ON CENTRAL LIMIT THEOREMS
FOR SHRUNKEN RANDOM VARIABLES

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ABSTRACT. We discuss Central Limit Theorems and absence of limiting distributions for shrunken random variables.

1. INTRODUCTION

For $r > 0$ let $U_r$ be the shrinking operator: $U_r(x) = \max(|x| - r, 0)\text{sgn}(x)$, or, equivalently,

$$U_r(x) = \begin{cases} 
    x + r & \text{for } -\infty < x < -r, \\
    0 & \text{for } -r \leq x \leq r, \\
    x - r & \text{for } r < x < \infty.
\end{cases}$$

This non-linear operator was studied by Jurek [2], [3] in relation to his work with s-self-decomposable distributions. Jurek showed that s-self-decomposable distributions were the limiting distributions of sums of the form

$$U_{r_n}(X_1) + U_{r_n}(X_2) + \cdots + U_{r_n}(X_n) + x_n$$

where $X_1, X_2, \ldots$ are independent random variables. Such distributions were also studied in a different context in [4] and [5].

If we let $G$ be the class of Gaussian distributions, $S$ the class of stable distributions, $L$ the class of self-decomposable distributions, $U$ the class of s-self-decomposable distributions, and $ID$ the class of infinitely divisible distributions, then we have the following hierarchy: $G \subset S \subset L \subset U \subset ID$. Thus the following question recently posed by Jurek naturally arises:

If $X, X_1, X_2, \ldots$ are independent and identically distributed (i.i.d.) random variables, what are the conditions on the distribution and the canonical form of $r_n$ so that there exist $x_n$ with (1.1) converging weakly to a standard normal distribution?

We have found that the condition on the distribution is that the tail must decay rapidly. We have:

**Theorem 1.** Let $X, X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) random variables. Let

$$G(x) = \int_0^\infty tP(|X| > t + x)dt.$$

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If

(i) \( 0 < G(x) < \infty \) for all \( x > 0 \) and

(ii) \( \lim_{x \to -\infty} G(x + \epsilon)/G(x) = 0 \) for all \( \epsilon > 0 \)

then for any \( r_n \to \infty \) satisfying

\[
(1.3) \quad \lim_{n \to \infty} 2nG(r_n) = 1
\]

and \( x_n = -nE[\max(|X| - r_n, 0)\operatorname{sgn}(X)] \) we have

\[
U_{r_n}(X_1) + U_{r_n}(X_2) + \cdots + U_{r_n}(X_n) + x_n \Rightarrow N(0, 1)
\]

where \( \Rightarrow \) denotes weak convergence.

Remark 1. It is easy to see that \( G(x) < \infty \) for some \( x > 0 \) if and only if \( EX^2 < \infty \).

When \( EX^2 < \infty \), \( G(x) \) is a non-increasing and continuous function. Therefore the solution \( r_n \) to (1.3) exists. Moreover, \( r_n \) must tend to \( \infty \) under condition (i).

Remark 2. For \( X \) with bounded support, one can use similar techniques as given below to show that the conclusion to Theorem 1 remains valid although one has to modify the choice of \( r_n \) so that \( n\operatorname{Var}\left(\max(|X| - r_n, 0)\operatorname{sgn}(X)\right) \to 1 \).

It is easy to check that (ii) is satisfied for the normal distribution. However, many other standard distributions, such as the exponential distribution, do not satisfy (ii). On the other hand, as a partial converse, our second theorem shows that one cannot have a central limit theorem type result for (1.1) for such distributions:

**Theorem 2.** Let \( X, X_1, X_2, \ldots \) be i.i.d. random variables. If there exists an \( \varepsilon_0 > 0 \) such that

\[
(1.4) \quad \lim_{x \to -\infty} \frac{P(|X| > x + \varepsilon_0)/P(|X| > x)}{c_0 > 0},
\]

then there do not exist \( r_n \to \infty \) and real numbers \( x_n \) such that

\[
U_{r_n}(X_1) + U_{r_n}(X_2) + \cdots + U_{r_n}(X_n) + x_n \Rightarrow N(0, 1).
\]

Thus, for the Weibull family, with density functions:

\[
f(x; a) = ax^{a-1}\exp(x^{-a}), \quad x > 0, \quad a > 0,
\]

we see that for \( a > 1 \), (1.1) converges weakly to a normal distribution with the proper choice of \( r_n \) given in Theorem 1, but no such Central Limit Theorem result holds for \( 0 < a \leq 1 \).

