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# NONCONTINUITY OF SPECTRUM FOR THE ADJOINT OF AN OPERATOR

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ABSTRACT. This paper deals with the connection between continuity of spectrum at an element $T$ of the Banach algebra of all bounded linear operators on a Banach space $X$ and at the adjoint $T^*$ of $T$. In particular, we show that, if $X$ is not reflexive, the spectrum function may be continuous at $T$ and discontinuous at $T^*$.

## 1. Introduction

We will denote by $\mathbb{K}_{\mathbb{C}}$ the set of all compact nonempty subsets of the complex plane $\mathbb{C}$, endowed with the Hausdorff metric $\Delta$. Recall that

$$\Delta(K_1, K_2) = \max\bigl\{\max\{\operatorname{dist}(\lambda, K_2) : \lambda \in K_1\}, \max\{\operatorname{dist}(\mu, K_1) : \mu \in K_2\}\bigr\}$$

for each $(K_1,\, K_2) \in \mathbb{K}_{\mathbb{C}} \times \mathbb{K}_{\mathbb{C}}$. If $\mathfrak{A}$ is a complex Banach algebra with identity, let $\sigma_{\mathfrak{A}} : \mathfrak{A} \to \mathbb{K}_{\mathbb{C}}$ and $r_{\mathfrak{A}} : \mathfrak{A} \to \mathbb{R}$ denote respectively the spectrum function on $\mathfrak{A}$ and the spectral radius function on $\mathfrak{A}$ (namely, the functions mapping respectively every $\mathfrak{a} \in \mathfrak{A}$ into its spectrum $\sigma_{\mathfrak{A}}(\mathfrak{a})$ and into its spectral radius $r_{\mathfrak{A}}(\mathfrak{a})$). Then the problem of continuity can be considered for each of $\sigma_{\mathfrak{A}}$ and $r_{\mathfrak{A}}$. Notice that

$$|r_{\mathfrak{A}}(\mathfrak{a}) - r_{\mathfrak{A}}(\mathfrak{b})| \le \Delta\bigl(\sigma_{\mathfrak{A}}(\mathfrak{a}), \sigma_{\mathfrak{A}}(\mathfrak{b})\bigr) \qquad \text{for every } \mathfrak{a}, \mathfrak{b} \in \mathfrak{A}.$$

Hence, if $\sigma_{\mathfrak{A}}$ is continuous at $\mathfrak{a} \in \mathfrak{A}$, it follows that $r_{\mathfrak{A}}$ also is continuous at $\mathfrak{a}$.

It is well known that $\sigma_{\mathfrak{A}}$ is continuous on $\mathfrak{A}$ if $\mathfrak{A}$ is commutative or finite-dimensional modulo the radical. An example—due to Kakutani—in the Banach algebra of all bounded linear operators on $\ell_2$ shows that even $r_{\mathfrak{A}}$ need not be continuous on the whole of $\mathfrak{A}$ if $\mathfrak{A}$ is neither commutative nor finite-dimensional modulo the radical (see [R], pp. 282–283).

If $X$ and $Y$ are Banach spaces over the same scalar field ($\mathbb{R}$ or $\mathbb{C}$), let $L(X,Y)$ stand for the Banach space of all bounded linear operators from $X$ into $Y$. For each $T \in L(X,Y)$, we will denote by $T^*$ the adjoint of $T$. Thus, if we use the symbol $E^*$ to denote the dual space of any Banach space $E$, we have $T^* \in L(Y^*, X^*)$. For simplicity of notation, we will write $L(X)$ instead of $L(X,X)$. Also, we will denote by $I_X$ the identity operator on $X$. We remark that, if the Banach space $X$ is nonzero, then $L(X)$ is a Banach algebra with identity $I_X$. We will denote by $K(X)$ the ideal of compact operators on $X$.

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Notice that, if $X$ is a finite-dimensional nonzero complex Banach space, then the Banach algebras $L(X)$ and $L(X^*)$ have finite dimension. Hence $\sigma_{L(X)}$ and $\sigma_{L(X^*)}$ are continuous on $L(X)$ and $L(X^*)$, respectively.

The problem of characterizing the points of continuity of $\sigma_{\mathfrak{A}}$ and of $r_{\mathfrak{A}}$ is still open in the case of a general Banach algebra $\mathfrak{A}$. In the special cases of the Banach algebra $L(H)$ and of the quotient algebra $L(H)/_{K(H)}$ for an infinite-dimensional separable complex Hilbert space $H$, this problem was solved by J. B. Conway and B. B. Morrel (see [CM1], [CM2], [CM3]; notice that the Hilbert spaces in [CM3] are understood to be separable). Conway and Morrel's conditions for continuity of spectrum and spectral radius turn out to be sufficient for continuity of the corresponding functions also in the more general cases of the Banach algebras $L(X)$ and $L(X)/_{K(X)}$, where $X$ is an infinite-dimensional complex Banach space.

