equiv 0$ on $\tau(\rho)$, and the conclusions of Theorem 1 will then follow easily.

Throughout this paragraph x denotes a point of $\alpha(\rho)$ with coordinates $(0,\dots,0,t)$. By (9) and (11),

$$\begin{aligned} t^{-1}h(x) &= \int_{\tau(\rho)} (t^2 + ||y||^2)^{-N/2} d\mu(y) + a_1 + O(t^2) \\ &\to \int_{\tau(\rho)} ||y||^{-N} d\mu(y) + a_1 \qquad (t \to 0+), \end{aligned}$$

by monotone convergence. Hypothesis (3) with $m = 1$ now implies that (12) holds with $j = 0$. To proceed with the inductive proof of (12), we introduce the function ϕ, defined by

$$\phi(\theta) = (1 + \theta)^{-N/2} \qquad (\theta > -1),$$

and note that

$$\frac{\phi^{(j)}(0)}{j!} = (-1)^j \binom{j + \frac{N}{2} - 1}{j} \qquad (j = 0, 1, 2, \dots) \tag{13}$$

and that by Taylor's theorem,

(14)
$$\phi(\theta) - \sum_{j=0}^{k} \frac{\phi^{(j)}(0)}{j!} \theta^j = \frac{\theta^{k+1}}{k!} \int_0^1 (1 - \zeta)^k \phi^{(k+1)}(\theta\zeta) d\zeta \quad (\theta > -1, k = 0, 1, 2, \dots).$$

It is easy to see that $(-1)^{k+1}\phi^{(k+1)}$ is positive and decreasing on $(-1,+\infty)$. Hence it follows from (14) that if Φ_k is defined by

$$\Phi_k(\theta) = \frac{(-1)^{k+1}}{\theta^{k+1}} \left(\phi(\theta) - \sum_{j=0}^{k} \frac{\phi^{(j)}(0)}{j!}\theta^j \right),$$

then Φ_k is positive and decreasing on $(0,+\infty)$ and

$$\lim_{\theta\to 0+} \Phi_k(\theta) = (-1)^{k+1}\frac{\phi^{(k+1)}(0)}{(k+1)!}. \tag{15}$$

Suppose now that (12) holds for $j=0,\dots,k$. Then by (9) and (11),

$$\begin{aligned} h(x) = t &\int_{\tau(\rho)} (t^2+||y||^2)^{-N/2} d\mu(y) \\ &- \sum_{j=0}^{k} (-1)^j \binom{j+\frac{N}{2}-1}{j} t^{2j+1} \int_{\tau(\rho)} ||y||^{-N-2j} d\mu(y) \\ &+ \sum_{j=k+1}^{\infty} a_{2j+1} t^{2j+1}. \end{aligned}$$

Using (13), we can write this equation in the form

$$\begin{aligned} \frac{h(x)}{t^{2k+3}} &= \frac{1}{t^{2k+2}} \left\{ \int_{\tau(\rho)} ||y||^{-N} \left(\phi\left(\frac{t^2}{||y||^2}\right) - \sum_{j=0}^{k} \frac{\phi^{(j)}(0)}{j!} \left(\frac{t^2}{||y||^2}\right)^j \right) d\mu(y) \right\} \\ &\quad + a_{2k+3} + O(t^2) \\ &= (-1)^{k+1} \int_{\tau(\rho)} ||y||^{-N-2k-2} \Phi_k\left(\frac{t^2}{||y||^2}\right) d\mu(y) + a_{2k+3} + O(t^2). \end{aligned}$$

Since Φ_k is positive and decreasing on $(0,+\infty)$, it now follows from (15) that

$$\lim_{t\to 0+} t^{-2k-3}h(x) = \frac{\phi^{(k+1)}(0)}{(k+1)!} \int_{\tau(\rho)} ||y||^{-N-2k-2} d\mu(y) + a_{2k+3}.$$

This, together with (13) and hypothesis (3), implies that (12) holds for $j=k+1$, and the inductive proof of (12) is complete.

Now suppose that $\mu \not\equiv 0$ on $\tau(\rho)$. Choose $\sigma \in (0,\rho)$ such that $\mu(\tau(\sigma)) > 0$. Defining

$$\lambda = \int_{\tau(\sigma)} ||y||^{-N} d\mu(y),$$

we have $\lambda > 0$ and

$$\int_{\tau(\sigma)} ||y||^{-N-2j} d\mu(y) \geq \lambda\sigma^{-2j} \qquad (j=0,1,\dots).$$

Hence, by (12),

$$|a_{2j+1}| \geq \lambda \binom{j+\frac{N}{2}-1}{j} \sigma^{-2j} \geq \lambda\sigma^{-2j},$$

so that $\sum a_{2j+1}t^{2j+1}$ diverges when $t \geq \sigma$. As this series converges to $H(0,\dots,0,t)$ when $-\rho < t < \rho$, we have arrived at a contradiction. This shows that $\mu \equiv 0$ on $\tau(\rho)$.