Our final result is that there is no non-degenerate limiting distribution for (1.1) for regular distributions:

**Theorem 3.** Let \( X, X_1, X_2, \ldots \) be i.i.d. random variables with density function \( f(x) \). Assume that there exist \( p > 1 \), \( c_1 \geq 0 \), \( c_2 \geq 0 \), \( c_1 + c_2 > 0 \) and a slowly varying function \( l(x) \) (at \( \infty \)) such that

\[
(1.5) \quad \lim_{x \to +\infty} \frac{f(x)}{(l(x)/x^p)} = c_1, \quad \lim_{x \to +\infty} \frac{f(-x)}{(l(x)/x^p)} = c_2.
\]

Then for any \( r_n \to \infty \) and real numbers \( x_n \), \( U_{r_n}(X_1) + U_{r_n}(X_2) + \cdots + U_{r_n}(X_n) + x_n \) does not have a non-degenerate limiting distribution. If \( r_n \) satisfies

\[
(1.6) \quad \frac{n(l(r_n))}{r_n^{p-1}} \to 0,
\]

then

\[
U_{r_n}(X_1) + U_{r_n}(X_2) + \cdots + U_{r_n}(X_n) \to 0 \quad \text{in probability.}
\]
2. Proofs

Throughout our proofs we make extensive use of the following theorem in [1], Theorem 18, p. 95. The statement of the theorem is modified slightly here to match our problem:

**Theorem (A).** Let \( \{Y_n, Y_{n,1}, Y_{n,2}, \ldots, Y_{n,n}\} \) be a sequence of series of random variables i.i.d. within each series satisfying that for all \( \varepsilon > 0 \), \( P(Y_n \geq \varepsilon) \to 0 \). There will exist a sequence of real constants \( x_n \) such that the distribution of the sums \( \sum_{k=1}^{n} Y_{nk} + x_n \) converges weakly to the standard normal distribution if and only if the following conditions are satisfied:

\[
(2.1) \quad nP(|Y_n| \geq \varepsilon) \to 0 \quad \text{for every } \varepsilon > 0
\]

and

\[
(2.2) \quad n \text{Var} \left( Y_n I_{\{|Y_n| \leq 1\}} \right) \to 1.
\]

If these conditions are satisfied, then we may write

\[
x_n = -nE(Y_n I_{\{|Y_n| \leq 1\}}) + o(1).
\]

**Proof of Theorem 1.** Let \( Y_{n,k} = U_{r_n}(X_k) = \max(|X_k| - r_n, 0) \text{sgn}(X_k) \) and \( Y_n = U_{r_n}(X) \). In order to show that we may choose our \( x_n \) as given in Theorem 1 without truncating the tail as in Theorem A, we will show

\[
(2.3) \quad nE|Y_n|I_{\{|Y_n| > 1\}} \to 0.
\]

In order to show that the variance term (2.2) tends to 1, we will show

\[
(2.4) \quad nEY_n^2 I_{\{|Y_n| \leq 1\}} \to 1
\]

and

\[
(2.5) \quad n \left( E|Y_n|I_{\{|Y_n| \leq 1\}} \right)^2 \to 0.
\]

First, in order to show (2.1), observe that \( P(|Y_n| > \varepsilon) = P(|X| > r_n + \varepsilon) \). Since \( P(|X| > r_n + \varepsilon) \leq P(|X| > t + r_n + \varepsilon/2) \) for all \( 0 \leq t \leq \varepsilon/2 \), we see

\[
P(|X| > r_n + \varepsilon) \leq \frac{8}{\varepsilon^2} \int_{0}^{\varepsilon/2} tP(|X| > t + r_n + \varepsilon/2)dt \leq \frac{8}{\varepsilon^2} G(r_n + \varepsilon/2).
\]

Now (2.1) follows by our choice of \( r_n \) and condition (ii).

To prove (2.3), we have

\[
E|Y_n|I_{\{|Y_n| > 1\}} = E(|X| - r_n)I_{\{|X| > r_n + 1\}} = P(|X| > r_n + 1) + \int_{1/2}^{\infty} P(|X| > r_n + 1/2 + t)dt \leq P(|X| > r_n + 1) + 2 \int_{1/2}^{\infty} tP(|X| > r_n + 1/2 + t)dt \leq P(|X| > r_n + 1) + 2G(r_n + 1/2).
\]