We also have been interested in continuity of spectrum and spectral radius. In particular, in [B1], 2.7, and in [B3], 1.1, we provided respectively two equivalent sufficient conditions for continuity of $\sigma_{\mathfrak{A}}$ and a sufficient condition for continuity of $r_{\mathfrak{A}}$ at a point $\mathfrak{a}$ of a complex Banach algebra $\mathfrak{A}$. In the special cases $\mathfrak{A} = L(H)$ and $\mathfrak{A} = L(H)/_{K(H)}$ (where $H$ is an infinite-dimensional separable complex Hilbert space), our conditions turn out to be equivalent to Conway and Morrel's, and thus necessary, as well as sufficient, for continuity of the corresponding functions. If $H$ is replaced by a general Banach space $X$, our conditions turn out to be less restrictive than Conway and Morrel's, both in $L(X)$ and in $L(X)/_{K(X)}$. A general reference for these results is Section 3 of the survey article [B4].

Our main concern here is to investigate the relationship between continuity of $\sigma_{L(X)}$ at $T$ and continuity of $\sigma_{L(X^*)}$ at $T^*$ (where $X$ is a nonzero complex Banach space and $T \in L(X)$). The corresponding problem for the spectral radius function is also considered. Our interest in this problem was stimulated by an assertion we found in [BE]: indeed, in [BE], 3.4, the following is claimed to be a consequence of continuity of the function mapping each operator into its adjoint (actually, [BE] is in the context of unbounded operators; the assertion we quote here is derived from [BE], 3.4, by restricting the hypotheses to the case of a bounded operator).

**Assertion 1.1.** *If $X$ is a nonzero complex Banach space and $\sigma_{L(X)}$ is continuous at $T \in L(X)$, then $\sigma_{L(X^*)}$ is continuous at $T^*$.*

This seems clearly to be a misprint. Indeed, what follows easily from continuity of the adjoint function from $L(X)$ into $L(X^*)$ is actually the converse of Assertion 1.1 (see Proposition 3.1 below). However, one may ask whether Assertion 1.1 is nevertheless true. In this paper this question is answered in the negative.

In Section 2 we collect some preliminaries, in order to make our paper as self-contained as possible.

Section 3 contains the results of this paper. We begin by observing that continuity of $\sigma_{L(X^*)}$ (respectively, $r_{L(X^*)}$) at $T^*$ implies continuity of $\sigma_{L(X)}$ (respectively, $r_{L(X)}$) at $T$, and the converse also holds if the Banach space $X$ is supposed to be reflexive (Proposition 3.1 and Corollary 3.2). Then we turn to our main result; namely, we prove that Assertion 1.1 is false: in Example 3.3 we construct a bounded linear operator $T$ on a nonreflexive Banach space, such that the spectrum function is continuous at $T$ and discontinuous at $T^*$.

In order to prove continuity of spectrum at $T$, we appeal to a special case of [B1], 2.7, which is stated here in Section 2. We point out that Conway and Morrel's conditions for continuity of spectrum cannot be helpful in a situation like the one of Example 3.3. Indeed, they are "self-adjoint", in the sense that they are satisfied either by both an operator $A$ and its adjoint (in which case $\sigma_{L(X)}$ and $\sigma_{L(X^*)}$ are continuous at $A$ and $A^*$, respectively) or by none of them: this is apparent if you look at one of the several equivalent versions of these conditions, which requires the union of the set of all points of nonzero index of the semi-Fredholm domain and of the set of all complex numbers $\lambda$ such that $\{\lambda\}$ is a component of the spectrum to be dense in the spectrum (see Section 2 of [B4] for a discussion about Conway and Morrel's conditions in the Banach space setting).

At present, as we note in the concluding remarks of this paper, we do not know whether the analogue of Assertion 1.1 for the spectral radius is true or not.

## 2. Preliminaries

If $E$, $F$ are Banach spaces and $T \in L(E, F)$, we denote by $\mathcal{R}(T)$ and $\mathcal{N}(T)$ the range and the kernel of $T$, respectively.

Now let $X$ be a Banach space. We will call an element $P$ of $L(X)$ a *projection of* $X$ if $P^2 = P$. It is well known that, if $P \in L(X)$ is a projection of $X$, then $\mathcal{R}(P)$ is a closed subspace of $X$ and moreover $X = \mathcal{R}(P) \oplus \mathcal{N}(P)$. Conversely, for each direct-sum decomposition $X = Y \oplus Z$, where $Y$ and $Z$ are closed subspaces of $X$, there exists a unique projection $P$ of $X$ satisfying $\mathcal{R}(P) = Y$ and $\mathcal{N}(P) = Z$. This projection will be called the *projection of* $X$ *onto* $Y$ *along* $Z$. Notice that $I_X - P$ is the projection of $X$ onto $Z$ along $Y$. Finally, we recall that a linear subspace $Y$ of $X$ is said to be *complemented* if it is the range of a projection of $X$ (or, equivalently, if $Y$ is closed and there exists a closed subspace $Z$ of $X$ such that $X = Y \oplus Z$).

The following characterization of the operators having complemented kernel and range will be useful in the sequel.

**Theorem 2.1** (see [C], Chapter 1, Theorem 1). *Let $X$ be a Banach space, and let $T \in L(X)$. Then the following two conditions are equivalent:*

(i) *both $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are complemented subspaces of $X$;*
(ii) *there exists $S \in L(X)$ such that $TST = T$ and $STS = S$.*

*Moreover, if $T$ satisfies the two equivalent conditions above and $S \in L(X)$ is such that $TST = T$ and $STS = S$, then $\mathcal{N}(S)$ and $\mathcal{R}(S)$ are complemented subspaces of $X$, $X = \mathcal{R}(T) \oplus \mathcal{N}(S) = \mathcal{N}(T) \oplus \mathcal{R}(S)$, $TS$ is the projection of $X$ onto $\mathcal{R}(T)$ along $\mathcal{N}(S)$ and $ST$ is the projection of $X$ onto $\mathcal{R}(S)$ along $\mathcal{N}(T)$.*

We will also apply the following perturbation result.

