

AN EXISTENCE THEOREM FOR SYSTEMS OF BOUNDARY VALUE PROBLEMS

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ABSTRACT. We prove an existence theorem for $u''(t) = f(t, u(t), u'(t))$, $u(0) = u_0$, $u(1) = u_1$ in \mathbb{R}^n , using the shooting method. The function f is supposed to be asymptotically linear.

1. INTRODUCTION

Let $\|\cdot\|$ denote the Euclid norm on \mathbb{R}^n . By $\|\|\cdot\|\|$ we denote the corresponding maximum norm on $C([0, 1], \mathbb{R}^n)$. The set of all real $n \times n$ matrices is denoted by $M^{n \times n}(\mathbb{R})$, and let $\|A\|$, $A \in M^{n \times n}(\mathbb{R})$ be the matrix norm corresponding to the Euclid norm.

Let $f : [0, 1] \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ be a continuous function. We consider the Dirichlet boundary value problem

$$(1) \quad u''(t) = f(t, u(t), u'(t)), \quad u(0) = u_0, \quad u(1) = u_1.$$

If f is such that the initial value problem

$$(2) \quad v''(t) = f(t, v(t), v'(t)), \quad v(0) = u_0, \quad v'(0) = s,$$

is uniquely solvable on $[0, 1]$ for each $s \in \mathbb{R}^n$ with solution $v(t, s)$, the function $q : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $q(s) = v(1, s)$ is called the *shooting function* of $u'' = f(t, u, u')$ (with respect to u_0), and obviously the solution of (2) is a solution of (1) if and only if $q(s) = u_1$.

We will prove the following theorem:

Theorem 1. *Let $f : [0, 1] \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ be continuous and let $A, B : [0, 1] \rightarrow M^{n \times n}(\mathbb{R})$ be continuous matrix functions such that*

1. $\lim_{\|x\| + \|p\| \rightarrow \infty} \frac{\|\|f(\cdot, x, p) - A(\cdot)x - B(\cdot)p\|\|}{\|x\| + \|p\|} = 0$;
2. *The boundary value problem $w''(t) = A(t)w(t) + B(t)w'(t)$, $w(0) = w(1) = 0$ has only the trivial solution.*

Then problem (1) is solvable for each $u_0, u_1 \in \mathbb{R}^n$.

Note that 1. is more general than it might look at first glance; e.g., if f satisfies 1. and g is bounded, then $f + g$ satisfies 1. as well.

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Theorem 1 is well known for the case $n = 1$ and is due to Perow; see [6]. The proof for this case uses typical methods of one and two dimensional analysis, for example, it involves polar coordinates, and cannot be transferred in an obvious way to systems of boundary value problems. To prove Theorem 1 we will approximate problem (1) by boundary value problems with surjective shooting function in a uniform way to obtain a solution of problem (1) itself. For further investigations of existence results with the shooting method see [4], [5].

Proof. To prove Theorem 1 we will use the following propositions.

Proposition 1. *Let $A, B : [0, 1] \rightarrow M^{n \times n}(\mathbb{R})$ be continuous matrix functions such that the boundary value problem $w''(t) = A(t)w(t) + B(t)w'(t)$, $w(0) = w(1) = 0$, is uniquely solvable (by $w = 0$). Then:*

1.) *The boundary value problem $w''(t) = A(t)w(t) + B(t)w'(t) + h(t)$, $w(0) = w(1) = 0$ is uniquely solvable for each $h \in C([0, 1], \mathbb{R}^n)$, and there is a constant $\alpha \geq 0$ such that*

$$\max\{\|w(t)\|, \|w'(t)\|\} \leq \alpha \|h\|, \quad t \in [0, 1].$$

2.) *The boundary value problem $w''(t) = A(t)w(t) + B(t)w'(t)$, $w(0) = w_0$, $w(1) = w_1$ is uniquely solvable for each $w_0, w_1 \in \mathbb{R}^n$ and the solution w depends continuously on w_0, w_1 .*

For a proof of Proposition 1 see for example [3], p. 418ff.

Proposition 2. *Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuous, $\beta \in (0, 1)$, $\gamma \geq 0$ and*

$$\|x - F(x)\| \leq \beta \|x\| + \gamma, \quad x \in \mathbb{R}^n.$$

Then F is surjective.

Proposition 2 is a consequence of [2], p. 19, Theorem 3.3.

Proof of Theorem 1. Fix $u_0 \in \mathbb{R}^n$.

For each $k \in \mathbb{N}$ there exists a locally Lipschitz continuous function $f_k : [0, 1] \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ such that $\|f_k(t, x, p) - f(t, x, p)\| \leq 1/k$, $(t, x, p) \in [0, 1] \times \mathbb{R}^{2n}$; see, for example, [1], p. 5. For each $k \in \mathbb{N}$ and $\|x\| + \|p\| > 0$ we have

$$\frac{\|f_k(\cdot, x, p) - A(\cdot)x - B(\cdot)p\|}{\|x\| + \|p\|} \leq \frac{1 + \|f(\cdot, x, p) - A(\cdot)x - B(\cdot)p\|}{\|x\| + \|p\|}.$$

Let $0 < \varepsilon \leq 1$. We can choose $r > 0$ such that for each $k \in \mathbb{N}$,

$$\|f_k(t, x, p) - A(t)x - B(t)p\| \leq \varepsilon(\|x\| + \|p\|), \quad t \in [0, 1], \quad \|x\| + \|p\| \geq r.$$

