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ON CENTRAL LIMIT THEOREMS FOR SHRUNKEN RANDOM VARIABLES

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ABSTRACT. We discuss Central Limit Theorems and absence of limiting distributions for shrunken random variables.

1. INTRODUCTION

For $r > 0$ let U_r be the shrinking operator: $U_r(x) = \max(|x| - r, 0)\mathrm{sgn}(x)$, or, equivalently,

$$U_r(x) = \begin{cases} x + r & \text{for} \quad -\infty < x < -r, \\ 0 & \text{for} \quad -r \le x \le r, \\ x - r & \text{for} \quad r < x < \infty. \end{cases}$$

This non-linear operator was studied by Jurek [2], [3] in relation to his work with s-self-decomposable distributions. Jurek showed that s-self-decomposable distributions were the limiting distributions of sums of the form

$$U_{r_n}(X_1) + U_{r_n}(X_2) + \cdots + U_{r_n}(X_n) + x_n \tag{1.1}$$

where $X_1, X_2, \ldots$ are independent random variables. Such distributions were also studied in a different context in [4] and [5].

If we let $\mathcal{G}$ be the class of Gaussian distributions, $\mathcal{S}$ the class of stable distributions, $\mathcal{L}$ the class of self-decomposable distributions, $\mathcal{U}$ the class of s-self-decomposable distributions, and $\mathcal{ID}$ the class of infinitely divisible distributions, then we have the following hierarchy: $\mathcal{G} \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{U} \subset \mathcal{ID}$. Thus the following question recently posed by Jurek naturally arises:

If $X, X_1, X_2, \ldots$ are independent and identically distributed (i.i.d.) random variables, what are the conditions on the distribution and the canonical form of r_n so that there exist x_n with (1.1) converging weakly to a standard normal distribution?

We have found that the condition on the distribution is that the tail must decay rapidly. We have:

Theorem 1. *Let $X, X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) random variables. Let*

$$G(x) = \int_0^\infty tP(|X| > t + x)dt. \tag{1.2}$$

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If

(i) $0 < G(x) < \infty$ *for all* $x > 0$ *and*

(ii) $\lim_{x\to\infty} G(x+\epsilon)/G(x) = 0$ *for all* $\epsilon > 0$

then for any $r_n \to \infty$ *satisfying*

$$\lim_{n\to\infty} 2nG(r_n) = 1 \tag{1.3}$$

and $x_n = -nE[\max(|X| - r_n, 0)sgn(X)]$ *we have*

$$U_{r_n}(X_1) + U_{r_n}(X_2) + \cdots + U_{r_n}(X_n) + x_n \Rightarrow N(0,1)$$

where $\Rightarrow$ *denotes weak convergence.*

Remark 1. It is easy to see that $G(x) < \infty$ for some $x > 0$ if and only if $EX^2 < \infty$. When $EX^2 < \infty$, $G(x)$ is a non-increasing and continuous function. Therefore the solution r_n to (1.3) exists. Moreover, r_n must tend to ∞ under condition (i).

Remark 2. For X with bounded support, one can use similar techniques as given below to show that the conclusion to Theorem 1 remains valid although one has to modify the choice of r_n so that $n\mathrm{Var}\Big(\max(|X| - r_n, 0)\mathrm{sgn}(X)\Big) \to 1$.

It is easy to check that (ii) is satisfied for the normal distribution. However, many other standard distributions, such as the exponential distribution, do not satisfy (ii). On the other hand, as a partial converse, our second theorem shows that one cannot have a central limit theorem type result for (1.1) for such distributions:

Theorem 2. *Let* $X, X_1, X_2, \ldots$ *be i.i.d. random variables. If there exists an* $\varepsilon_0 > 0$ *such that*

$$\liminf_{x\to\infty} P(|X| > x + \varepsilon_0)/P(|X| > x) > 0, \tag{1.4}$$

then there do not exist $r_n \to \infty$ *and real numbers* x_n *such that*

$$U_{r_n}(X_1) + U_{r_n}(X_2) + \cdots + U_{r_n}(X_n) + x_n \Rightarrow N(0,1).$$

Thus, for the Weibull family, with density functions:

$$f(x;a) = ax^{a-1}\exp(x^{-a}),\ x > 0,\ a > 0,$$

we see that for $a > 1$, (1.1) converges weakly to a normal distribution with the proper choice of r_n given in Theorem 1, but no such Central Limit Theorem result holds for $0 < a \le 1$.