Henceforth, the symbol $\sim$ will denote isomorphism between Banach spaces. We will write $\| \, \|_X$ for the norm of a Banach space $X$.

**Theorem 2.2** (see [C], Chapter 5, Theorem 9 and Corollary). *Let $X$ be a nonzero Banach space, and let $S$, $T \in L(X) \setminus \{0\}$ be such that $TST = T$ and $STS = S$ (which, by Theorem 2.1, forces $\mathcal{N}(T)$, $\mathcal{R}(T)$, $\mathcal{N}(S)$ and $\mathcal{R}(S)$ to be complemented subspaces of $X$). If $\mathcal{N}(T) \subset \mathcal{R}(T^n)$ for every $n \in \mathbb{N}$, and $A \in L(X)$ is such that*

*$A\Big(\bigcap_{n\in\mathbb{N}}\mathcal{R}(T^n)\Big)\subset\bigcap_{n\in\mathbb{N}}\mathcal{R}(T^n)$ and $\|A\|_{L(X)}<\dfrac{1}{\|S\|_{L(X)}}$, then $T-A$ has complemented kernel and range and satisfies $X=\mathcal{R}(T-A)\oplus\mathcal{N}(S)=\mathcal{N}(T-A)\oplus\mathcal{R}(S)$ (which gives $\mathcal{N}(T-A)\sim\mathcal{N}(T)$ and $X/_{\mathcal{R}(T-A)}\sim\mathcal{N}(S)$).*

From now on, the closure and the boundary of a subset $\Sigma$ of a topological space will be denoted by $\overline{\Sigma}$ and $\partial\Sigma$, respectively.

For each Banach algebra $\mathfrak{A}$ with identity, let $\mathcal{G}_{\mathfrak{A}}$ denote the group of invertible elements of $\mathfrak{A}$. The following characterization of membership in $\overline{\mathcal{G}}_{L(X)}$ for a bounded linear operator—having complemented kernel and range—on a Banach space $X$ was obtained by M. Gonzalez in [G].

**Theorem 2.3** (see [G], §1). *Let $X$ be a Banach space, and let $T\in L(X)$ have complemented kernel and range. Then $T\in\overline{\mathcal{G}}_{L(X)}$ if and only if $\mathcal{N}(T)\sim X/_{\mathcal{R}(T)}$.*

Let $\mathfrak{A}$ be a complex Banach algebra with identity $\mathfrak{e}$. For each $\mathfrak{a}\in\mathfrak{A}$, we set

$$\mathcal{S}(\mathfrak{a})=\{\lambda\in\mathbb{C}:\ \lambda\mathfrak{e}-\mathfrak{a}\notin\overline{\mathcal{G}}_{\mathfrak{A}}\}.$$

Since $\mathfrak{A}\setminus\overline{\mathcal{G}}_{\mathfrak{A}}$ is an open subset of $\mathfrak{A}$, it follows that $\mathcal{S}(\mathfrak{a})$ is an open subset of $\mathbb{C}$. Furthermore, we have $\overline{\mathcal{S}(\mathfrak{a})}\subset\sigma_{\mathfrak{A}}(\mathfrak{a})$.

For our purposes here, it is sufficient to state the following straightforward consequence of [B1], 2.7.

**Theorem 2.4.** *Let $\mathfrak{A}$ be a complex Banach algebra with identity. If $\mathfrak{a}\in\mathfrak{A}$ satisfies $\sigma_{\mathfrak{A}}(\mathfrak{a})=\overline{\mathcal{S}(\mathfrak{a})}$, then $\sigma_{\mathfrak{A}}$ is continuous at $\mathfrak{a}$.*

Finally, we will more than once make use of the following well known result, relating the kernels and ranges of an operator and of its adjoint.

Henceforth, the annihilator of a subspace $M$ of a Banach space $X$ in $X^*$ (see [TL], Definition on page 163) will be denoted by $M^\circ$.

**Theorem 2.5** (see [TL], IV, 8.4 and 10.1). *Let $X$, $Y$ be Banach spaces, and let $T\in L(X,Y)$. Then:*

(i) *$\mathcal{N}(T^*)=\big(\mathcal{R}(T)\big)^\circ$;*
(ii) *$\mathcal{R}(T)$ is closed if and only if $\mathcal{R}(T^*)$ is closed;*
(iii) *if $\mathcal{R}(T)$ is closed, we have $\mathcal{R}(T^*)=\big(\mathcal{N}(T)\big)^\circ$.*

## 3. The example

We begin by establishing the relationship between continuity of spectrum (respectively, spectral radius) at a bounded linear operator $T$ on a Banach space and continuity of spectrum (respectively, spectral radius) at $T^*$.