The representation (9) now reduces to $h = H$ on $\beta(\rho)$, and since $H = 0$ on $\tau(\rho)$, (4) holds for each $y \in \tau(\rho)$. Also, (12) now gives $a_{2j+1} = 0$ for each j, so that by (11), $h = H = 0$ on $\alpha(\rho)$. Since ρ is an arbitrary number in $(0, r)$, this completes the proof.

4. Proofs of the corollaries

Note first that $(0, \dots, 0, 1)$ lies in the spherical cap $C(d)$ for each $d \in (0, 1)$. Hence if the hypotheses of Corollary 1 hold, then so do the hypotheses of Theorem 1, and therefore h has a harmonic continuation $\overline{h}$ to $B(r)$ such that $\overline{h} = 0$ on $\tau(r)$. The function $\overline{h}$ is given on $B(r)$ by a series $\sum_{j=0}^{\infty} H_j$, where H_j is a homogeneous harmonic polynomial of degree j on R^N. We show by induction that $H_j \equiv 0$ for each j. First we have $H_0 \equiv H_0(0) = \overline{h}(0) = 0$. Now suppose that $H_j \equiv 0$ for each $j = 0, \dots, k$. If $z \in C(d_{k+1})$ and $0 < t < r$, then

$$t^{-k-1} h(tz) = H_{k+1}(z) + O(t),$$

and hypothesis (6) implies that $H_{k+1}(z) = 0$. Hence, by homogeneity, $H_{k+1} = 0$ on the truncated cone $\{tz : z \in C(d_{k+1}), 0 < t < r\}$, and therefore $H_{k+1} \equiv 0$. This completes the induction.

We note in passing that the hypotheses of Corollary 1 can be relaxed a little: instead of assuming (6) for each $z \in C(d_m)$, we need only suppose that (6) holds for each $z \in S_m$, where S_m is a subset of the hemisphere $C(0)$ such that $(0, \dots, 0, 1) \in S_m$ and the closure of S_m has non-empty interior in the topology of $C(0)$.

To prove Corollary 2, note first that (7) implies (2), so that by Theorem 1 a function h satisfying the hypotheses of Corollary 2 has a harmonic continuation $\overline{h}$ to $B(r)$ satisfying (5). The function $\partial\overline{h}/\partial x_N$ is also harmonic on $B(r)$ and if $y \in \tau(r)$, then

$$\begin{aligned}\frac{\partial \overline{h}}{\partial x_N}(y) &= \lim_{t \to 0+} h(y_1, \dots, y_{N-1}, t)/t \\ &\geq \liminf_{x \to y, x \in \beta(r)} h(x)/x_N \geq 0.\end{aligned}$$

It also follows from Theorem 1 that $h = 0$ on $\alpha(r)$ and hence $\partial h/\partial x_N = 0$ on $\alpha(r)$. We have now shown that the hypotheses of Theorem 1 are satisfied with $\partial h/\partial x_N$ in place of h. Hence $\partial h/\partial x_N$ has a harmonic continuation H to $B(r)$ satisfying

$$H(x_1, \dots, x_{N-1}, -x_N) = -\frac{\partial h}{\partial x_N}(x_1, \dots, x_N) \quad (x \in \beta(r)). \tag{16}$$

The harmonic functions H and $\partial\overline{h}/\partial x_N$ are both equal to $\partial h/\partial x_N$ on $\beta(r)$, and hence $H = \partial\overline{h}/\partial x_N$ on $B(r)$. Therefore, differentiating (5), we obtain

$$H(x_1, \dots, x_{N-1}, -x_N) = \frac{\partial h}{\partial x_N}(x_1, \dots, x_N) \quad (x \in \beta(r)). \tag{17}$$

Equations (16) and (17) imply that $H \equiv 0$, so that $\overline{h}(x)$ is independent of x_N. Since $\overline{h} = 0$ on $\tau(r)$, it follows that $h (= \overline{h}) = 0$ on $\beta(r)$.

5. Other domains

We briefly indicate how results analogous to Theorem 1 and its corollaries can be proved with a spherical cap in place of the $(N-1)$-dimensional ball $\tau(r)$. In this section, the open ball of centre x and radius r in R^N is denoted by $B(x, r)$, and its boundary is denoted by $S(x,r)$. Let ρ, s be numbers such that $0 < s < \rho$, let $x_0 = (0,\ldots,0,-\rho)$, and let $\Omega = B(x_0,\rho) \cap B(0,s)$. Suppose that h is harmonic on Ω and let h^* be the image of h under the Kelvin transform relative to $S(2x_0, 2\rho)$, that is,

$$h^*(x) = \left(\frac{2\rho}{||x-2x_0||}\right)^{N-2} h(x^*),$$

where

$$x^* = \frac{4\rho^2(x-2x_0)}{||x-2x_0||^2} + 2x_0.$$

Then h^* is harmonic on the domain $\Omega^* = \{x^* : x \in \Omega\}$ (see, e.g., [4, p. 36]). It is easy to check that $\beta(r) \subseteq \Omega^* \subseteq \beta(r')$ for some $r, r' > 0$. Suppose now that

$$\liminf_{x \to y, x \in \Omega} h(x) \geq 0 \qquad (y \in S(x_0,\rho) \cap B(0,s))$$

and

$$\liminf_{t \to 0-} |t|^{-m} |h(0,\ldots,0,t)| = 0 \qquad (m = 1,2,\ldots).$$

Then h^* satisfies the hypotheses of Theorem 1 and hence $h^* = 0$ on $\alpha(r)$ and h^* has limit 0 at each point of $\tau(r)$. These conclusions imply that $h = 0$ on the line segment $\{(0,\ldots,0,t) : -s < t < 0\}$ and that h has limit 0 at each point of the spherical cap $\{x^* : x \in \tau(r)\} \subset S(x_0,\rho)$.