Our final result is that there is no non-degenerate limiting distribution for (1.1) for regular distributions:

Theorem 3. *Let* $X, X_1, X_2, \ldots$ *be i.i.d. random variables with density function* $f(x)$. *Assume that there exist* $p > 1$, $c_1 \ge 0$, $c_2 \ge 0$, $c_1 + c_2 > 0$ *and a slowly varying function* $l(x)$ *(at* ∞*) such that*

$$\lim_{x\to+\infty} \frac{f(x)}{(l(x)/x^p)} = c_1, \quad \lim_{x\to+\infty} \frac{f(-x)}{(l(x)/x^p)} = c_2. \tag{1.5}$$

Then for any $r_n \to \infty$ *and real numbers* x_n, $U_{r_n}(X_1)+U_{r_n}(X_2)+\cdots+U_{r_n}(X_n)+x_n$ *does not have a non-degenerate limiting distribution. If* r_n *satisfies*

$$\frac{nl(r_n)}{r_n^{p-1}} \to 0, \tag{1.6}$$

then

$$U_{r_n}(X_1) + U_{r_n}(X_2) + \cdots + U_{r_n}(X_n) \to 0 \quad \textit{in probability.}$$

2. Proofs

Throughout our proofs we make extensive use of the following theorem in [1], Theorem 18, p. 95. The statement of the theorem is modified slightly here to match our problem:

Theorem (A). *Let* $\{Y_n, Y_{n,1}, Y_{n,2}, \ldots, Y_{n,n}\}$ *be a sequence of series of random variables i.i.d. within each series satisfying that for all* $\varepsilon > 0$, $P(Y_n \geq \epsilon) \to 0$. *There will exist a sequence of real constants* x_n *such that the distribution of the sums* $\sum_{k=1}^{n} Y_{nk} + x_n$ *converges weakly to the standard normal distribution if and only if the following conditions are satisfied:*

$$nP(|Y_n| \geq \varepsilon) \to 0 \quad \textit{for every} \quad \varepsilon > 0 \tag{2.1}$$

and

$$n \operatorname{Var}\left(Y_n I_{\{|Y_n| \leq 1\}}\right) \to 1. \tag{2.2}$$

If these conditions are satisfied, then we may write

$$x_n = -nE\left(Y_n I_{\{|Y_n| \leq 1\}}\right) + o(1).$$

Proof of Theorem 1. Let $Y_{n,k} = U_{r_n}(X_k) = \max(|X_k| - r_n, 0)\operatorname{sgn}(X_k)$ and $Y_n = U_{r_n}(X)$. In order to show that we may choose our x_n as given in Theorem 1 without truncating the tail as in Theorem A, we will show

$$nE|Y_n| I_{\{|Y_n| > 1\}} \to 0. \tag{2.3}$$

In order to show that the variance term (2.2) tends to 1, we will show

$$nEY_n^2 I_{\{|Y_n| \leq 1\}} \to 1 \tag{2.4}$$

and

$$n\left(E|Y_n| I_{\{|Y_n| \leq 1\}}\right)^2 \to 0. \tag{2.5}$$

First, in order to show (2.1), observe that $P(|Y_n| > \varepsilon) = P(|X| > r_n + \varepsilon)$. Since $P(|X| > r_n + \varepsilon) \leq P(|X| > t + r_n + \varepsilon/2)$ for all $0 \leq t \leq \varepsilon/2$, we see

$$\begin{aligned} P(|X| > r_n + \varepsilon) &\leq \frac{8}{\varepsilon^2} \int_0^{\varepsilon/2} tP(|X| > t + r_n + \varepsilon/2)dt \\ &\leq \frac{8}{\varepsilon^2} G(r_n + \varepsilon/2). \end{aligned}$$

Now (2.1) follows by our choice of r_n and condition (ii).