**Proposition 3.1.** *Let $X$ be a nonzero complex Banach space, and let $T\in L(X)$.*

(i) *If $\sigma_{L(X^*)}$ is continuous at $T^*$, then $\sigma_{L(X)}$ is continuous at $T$.*
(ii) *If $r_{L(X^*)}$ is continuous at $T^*$, then $r_{L(X)}$ is continuous at $T$.*

*Proof.* It suffices to remark that $\sigma_{L(X)}=\sigma_{L(X^*)}\circ\Phi_X$ and $r_{L(X)}=r_{L(X^*)}\circ\Phi_X$, where

$$\Phi_X: L(X)\ni A\mapsto A^*\in L(X^*).$$

Since the linear map $\Phi_X$ is an isometry, and consequently is continuous, we get the desired result. $\square$

If the Banach space $X$ is assumed to be reflexive, then the converse of each of assertions (i) and (ii) of Proposition 3.1 also holds, as the following result shows.

**Corollary 3.2.** *Let $X$ be a nonzero reflexive complex Banach space, and let $T \in L(X)$.*

(i) *If $\sigma_{L(X)}$ is continuous at $T$, then $\sigma_{L(X^*)}$ is continuous at $T^*$.*
(ii) *If $r_{L(X)}$ is continuous at $T$, then $r_{L(X^*)}$ is continuous at $T^*$.*

*Proof.* Since $X$ is reflexive, the map

$$\Psi_X : L(X) \ni A \mapsto A^{**} \in L(X^{**})$$

is an isometric isomorphism of Banach algebras. Hence continuity of $\sigma_{L(X)}$ (respectively, $r_{L(X)}$) at $T$ implies continuity of $\sigma_{L(X^{**})}$ (respectively, $r_{L(X^{**})}$) at $T^{**}$, which gives continuity of $\sigma_{L(X^*)}$ (respectively, $r_{L(X^*)}$) at $T^*$ by Proposition 3.1. $\square$

The following example shows that, at least as concerns assertion (i), reflexivity of $X$ cannot be removed from the hypotheses of Corollary 3.2.

**Example 3.3.** We begin by considering the bounded linear operators $V$ and $W$ on the complex Banach space $\ell_\infty$, defined by

$$V(y_n)_{n\in\mathbb{N}} = (y_{2n})_{n\in\mathbb{N}} \qquad \text{for each } (y_n)_{n\in\mathbb{N}} \in \ell_\infty$$

and

$$W(z_n)_{n\in\mathbb{N}} = (\zeta_n)_{n\in\mathbb{N}} \ \left(\text{where } \zeta_k = \begin{cases} z_{\frac{k}{2}} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases} \quad \text{for each } k \in \mathbb{N}\right)$$

for each $(z_n)_{n\in\mathbb{N}} \in \ell_\infty$. Notice that each of $V$ and $W$ leaves $c_0$ invariant. Then two bounded linear operators $S$ and $U$ on $c_0$ can be defined as follows:

$$S(x_n)_{n\in\mathbb{N}} = V(x_n)_{n\in\mathbb{N}} \quad \text{and} \quad U(x_n)_{n\in\mathbb{N}} = W(x_n)_{n\in\mathbb{N}} \qquad \text{for each } (x_n)_{n\in\mathbb{N}} \in c_0.$$

Now let $E$ denote the complex Banach space $c_0 \times \ell_\infty \times \ell_\infty$, endowed with the norm $\| \, \|_E$ defined by

$$\|(x,y,z)\|_E = \max\left\{\|x\|_{c_0}, \|y\|_{\ell_\infty}, \|z\|_{\ell_\infty}\right\} \qquad \text{for each } (x,y,z) \in c_0 \times \ell_\infty \times \ell_\infty.$$

We define a bounded linear operator $T$ on $E$ as follows:

$$T(x,y,z) = (Sx, Vy, Wz) \qquad \text{for each } (x,y,z) \in E.$$

We prove that the spectrum function is continuous at $T$ and discontinuous at $T^*$.

We first prove that $\sigma_{L(E)}$ is continuous at $T$.

We begin by remarking that $SU = I_{c_0}$ and $VW = I_{\ell_\infty}$. Then $\mathcal{R}(S) = c_0$, $\mathcal{N}(U) = \{0\}$, $\mathcal{R}(V) = \ell_\infty$ and $\mathcal{N}(W) = \{0\}$. Furthermore, if $R \in L(E)$ is defined by

$$R(x,y,z) = (Ux, Wy, Vz) \qquad \text{for each } (x,y,z) \in E,$$

we see that for every $(x,y,z) \in E$ we have

$$TRT(x,y,z) = (SUSx, VWVy, WVWz) = (Sx, Vy, Wz) = T(x,y,z)$$

and

$$RTR(x,y,z) = (USUx, WVWy, VWVz) = (Ux, Wy, Vz) = R(x,y,z).$$

Hence $TRT = T$ and $RTR = R$. Consequently, by Theorem 2.1, each of $\mathcal{N}(T)$, $\mathcal{R}(T)$, $\mathcal{N}(R)$ and $\mathcal{R}(R)$ is a complemented subspace of $E$. From now on, to shorten notation, we will let $F$ and $G$ stand for $\mathcal{R}(R)$ and $\mathcal{N}(R)$, respectively.