Note that \( P(|X| > r_n + 1) \leq 8G(r_n + 1/2) \) by the proof of (2.1) above with \( \varepsilon = 1 \). Thus, \( E|Y_n|I_{\{|Y_n| > 1\}} \leq 10G(r_n + 1/2) \) which yields (2.3) immediately by our choice of \( r_n \) and condition (ii).
To prove (2.4), let $G_1(x) = \int_x^\infty P(|X| > t)dt$. Note that

$$EY_n^2 I_{\{|Y_n| \leq 1\}} = E(|X| - r_n)^2 I_{[r_n < |X| \leq r_n + 1]}$$

$$= \int_0^1 t^2d(-P(|X| > r_n + t))$$

$$= -P(|X| > 1 + r_n) + 2 \int_0^1 tP(|X| > r_n + t)dt$$

$$= -P(|X| > 1 + r_n) + 2 \int_0^1 td(-G_1(t + r_n))$$

$$= -P(|X| > 1 + r_n) - 2G_1(1 + r_n) + 2 \int_0^1 \int_t^\infty P(|X| > s + r_n)dsdt$$

$$= -P(|X| > 1 + r_n) - 2 \int_{1+r_n}^{\infty} P(|X| > t)dt$$

$$+ 2 \int_0^1 \int_t^\infty P(|X| > s + r_n)dsdt + 2 \int_1^{\infty} P(|X| > s + r_n)ds$$

$$= -P(|X| > 1 + r_n) + 2 \int_0^1 sP(|X| > s + r_n)ds$$

$$= -P(|X| > 1 + r_n) + 2G(r_n) - 2 \int_1^{\infty} sP(|X| > s + r_n)ds.$$

This proves (2.4) by (2.1), our choice of $r_n$, and by noting that

$$\int_1^{\infty} sP(|X| > s + r_n)ds \leq \int_{1/2}^{\infty} 2sP(|X| > s + r_n + 1/2)ds \leq 2G(r_n + 1/2)$$

so that $n \int_1^{\infty} sP(|X| > s + r_n)ds \to 0$.

Finally, we prove (2.5). It follows from the Hölder inequality that

$$n \left( E|Y_n|I_{\{|Y_n| \leq 1\}} \right)^2 = n \left( E(|X| - r_n)I_{[r_n < |X| \leq 1 + r_n]} \right)^2$$

$$\leq n P(r_n < |X| \leq 1 + r_n)E(|X| - r_n)^2 I_{[r_n < |X| \leq 1 + r_n]} \to 0$$

by (2.4) and the fact that $P(|X| > r_n) \to 0$ since $r_n \to \infty$.

This completes the proof of Theorem 1. \qed

**Proof of Theorem 2.** Since $r_n \to \infty$, we have

$$\forall \varepsilon > 0, \quad P(\max(|X| - r_n, 0) > \varepsilon) \to 0.$$

Assume that (1.2) converges weakly to a standard normal distribution. Then, by Theorem A again, we have

$$\forall \varepsilon > 0, \quad n P(|X| \geq r_n + \varepsilon) = n P(\max(|X| - r_n, 0) > \varepsilon) \to 0$$

and

$$n \Var \left( \max(|X| - r_n, 0)\mathrm{sgn}(X)I_{\{\max(|X| - r_n, 0)\mathrm{sgn}(X) < 1\}} \right) \to 1.$$

Thus, for sufficiently large $n$

$$1/2 \leq n \left( E(|X| - r_n)^2 I_{[r_n < |X| \leq 1 + r_n]} \right)$$

$$\leq n P(|X| > r_n),$$
which together with (2.6) yields
\[ \forall \varepsilon > 0, \quad \lim_{n \to \infty} P(|X| > r_n + \varepsilon) / P(|X| > r_n) = 0. \]
This contradicts the assumption (1.4) and completes the proof of Theorem 2.

**Proof of Theorem 3.** Assume that (1.1) has a non-degenerate limiting distribution for some \( r_n \to \infty \) and \( \{x_n\} \). Let \( Y_n = U_{r_n}(X) = \max(|X| - r_n, 0) \text{sgn}(X) \). Then
\[ (2.7) \quad e^{itY_n} (Ee^{itY_n})^n \to g(t), \]
where \( g(t) \) is a non-degenerate characteristic function. Write
\[ Ee^{itY_n} = E \cos(t \max(|X| - r_n, 0)) + iE \sin(t \max(|X| - r_n, 0) \text{sgn}(X)). \]

For \( t \neq 0 \), we have
\[
E \cos(t \max(|X| - r_n, 0)) = 1 - P(|X| > r_n) + \int_0^\infty \cos(x) d(-P(|X| > r_n + x/|t|))
\]
\[ = 1 - \int_0^\infty \sin(x) P(|X| > r_n + x/|t|) dx. \]

Put
\[ I_n(t) = \int_0^\infty \sin(x) P(|X| > r_n + x/|t|) dx. \]