To prove (2.3), we have

$$\begin{aligned} E|Y_n| I_{\{|Y_n| > 1\}} &= E(|X| - r_n) I_{\{|X| > r_n + 1\}} \\ &= P(|X| > r_n + 1) + \int_{1/2}^{\infty} P(|X| > r_n + 1/2 + t)dt \\ &\leq P(|X| > r_n + 1) + 2\int_{1/2}^{\infty} tP(|X| > r_n + 1/2 + t)dt \\ &\leq P(|X| > r_n + 1) + 2G(r_n + 1/2). \end{aligned}$$

Note that $P(|X| > r_n + 1) \leq 8G(r_n + 1/2)$ by the proof of (2.1) above with $\epsilon = 1$. Thus, $E|Y_n| I_{\{|Y_n| > 1\}} \leq 10G(r_n + 1/2)$ which yields (2.3) immediately by our choice of r_n and condition (ii).

To prove (2.4), let $G_1(x) = \int_x^\infty P(|X| > t)dt$. Note that

$$\begin{aligned}
EY_n^2 I_{\{|Y_n|\le 1\}} &= E(|X| - r_n)^2 I_{\{r_n<|X|\le r_n+1\}} \\
&= \int_0^1 t^2 d(-P(|X| > r_n + t)) \\
&= -P(|X| > 1 + r_n) + 2\int_0^1 tP(|X| > r_n + t)dt \\
&= -P(|X| > 1 + r_n) + 2\int_0^1 td(-G_1(t + r_n)) \\
&= -P(|X| > 1 + r_n) - 2G_1(1 + r_n) \\
&\quad +2\int_0^1 \int_t^\infty P(|X| > s + r_n)dsdt \\
&= -P(|X| > 1 + r_n) - 2\int_{1+r_n}^\infty P(|X| > t)dt \\
&\quad +2\int_0^1 \int_t^1 P(|X| > s + r_n)dsdt + 2\int_1^\infty P(|X| > s + r_n)ds \\
&= -P(|X| > 1 + r_n) + 2\int_0^1 sP(|X| > s + r_n)ds \\
&= -P(|X| > 1 + r_n) + 2G(r_n) - 2\int_1^\infty sP(|X| > s + r_n)ds.
\end{aligned}$$

This proves (2.4) by (2.1), our choice of r_n, and by noting that

$$\int_1^\infty sP(|X| > s + r_n)ds \le \int_{1/2}^\infty 2sP(|X| > s + r_n + 1/2)ds \le 2G(r_n + 1/2)$$

so that $n\int_1^\infty sP(|X| > s + r_n)ds \to 0$.

Finally, we prove (2.5). It follows from the Hölder inequality that

$$\begin{aligned}
n\left(E|Y_n|I_{\{|Y_n|\le 1\}}\right)^2 &= n\left(E(|X| - r_n)I_{\{r_n<|X|\le 1+r_n\}}\right)^2 \\
&\le n\,P(r_n < |X| \le 1 + r_n)E(|X| - r_n)^2 I_{\{r_n<|X|\le 1+r_n\}} \to 0
\end{aligned}$$

by (2.4) and the fact that $P(|X| > r_n) \to 0$ since $r_n \to \infty$.

This completes the proof of Theorem 1. □

Proof of Theorem 2. Since $r_n \to \infty$, we have

$$\forall\, \varepsilon > 0, \quad P(\max(|X| - r_n, 0) > \varepsilon) \to 0.$$

Assume that (1.2) converges weakly to a standard normal distribution. Then, by Theorem A again, we have

$$\forall\ \varepsilon > 0,\ n\,P(|X| \ge r_n + \varepsilon) = n\,P(\max(|X| - r_n, 0) > \varepsilon) \to 0 \tag{2.6}$$

and

$$n\mathrm{Var}\left(\max(|X| - r_n, 0)\mathrm{sgn}(X)I_{\{|\max(|X|-r_n,0)\,\mathrm{sgn}(X)|<1\}}\right) \to 1.$$

Thus, for sufficiently large n

$$\begin{aligned}
1/2 &\le n\left(E(|X| - r_n)^2 I_{\{r_n<|X|\le 1+r_n\}}\right) \\
&\le nP(|X| > r_n),
\end{aligned}$$

which together with (2.6) yields

$$\forall\, \varepsilon > 0, \quad \lim_{n\to\infty} P(|X| > r_n + \varepsilon)/P(|X| > r_n) = 0.$$

This contradicts the assumption (1.4) and completes the proof of Theorem 2.