It is easily seen that

$$T^n(x,y,z) = (S^n x, V^n y, W^n z) \qquad \text{for each } (x,y,z) \in E \text{ and for each } n \in \mathbb{N}.$$

Hence for every $n \in \mathbb{N}$ we obtain

$$\begin{aligned}\mathcal{R}(T^n) &= \mathcal{R}(S^n) \times \mathcal{R}(V^n) \times \mathcal{R}(W^n) \\ &= c_0 \times \ell_\infty \times \mathcal{R}(W^n) \supset \mathcal{N}(S) \times \mathcal{N}(V) \times \{0\} = \mathcal{N}(S) \times \mathcal{N}(V) \times \mathcal{N}(W) = \mathcal{N}(T).\end{aligned}$$

We remark that $\|S\|_{L(c_0)} = \|U\|_{L(c_0)} = \|V\|_{L(\ell_\infty)} = \|W\|_{L(\ell_\infty)} = 1$. Then it is not difficult to check that $\|T\|_{L(E)} = \|R\|_{L(E)} = 1$.

Since $\|R\|_{L(E)} = 1$, applying Theorem 2.2 with $A = \lambda I_E$ we conclude that, for each $\lambda$ in the open unit ball $\mathbf{B}$ of the complex plane, both $\mathcal{N}(\lambda I_E - T)$ and $\mathcal{R}(\lambda I_E - T)$ are complemented subspaces of $E$, and moreover we have

$$\mathcal{N}(\lambda I_E - T) \sim \mathcal{N}(T) = \mathcal{N}(S) \times \mathcal{N}(V) \times \{0\} \sim \mathcal{N}(S) \times \mathcal{N}(V)$$

and

$$E/_{\mathcal{R}(\lambda I_E - T)} \sim G = \mathcal{N}(U) \times \mathcal{N}(W) \times \mathcal{N}(V) = \{0\} \times \{0\} \times \mathcal{N}(V) \sim \mathcal{N}(V).$$

Since

$$\mathcal{N}(S) = \{(x_n)_{n\in\mathbb{N}} \in c_0 : \ x_{2k} = 0 \text{ for each } k \in \mathbb{N}\} \sim c_0$$

and

$$\mathcal{N}(V) = \{(y_n)_{n\in\mathbb{N}} \in \ell_\infty : \ y_{2k} = 0 \text{ for each } k \in \mathbb{N}\} \sim \ell_\infty,$$

it follows that

$$\mathcal{N}(\lambda I_E - T) \sim c_0 \times \ell_\infty \qquad \text{and} \qquad E/_{\mathcal{R}(\lambda I_E - T)} \sim \ell_\infty \qquad \text{for each } \lambda \in \mathbf{B}.$$

Now let $\lambda \in \mathbf{B}$. Since $c_0$ is isomorphic to no complemented subspace of $\ell_\infty$ (see [LT], 2.a.7), it follows that $\mathcal{N}(\lambda I_E - T)$ is not isomorphic to $E/_{\mathcal{R}(\lambda I_E - T)}$. Now we can apply Theorem 2.3 and concude that $\lambda I_E - T \notin \overline{\mathcal{G}}_{L(E)}$.

We have thus proved that $\mathbf{B} \subset \mathcal{S}(T)$. On the other hand, since $\|T\|_{L(E)} = 1$, it follows that $\sigma_{L(E)}(T) \subset \overline{\mathbf{B}}$. Since $\mathcal{S}(T)$ is an open subset of $\mathbb{C}$, contained in $\sigma_{L(E)}(T)$, we obtain

$$\mathcal{S}(T) = \mathbf{B} \qquad \text{and} \qquad \sigma_{L(E)}(T) = \overline{\mathbf{B}}.$$

Hence $\sigma_{L(E)}(T) = \overline{\mathcal{S}(T)}$, and consequently, by Theorem 2.4, $\sigma_{L(E)}$ is continuous at $T$.

We point out that $T^*$ does not satisfy the sufficient condition for continuity of spectrum given in Theorem 2.4. Indeed, since $\sigma_{L(E^*)}(T^*) = \sigma_{L(E)}(T) = \overline{\mathbf{B}}$, it follows that $\mathcal{S}(T^*) \subset \mathbf{B}$. Furthermore, for each $\lambda \in \mathbf{B}$, since $\mathcal{N}(\lambda I_E - T)$ and $\mathcal{R}(\lambda I_E - T)$ are complemented subspaces of $E$, by applying Theorem 2.1 (see also [C], §1.2, Remark (3)) we can assert that $\mathcal{N}(\lambda I_{E^*} - T^*)$ and $\mathcal{R}(\lambda I_{E^*} - T^*)$ are complemented subspaces of $E^*$. Finally, by Theorem 2.5 and [TL], III, 3.3, we have

$$\mathcal{N}(\lambda I_{E^*} - T^*) = \big(\mathcal{R}(\lambda I_E - T)\big)^\circ \sim \Big(E/_{\mathcal{R}(\lambda I_E - T)}\Big)^* \sim \ell_\infty^*$$

and

$$E^*/_{\mathcal{R}(\lambda I_{E^*}-T^*)} = E^*/_{\left(\mathcal{N}(\lambda I_E-T)\right)^\circ} \sim \left(\mathcal{N}(\lambda I_E-T)\right)^* \sim (c_0\times\ell_\infty)^* \sim \ell_1\times\ell_\infty^*.$$