We define $g_k, h_k : [0, 1] \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ by

$$g_k(t, x, p) = f_k(t, x, p), \quad \|x\| + \|p\| > r; \quad g_k(t, 0, 0) = 0;$$

$$g_k(t, x, p) = \frac{\|x\| + \|p\|}{r} f_k\left(t, \frac{rx}{\|x\| + \|p\|}, \frac{rp}{\|x\| + \|p\|}\right), \quad 0 < \|x\| + \|p\| \leq r,$$

and $h_k = f_k - g_k$. For each $k \in \mathbb{N}$ the functions g_k, h_k are continuous,

$$\|g_k(t, x, p) - A(t)x - B(t)p\| \leq \varepsilon(\|x\| + \|p\|), \quad (t, x, p) \in [0, 1] \times \mathbb{R}^{2n},$$

and

$$\|h_k(t, x, p)\| \leq 2\left(\max_{\|\xi\| + \|\eta\| \leq r} \|f(\cdot, \xi, \eta)\| + 1\right) =: K(r), \quad (t, x, p) \in [0, 1] \times \mathbb{R}^{2n}.$$

We will use the decompositions $f_k = g_k + h_k$ twice. First set $\varepsilon = 1$ with corresponding $r = r_1$. Consider

$$(3) \quad v_k''(t) = f_k(t, v_k(t), v_k'(t)), \quad v_k(0) = u_0, \quad v_k'(0) = s.$$

From the properties of g_k and h_k we get for $(t, x, p) \in [0, 1] \times \mathbb{R}^{2n}$

$$\|f_k(t, x, p)\| \leq (1 + \max_{t \in [0,1]} \|A(t)\| + \max_{t \in [0,1]} \|B(t)\|)(\|x\| + \|p\|) + K(r_1).$$

Therefore (3) is uniquely solvable on $[0, 1]$ (hence we have a continuous shooting function $q_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $q_k(s) = v_k(1, s)$ for each $k \in \mathbb{N}$), and there are constants $K_1, K_2 > 0$ independent of k such that

$$(4) \quad \|v_k(t, s)\| + \|v_k'(t, s)\| \leq K_1\|s\| + K_2, \quad t \in [0, 1], \quad s \in \mathbb{R}^n.$$

Now we choose $\varepsilon \in (0, 1)$ (with corresponding g_k, h_k, r) such that $\beta := \alpha\varepsilon K_1 < 1$, and we set $\gamma := \alpha\varepsilon K_2 + \alpha K(r)$.

For each $k \in \mathbb{N}$ consider the solution $w_k(t, s)$ of the boundary value problem

$$w_k''(t) = A(t)w_k(t) + B(t)w_k'(t), \quad w_k(0) = u_0, \quad w_k(1) = q_k(s).$$

We have

$$\begin{aligned} v_k''(t, s) - w_k''(t, s) &= A(t)(v_k(t, s) - w_k(t, s)) + B(t)(v_k'(t, s) - w_k'(t, s)) \\ &+ g_k(t, v_k(t, s), v_k'(t, s)) - A(t)v_k(t, s) - B(t)v_k'(t, s) + h_k(t, v_k(t, s), v_k'(t, s)), \end{aligned}$$

and

$$v_k(0, s) - w_k(0, s) = v_k(1, s) - w_k(1, s) = 0.$$

According to Proposition 1

$$\begin{aligned} &\|v_k'(t, s) - w_k'(t, s)\| \\ &\leq \alpha\|g_k(\cdot, v_k(\cdot, s), v_k'(\cdot, s)) - A(\cdot)v_k(\cdot, s) - B(\cdot)v_k'(\cdot, s) + h_k(\cdot, v_k(\cdot, s), v_k'(\cdot, s))\| \\ &\leq \alpha(\varepsilon(\|v_k(\cdot, s)\| + \|v_k'(\cdot, s)\|) + K(r)) \leq \alpha\varepsilon K_1\|s\| + \alpha\varepsilon K_2 + \alpha K(r), \quad t \in [0, 1]. \end{aligned}$$

Setting $t = 0$ we obtain

$$(5) \quad \|s - w_k'(0, s)\| \leq \beta\|s\| + \gamma, \quad s \in \mathbb{R}^n.$$

According to Proposition 1 the function $w_k'(0, s)$, $s \in \mathbb{R}^n$ is continuous and according to Proposition 2 it is surjective. Since the shooting function of $w'' = A(t)w + B(t)w'$ (with respect to u_0) is bijective we have that q_k is surjective, $k \in \mathbb{N}$.

Now let s_k , $k \in \mathbb{N}$ be such that the shooting functions q_k hit, i.e. $q_k(s_k) = u_1$, $k \in \mathbb{N}$. Then $u_k(t) = v_k(t, s)$ solves $u_k'' = f_k(t, u_k(t), u_k'(t))$, $u_k(0) = u_0$, $u_k(1) = u_1$, and from (4) and (5) we obtain

$$\|u_k(t)\| + \|u_k'(t)\| \leq K_1\|s_k\| + K_2 \leq \frac{K_1}{1 - \beta}(\|w_k'(0, s_k)\| + \gamma) + K_2.$$

Note that $w_k'(0, s_k)$ is uniquely determined by u_1 , hence independent of k . Therefore, the sequences $(u_k)_{k=1}^\infty$ and $(u_k')_{k=1}^\infty$ are bounded and equicontinuous. The existence of a solution of problem (1) follows by Ascoli-Arzelà's theorem. \square

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