Proof of Theorem 3. Assume that (1.1) has a non-degenerate limiting distribution for some $r_n \to \infty$ and $\{x_n\}$. Let $Y_n = U_{r_n}(X) = \max(|X| - r_n, 0)\mathrm{sgn}(X)$. Then

$$e^{itx_n}\left(Ee^{itY_n}\right)^n \to g(t), \tag{2.7}$$

where $g(t)$ is a non-degenerate characteristic function. Write

$$Ee^{itY_n} = E\cos(t\max(|X| - r_n, 0)) + iE\sin(t\max(|X| - r_n, 0)\mathrm{sgn}(X)).$$

For $t \neq 0$, we have

$$\begin{aligned} &E\cos(t\max(|X| - r_n, 0)) \\ &= \ 1 - P(|X| > r_n) + \int_0^\infty \cos(x)d(-P(|X| > r_n + x/|t|)) \\ &= \ 1 - \int_0^\infty \sin(x)P(|X| > r_n + x/|t|)dx. \end{aligned}$$

Put

$$I_n(t) = \int_0^\infty \sin(x)P(|X| > r_n + x/|t|)dx.$$

Using the properties of regular distributions, one can check that $I_n(t)$ is equal to

$$\sum_{k=0}^\infty \Big\{ \int_{2k\pi}^{(2k+1)\pi} \sin(x)P(|X| > r_n + x/|t|)dx + \int_{(2k+1)\pi}^{(2k+2)\pi} \sin(x)P(|X| > r_n + x/|t|)dx \Big\}$$

$$\begin{aligned} &= \ \sum_{k=0}^\infty \int_0^\pi \sin(x)\Big\{P(|X| > r_n + (2k\pi + x)/|t|) \\ &\qquad\qquad -P(|X| > r_n + ((2k+1)\pi + x)/|t|)\Big\}dx \\ &= \ \int_0^\pi \sin(x)\left\{\sum_{k=0}^\infty P((2k\pi + x)/|t| + r_n < |X| \le r_n + ((2k+1)\pi + x)/|t|)\right\}dx \\ &\sim \ \int_0^\pi \sin(x)\left\{\sum_{k=0}^\infty \frac{((c_1 + c_2)\pi/|t|)l(r_n + (2k\pi + x)/|t|)}{(r_n + (2k\pi + x)/|t|)^p}\right\}dx \quad \text{[by (1.5)]} \\ &\sim \ \left\{\sum_{k=0}^\infty \frac{((c_1 + c_2)\pi/|t|)l(r_n + 2k\pi/|t|)}{(r_n + 2k\pi/|t|)^p}\right\}\int_0^\pi \sin(x)dx \\ &\sim \ 2(c_1 + c_2)(\pi/|t|)\int_0^\infty \frac{l(r_n + 2x\pi/|t|)}{(r_n + 2x\pi/|t|)^p}dx \\ &= \ (c_1 + c_2)\int_0^\infty \frac{l(r_n + x)}{(r_n + x)^p}dx \\ &\sim \ \frac{(c_1 + c_2)l(r_n)}{(p-1)r_n^{p-1}}. \end{aligned}$$

Therefore

$$E\cos(t\max(|X|-r_n,0))-1\sim-\frac{(c_1+c_2)l(r_n)}{(p-1)r_n^{p-1}}. \tag{2.8}$$

To estimate $E\sin(t\max(|X|-r_n,0)\mathrm{sgn}(X))$, write $E\sin(t\max(|X|-r_n,0)\mathrm{sgn}(X))$ $= E\sin(t(X-r_n))I_{\{X>r_n\}}-E\sin(t(-X-r_n))I_{\{-X>r_n\}}$.