Since every dual space is complemented in its bidual (see [H], 5.9.4), it follows that $\ell_1$ is isomorphic to a complemented subspace of $\ell_\infty^*$. From this, since $\ell_1$ is easily seen to be isomorphic to its square, we can derive that $\ell_1\times\ell_\infty^*$ is isomorphic to $\ell_\infty^*$, and consequently

$$\mathcal{N}(\lambda I_{E^*}-T^*) \sim E^*/_{\mathcal{R}(\lambda I_{E^*}-T^*)} \qquad \text{for each } \lambda\in\mathbf{B}.$$

Now Theorem 2.3 yields $\mathcal{S}(T^*)\cap\mathbf{B}=\emptyset$, which, since $\mathcal{S}(T^*)\subset\mathbf{B}$, gives $\mathcal{S}(T^*)=\emptyset$. Hence $T^*$ does not satisfy the condition of Theorem 2.4.

Now we prove that the spectrum function is actually discontinuous at $T^*$.

From Theorem 2.2 it follows also that

$$E=\mathcal{N}(\lambda I_E-T)\oplus F=\mathcal{R}(\lambda I_E-T)\oplus G \qquad \text{for each } \lambda\in\mathbf{B}.$$

For each $\lambda\in\mathbf{B}$, let $P_\lambda$ and $Q_\lambda$ denote the projection of $E$ onto $\mathcal{N}(\lambda I_E-T)$ along $F$ and the projection of $E$ onto $G$ along $\mathcal{R}(\lambda I_E-T)$, respectively. From Theorem 2.5 it follows that, for each $\lambda\in\mathbf{B}$, $P_\lambda^*$ is the projection of $E^*$ onto $F^\circ$ along $\mathcal{R}(\lambda I_{E^*}-T^*)$ and $Q_\lambda^*$ is the projection of $E^*$ onto $\mathcal{N}(\lambda I_{E^*}-T^*)$ along $G^\circ$. Consequently, we have

$$E^*=\mathcal{N}(\lambda I_{E^*}-T^*)\oplus G^\circ=\mathcal{R}(\lambda I_{E^*}-T^*)\oplus F^\circ \qquad \text{for each } \lambda\in\mathbf{B}.$$

In particular, we deduce that

$$E^*=\mathcal{N}(T^*)\oplus G^\circ=\mathcal{R}(T^*)\oplus F^\circ,$$

$P_0^*$ is the projection of $E^*$ onto $F^\circ$ along $\mathcal{R}(T^*)$ and $Q_0^*$ is the projection of $E^*$ onto $\mathcal{N}(T^*)$ along $G^\circ$.

Since $\mathcal{N}(T^*)$ is isomorphic to $E^*/_{\mathcal{R}(T^*)}$, and consequently to $F^\circ$, there exists $\Gamma\in L\big(\mathcal{N}(T^*),F^\circ\big)$ such that $\mathcal{N}(\Gamma)=\{0\}$ and $\mathcal{R}(\Gamma)=F^\circ$. Furthermore, without loss of generality we can assume that $\|\Gamma\|_{L(\mathcal{N}(T^*),F^\circ)}=1$. For each $n\in\mathbb{N}$, let $A_n\in L(E^*)$ be defined by

$$A_n\xi^*=T^*\xi^*-\frac{1}{n+1}\,\Gamma Q_0^*\xi^* \qquad \text{for each } \xi^*\in E^*.$$

Clearly, $A_n$ converges to $T^*$ in $L(E^*)$ as $n\to+\infty$.

We prove that $\sigma_{L(E^*)}(A_n)=\partial\mathbf{B}$ for each $n\in\mathbb{N}$.

From Theorem 2.1 it follows that $RT=I_E-P_0$ and $TR=I_E-Q_0$. Since $SU=I_{c_0}$ and $VW=I_{\ell_\infty}$, for each $(x,y,z)\in E$ we obtain

$$(I_E-P_0)(x,y,z)=(USx,WVy,z) \qquad \text{and} \qquad (I_E-Q_0)(x,y,z)=(x,y,WVz).$$

We remark that, if we set $\delta_n=\begin{cases}1 & \text{if } n \text{ is even}\\ 0 & \text{if } n \text{ is odd}\end{cases}$ for each $n\in\mathbb{N}$, we have

$$US(x_n)_{n\in\mathbb{N}}=(\delta_nx_n)_{n\in\mathbb{N}} \qquad \text{for each } (x_n)_{n\in\mathbb{N}}\in c_0$$

and

$$WV(y_n)_{n\in\mathbb{N}}=(\delta_ny_n)_{n\in\mathbb{N}} \quad \text{for each } (y_n)_{n\in\mathbb{N}}\in\ell_\infty.$$