Finally we give an example to show that Theorems BR and 1 may fail in a smooth domain.

Example. *Let $\Omega = \{x : x_N > x_1^3, ||x|| < 1\}$. There exists a continuous, real-valued function h on $\overline{\Omega}$, harmonic on Ω, such that*

(i) *$h \geq 0$ on $\{x \in \partial\Omega : ||x|| < 1\}$,*

(ii) *$\lim_{t \to 0+} t^{-m} h(0,\ldots,0,t) = 0 \quad (m = 1,2,\ldots)$, but*

(iii) *$h > 0$ on $\alpha(1) \cup \{x \in \partial\Omega : ||x|| < 1, x_N \neq 0\}$,*

(iv) *h has no harmonic continuation to any neighbourhood of* 0.

It is enough to work in the plane, for if we produce an example h with $N = 2$, then the function $(x_1,\ldots,x_N) \mapsto h(x_1,x_N)$ will be a corresponding example in R^N.

We identify R^2 with the complex plane $\mathbb{C}$ in the usual way and first define a function h_1 on the cut plane $\mathbb{C}\backslash\{ix_2 : x_2 \leq 0\}$ by

$$h_1(re^{i\theta}) = \mathrm{Re}(\exp(r^{-2/3}e^{-2(\theta+\pi)i/3})) \quad (r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}).$$

Then h_1, being the real part of a regular function, is harmonic on its domain of definition. Also,

$$\begin{aligned} |h_1(re^{i\theta})| &\leq \exp(r^{-2/3}\cos\frac{2(\theta+\pi)}{3}) \\ &< \exp(-ar^{-2/3}) \qquad (r > 0, -\frac{\pi}{6} < \theta < \frac{7\pi}{6}), \end{aligned}$$

where $a = -\cos\frac{5\pi}{9} > 0$. This implies that

(a) if we define $h_1(0) = 0$, then the restriction of h_1 to $\overline{\Omega}$ is continuous on $\overline{\Omega}$,

(b) there exists a positive constant A such that

$$|h_1(z)| \leq A|z|^4 \qquad (z \in \overline{\Omega}),$$

(c) $\lim_{t\to 0+} t^{-m} h_1(it) = 0 \qquad (m = 1, 2, \dots)$.
Also,
(d) $h_1(it) = \exp(-t^{-2/3}) > 0 \qquad (t > 0)$.

Now define h_2 on $\mathbb{C}$ by $h_2(x_1 + ix_2) = x_1x_2$. Then h_2 is harmonic on $\mathbb{C}$ and if $z = x_1 + ix_2 \in \{z \in \partial\Omega : |z| < 1\}$, then

$$h_2(z) = x_1^4 \geq \frac{1}{4}(x_1^2 + x_1^6)^2 = \frac{1}{4}(x_1^2 + x_2^2)^2 = \frac{1}{4}|z|^4.$$

Finally define $h = h_1 + 5Ah_2$ on $\overline{\Omega}$. Then h is continuous on $\overline{\Omega}$ and harmonic on Ω, and using the above properties of h_1 and h_2 we find that $h \geq 0$ on $\{z \in \partial\Omega : |z| < 1\} \cup \{it : 0 < t < 1\}$ with equality only at 0 and

$$\lim_{t\to 0+} t^{-m} h(it) = 0 \qquad (m = 1, 2, \dots).$$

Also it is clear that h_1 has no harmonic continuation to any neighbourhood of 0, and therefore h has no such continuation.

References

[1] M. S. Baouendi, L. P. Rothschild, *A local Hopf lemma and unique continuation for harmonic functions*, Duke J. Math., Inter. Research Notices, 71 (1993), 245-251. MR **94i:**31008

[2] M. S. Baouendi, L. P. Rothschild, *Harmonic functions satisfying weighted sign conditions on the boundary*, Ann. Inst. Fourier, Grenoble, 43 (1993), 1311-1318. MR **95c:**35067

[3] M. Brelot, *Éléments de la théorie classique du potentiel*, Centre de documentation universitaire, Paris, 1965. MR **31:**2412

[4] L. L. Helms, *Introduction to potential theory*, Wiley, New York, 1969. MR **41:**5638

[5] A. Huber, *On functions subharmonic in a half-space*, Trans. Amer. Math. Soc., 82 (1956), 147-159. MR **17:**1197b

[6] Ü. Kuran, *Study of superharmonic functions in $R^n \times (0, +\infty)$ by a passage to R^{n+3}*, Proc. London Math. Soc. (3), 20 (1970), 276-392. MR **41:**5643

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