Following the proof of (2.8), we have

$$\begin{aligned}
E\sin(t(|X|-r_n)I_{\{X>r_n\}}) &= \int_0^\infty \sin(tx)d(-P(X>r_n+x))\\
&= t\int_0^\infty \cos(tx)P(X>r_n+x)dx\\
&= (t/|t|)\int_0^\infty \cos(x)P(X>r_n+x/|t|)dx\\
&= \mathrm{sgn}(t)\int_0^{\pi/2}\cos(x)P(X>r_n+x/|t|)dx\\
&\quad +\mathrm{sgn}(t)\sum_{k=0}^\infty\Big\{\int_{2k\pi+\pi/2}^{(2k+1)\pi+\pi/2}\cos(x)P(X>r_n+x/|t|)dx\\
&\qquad +\int_{(2k+1)\pi+\pi/2}^{(2k+2)\pi+\pi/2}\cos(x)P(X>r_n+x/|t|)dx\Big\}\\
&= \mathrm{sgn}(t)\int_0^{\pi/2}\cos(x)P(X>r_n+x/|t|)dx\\
&\quad +\mathrm{sgn}(t)\sum_{k=0}^\infty\Big\{\int_0^\pi \sin(x)\Big\{-P(X>r_n+(x+2k\pi+\pi/2)/|t|)\\
&\qquad +P(X>r_n+(x+(2k+1)\pi+\pi/2)/|t|)\Big\}dx\Big\}\\
&\sim \mathrm{sgn}(t)\frac{c_1 l(r_n)}{r_n^p}\int_0^{\pi/2}\cos(x)dx\\
&\quad -(c_1\pi/|t|)\mathrm{sgn}(t)\sum_{k=0}^\infty\frac{l(r_n+2k\pi/|t|)}{(r_n+2k\pi/|t|)^p}\int_0^\pi\sin(x)dx\\
&\sim -\frac{c_1\mathrm{sgn}(t)l(r_n)}{(p-1)r_n^{p-1}}.
\end{aligned}$$

Similarly,

$$E\sin(t(-X-r_n)I_{\{-X>r_n\}})\sim-\frac{c_2\mathrm{sgn}(t)l(r_n)}{(p-1)r_n^{p-1}}.$$

Hence

$$E\sin(t\max(|X|-r_n,0)\mathrm{sgn}(X))\sim(c_2-c_1)\mathrm{sgn}(t)\frac{l(r_n)}{(p-1)r_n^{p-1}}. \tag{2.9}$$

From (2.8) and (2.9) it follows that

(2.10)

$$\big(Ee^{itY_n}\big)^n=\Big(1-\frac{l(r_n)}{(p-1)r_n^{p-1}}(c_1+c_2+i\mathrm{sgn}(t)(c_1-c_2)+o(1))\Big)^n.$$

Consider four different cases:

Case 1. When $\liminf_{n\to\infty} \frac{nl(r_n)}{r_n^{p-1}} < \limsup_{n\to\infty} \frac{nl(r_n)}{r_n^{p-1}}$, (2.7) contradicts (2.10).

Case 2. When $\lim_{n\to\infty} \frac{nl(r_n)}{r_n^{p-1}} = \infty$, (2.10) implies that for every $t \neq 0$,

$$\left(Ee^{itY_n}\right)^n \to 0,$$

which is also in contradiction with (2.7).

Case 3. When $\lim_{n\to\infty} \frac{nl(r_n)}{r_n^{p-1}} = a$, where $0 < a < \infty$, we obtain from (2.10) that

$$\forall \ \ t \neq 0, \quad \left(Ee^{itY_n}\right)^n \to \exp\left(-\frac{a}{p-1}(c_1 + c_2 + i\operatorname{sgn}(t)(c_1 - c_2))\right).$$

Since $\left(Ee^{i0Y_n}\right)^n \equiv 1$, the limiting function of $\left(Ee^{itY_n}\right)^n$ is not continuous, which contradicts the assumption that $g(t)$ is a characteristic function.

Case 4. When $\lim_{n\to\infty} \frac{nl(r_n)}{r_n^{p-1}} = 0$, then we have

$$\left(Ee^{itY_n}\right)^n \to 1$$

and hence the limiting distribution is degenerate.

This completes the proof of the theorem.

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