Now let $\xi$, $\zeta \in E$. If $(x_n)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}} \in c_0$, $(y_n)_{n\in\mathbb{N}}$, $(z_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$, $(w_n)_{n\in\mathbb{N}} \in \ell_\infty$ satisfy $\xi = \Big((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}\Big)$ and $\zeta = \Big((u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}, (w_n)_{n\in\mathbb{N}}\Big)$, we obtain

$$\begin{aligned}
&\|P_0\xi + (I_E - P_0)\zeta\|_E \\
&\quad = \Big\|\Big(\big((1-\delta_n)x_n + \delta_n u_n\big)_{n\in\mathbb{N}}, \big((1-\delta_n)y_n + \delta_n v_n\big)_{n\in\mathbb{N}}, (w_n)_{n\in\mathbb{N}}\Big)\Big\|_E \\
&\quad = \max\Big\{\big\|\big((1-\delta_n)x_n + \delta_n u_n\big)_{n\in\mathbb{N}}\big\|_{c_0}, \big\|\big((1-\delta_n)y_n + \delta_n v_n\big)_{n\in\mathbb{N}}\big\|_{\ell_\infty}, \big\|(w_n)_{n\in\mathbb{N}}\big\|_{\ell_\infty}\Big\} \\
&\quad \le \max\big\{\|\xi\|_E, \|\zeta\|_E\big\}
\end{aligned}$$

and

$$\begin{aligned}
&\|Q_0\xi + (I_E - Q_0)\zeta\|_E \\
&\qquad = \Big\|\Big((u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}, \big((1-\delta_n)z_n + \delta_n w_n\big)_{n\in\mathbb{N}}\Big)\Big\|_E \\
&\qquad = \max\Big\{\big\|(u_n)_{n\in\mathbb{N}}\big\|_{c_0}, \big\|(v_n)_{n\in\mathbb{N}}\big\|_{\ell_\infty}, \big\|\big((1-\delta_n)z_n + \delta_n w_n\big)_{n\in\mathbb{N}}\big\|_{\ell_\infty}\Big\} \\
&\qquad \le \max\big\{\|\xi\|_E, \|\zeta\|_E\big\}.
\end{aligned}$$

From this, applying [HWW], I, comments following 1.1, we conclude that

$$\|\xi^*\|_{E^*} = \|P_0^*\xi^*\|_{E^*} + \|\xi^* - P_0^*\xi^*\|_{E^*} = \|Q_0^*\xi^*\|_{E^*} + \|\xi^* - Q_0^*\xi^*\|_{E^*}$$

for each $\xi^* \in E^*$. Consequently, for each $\xi^* \in E^*$ we have

$$\begin{aligned}
\|A_n\xi^*\|_{E^*} &= \|P_0^* A_n\xi^*\|_{E^*} + \|A_n\xi^* - P_0^* A_n\xi^*\|_{E^*} = \frac{\|\varGamma Q_0^*\xi^*\|_{E^*}}{n+1} + \|T^*\xi^*\|_{E^*} \\
&= \frac{\|\varGamma Q_0^*\xi^*\|_{E^*}}{n+1} + \|T^*(\xi^* - Q_0^*\xi^*)\|_{E^*} \le \|Q_0^*\xi^*\|_{E^*} + \|\xi^* - Q_0^*\xi^*\|_{E^*} = \|\xi^*\|_{E^*}.
\end{aligned}$$

Hence $\|A_n\|_{L(E^*)} \le 1$ for every $n \in \mathbb{N}$, which gives $\sigma_{L(E^*)}(A_n) \subset \overline{\mathbf{B}}$ for every $n \in \mathbb{N}$.

Now let $n \in \mathbb{N}$, $\lambda \in \mathbf{B}$. We remark that

$$(\lambda I_{E^*} - A_n)\xi^* = (\lambda I_{E^*} - T^*)\xi^* + \frac{1}{n+1}\,\varGamma Q_0^*\xi^* \qquad \text{for each } \xi^* \in E^*.$$

Since $\mathcal{R}(\lambda I_{E^*} - T^*) \cap \mathcal{R}(\varGamma) = \mathcal{R}(\lambda I_{E^*} - T^*) \cap F^\circ = \{0\}$, it follows that

$$\mathcal{N}(\lambda I_{E^*} - A_n) = \mathcal{N}(\lambda I_{E^*} - T^*) \cap \mathcal{N}(\varGamma Q_0^*) = \mathcal{N}(\lambda I_{E^*} - T^*) \cap G^\circ = \{0\}.$$

Since $G^\circ = \mathcal{N}(Q_0^*)$ and $E^* = \mathcal{N}(\lambda I_{E^*} - T^*) \oplus G^\circ$, we obtain

$$\mathcal{R}(\lambda I_{E^*} - A_n) \supset (\lambda I_{E^*} - A_n)(G^\circ) = (\lambda I_{E^*} - T^*)(G^\circ) = \mathcal{R}(\lambda I_{E^*} - T^*).$$

Hence

$$F^\circ = \mathcal{R}(\varGamma Q_0^*) \subset \mathcal{R}(\lambda I_{E^*} - A_n) + \mathcal{R}(\lambda I_{E^*} - T^*) = \mathcal{R}(\lambda I_{E^*} - A_n).$$

This, since $E^* = \mathcal{R}(\lambda I_{E^*} - T^*) \oplus F^\circ$, gives $\mathcal{R}(\lambda I_{E^*} - A_n) = E^*$. Therefore, $\lambda \notin \sigma_{L(E^*)}(A_n)$.

We have thus proved that $\sigma_{L(E^*)}(A_n) \subset \partial\mathbf{B}$ for every $n \in \mathbb{N}$ (notice that this, together with $\|A_n\|_{L(E^*)} \le 1$ for every $n \in \mathbb{N}$, yields $\|A_n\|_{L(E^*)} = 1$ for every $n \in \mathbb{N}$).

It remains to prove the opposite inclusion. To this end, we begin by showing that $\sigma_{L(\ell_\infty)}(W) = \overline{\mathbf{B}}$. We first remark that $\ell_\infty = \mathcal{R}(\mu I_{\ell_\infty} - W) \oplus \mathcal{N}(V)$ for every

$\mu \in \mathbf{B}$ (this can be obtained either from the decomposition $E = \mathcal{R}(\mu I_E - T) \oplus G$, or by applying Theorem 2.2 with $A = \mu I_{\ell_\infty}$). Since $\mathcal{N}(V) \neq \{0\}$, it follows that $\sigma_{L(\ell_\infty)}(W) \supset \mathbf{B}$. On the other hand, $\|W\|_{L(\ell_\infty)} = 1$ gives $\sigma_{L(\ell_\infty)}(W) \subset \overline{\mathbf{B}}$, from which we derive the desired equality.

Now let $\lambda \in \partial\mathbf{B}$. Since $\lambda \in \partial\sigma_{L(\ell_\infty)}(W)$, it follows that $\lambda I_{\ell_\infty} - W$ is not an injective operator with closed range (see [TL], V, 4.1). Hence, if we consider the bounded linear operator

$$J : \ell_\infty \ni z \mapsto (0, 0, z) \in E,$$

we can assert that $J(\lambda I_{\ell_\infty} - W)$ is not an injective operator with closed range, either. It is easily seen that $(\lambda I_E - T)J = J(\lambda I_{\ell_\infty} - W)$. Since $(\lambda I_E - T)J$ is not an injective operator with closed range, from Theorem 2.5 it follows that $J^*(\lambda I_{E^*} - T^*)$ is not surjective. On the other hand, since

$$\mathcal{R}(J) = \{0\} \times \{0\} \times \ell_\infty \subset \mathcal{R}(U) \times \mathcal{R}(W) \times \ell_\infty = \mathcal{R}(U) \times \mathcal{R}(W) \times \mathcal{R}(V) = F,$$

applying Theorem 2.5 we obtain

$$\mathcal{N}(J^*) = \big(\mathcal{R}(J)\big)^\circ \supset F^\circ = \mathcal{R}(\Gamma).$$

Hence

$$J^*(\lambda I_{E^*} - A_n) = J^*(\lambda I_{E^*} - T^*)$$

for every $n \in \mathbb{N}$. We conclude that $J^*(\lambda I_{E^*} - A_n)$ is surjective for no $n \in \mathbb{N}$. Since $J$ is injective and has closed range, and consequently, by Theorem 2.5, $J^*$ is surjective, it follows that $\lambda I_{E^*} - A_n$ is surjective for no $n \in \mathbb{N}$. Hence $\lambda \in \sigma_{L(E^*)}(A_n)$ for every $n \in \mathbb{N}$. The desired equality $\sigma_{L(E^*)}(A_n) = \partial\mathbf{B}$ is now established for every $n \in \mathbb{N}$.

Now, in order to prove that $\sigma_{L(E^*)}$ is discontinuous at $T^*$, it suffices to observe that

$$\Delta\big(\sigma_{L(E^*)}(A_n), \sigma_{L(E^*)}(T^*)\big) = \Delta(\partial\mathbf{B}, \overline{\mathbf{B}}) = 1 \qquad \text{for each } n \in \mathbb{N},$$

although $A_n \to T^*$ as $n \to +\infty$.

In [B2] we had already used the fact that $c_0 \times \ell_\infty$ and $\ell_\infty$ are nonisomorphic Banach spaces having isomorphic duals, in order to construct an operator $T \in L(c_0 \times \ell_\infty)$, having complemented kernel and range, such that $T \notin \overline{\mathcal{G}}_{L(c_0 \times \ell_\infty)}$ and $T^* \in \overline{\mathcal{G}}_{L((c_0 \times \ell_\infty)^*)}$ (see [B2], 1.6). We point out that the operator $T$ of [B2], 1.6 would not have been suitable for our purposes in Example 3.3 here. Indeed, it can be proved that $\sigma_{L(c_0 \times \ell_\infty)}(T) = \overline{\mathbf{B}}$ and $\mathcal{S}(T^*) = \mathbf{B} \setminus \{0\}$. Consequently, by Theorem 2.4 and Proposition 3.1, the spectrum function is continuous at $T^*$ as well as it is so at $T$.

We conclude by remarking that the sequence $(A_n)_{n \in \mathbb{N}}$, constructed in Example 3.3, does not yield an example of discontinuity of $r_{L(E^*)}$ at the adjoint of an operator at which $r_{L(E)}$ is continuous, not even if we add a multiple of the identity to its terms (notice that adding a multiple of the identity to $T$ preserves continuity of $\sigma_{L(E)}$, which in turn gives continuity of $r_{L(E)}$). Indeed, for each $\nu \in \mathbb{C}$, since $\sigma(A_n) = \partial\mathbf{B}$ for every $n \in \mathbb{N}$, it follows that

$$r_{L(E^*)}(A_n + \nu I_{E^*}) = |\nu| + 1 = r_{L(E^*)}(T^* + \nu I_{E^*}) \qquad \text{for every } n \in \mathbb{N}.$$

Therefore Example 3.3 leaves open the problem whether continuity of spectral radius at an operator $A$ implies continuity of spectral radius at $A^*$.

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