Using the properties of regular distributions, one can check that \( I_n(t) \) is equal to
\[
\sum_{k=0}^{\infty} \left\{ \int_{2k\pi}^{2(k+1)\pi} \sin(x) P(|X| > r_n + x/|t|) dx + \int_{(2k+1)\pi}^{2(k+2)\pi} \sin(x) P(|X| > r_n + x/|t|) dx \right\}
\]
\[ = \sum_{k=0}^{\infty} \int_0^\pi \sin(x) \left\{ P(|X| > r_n + (2k\pi + x)/|t|) \right. \]
\[ - P(|X| > r_n + ((2k+1)\pi + x)/|t|) \}
\[ \left. \right\} dx \]
\[ = \int_0^\pi \sin(x) \left\{ \sum_{k=0}^{\infty} P((2k\pi + x)/|t| + r_n < |X| \leq r_n + ((2k+1)\pi + x)/|t|) \right\} dx \]
\[ \sim \int_0^\pi \sin(x) \left\{ \sum_{k=0}^{\infty} \frac{(c_1 + c_2)\pi/|t| l(r_n + (2k\pi + x)/|t|)}{(r_n + (2k\pi + x)/|t|)^p} \right\} dx \quad [\text{by (1.5)}] \]
\[ \sim \left\{ \sum_{k=0}^{\infty} \frac{(c_1 + c_2)\pi/|t| l(r_n + 2k\pi/|t|)}{(r_n + 2k\pi/|t|)^p} \right\} \int_0^\pi \sin(x) dx \]
\[ \sim 2(c_1 + c_2)(\pi/|t|) \int_0^\infty \frac{l(r_n + 2x\pi/|t|)}{(r_n + 2x\pi/|t|)^p} dx \]
\[ = (c_1 + c_2) \int_0^\infty \frac{l(r_n + x)}{(r_n + x)^p} dx \]
\[ \sim \frac{(c_1 + c_2) l(r_n)}{(p - 1)r_n^{p-1}}. \]
Therefore
\begin{equation}
E \cos(t \max(|X| - r_n, 0)) - 1 \sim - \frac{(c_1 + c_2)l(r_n)}{(p - 1)r_n^{p-1}}.
\end{equation}

To estimate \( E \sin(t \max(|X| - r_n, 0) \text{sgn}(X)) \), write \( E \sin(t \max(|X| - r_n, 0) \text{sgn}(X)) = E \sin(t(X - r_n))I_{(X > r_n)} - E \sin(t(-X - r_n))I_{(-X > r_n)} \).

Following the proof of (2.8), we have
\[
E \sin(t(|X| - r_n)I_{(X > r_n)}) = \int_0^\infty \sin(tx)d(-P(X > r_n + x))
\]
\[
= t \int_0^\infty \cos(tx)P(X > r_n + x)dx
\]
\[
= (t/|t|) \int_0^\infty \cos(x)P(X > r_n + x/|t|)dx
\]
\[
= \text{sgn}(t) \int_0^{\pi/2} \cos(x)P(X > r_n + x/|t|)dx
\]
\[
+ \text{sgn}(t) \sum_{k=0}^{\infty} \left\{ \int_{(2k+1)\pi/2}^{\pi/2} \cos(x)P(X > r_n + x/|t|)dx
\right. 
\]
\[
+ \int_{(2k+1)\pi/2}^{\pi/2} \cos(x)P(X > r_n + x/|t|)dx \}
\]
\[
= \text{sgn}(t) \int_0^{\pi/2} \cos(x)P(X > r_n + x/|t|)dx
\]
\[
+ \text{sgn}(t) \sum_{k=0}^{\infty} \left\{ \int_0^\pi \sin(x)\left\{ -P(X > r_n + (x + 2k\pi + \pi/2)/|t|)
\right. 
\right. 
\]
\[
+ P(X > r_n + (x + (2k + 1)\pi + \pi/2)/|t|) \}
\right. 
\]
\[
\sim \text{sgn}(t) \frac{c_1l(r_n)}{r_n^p} \int_0^{\pi/2} \cos(x)dx
\]
\[
-(c_1\pi/|t|)\text{sgn}(t) \sum_{k=0}^{\infty} \frac{l(r_n + 2k\pi/|t|)}{(r_n + 2k\pi/|t|)^p} \int_0^\pi \sin(x)dx
\]
\[
\sim - \frac{c_1\text{sgn}(t)l(r_n)}{(p - 1)r_n^{p-1}}.
\]

Similarly,
\[
E \sin(t(-X - r_n)I_{(-X > r_n)}) \sim - \frac{c_2\text{sgn}(t)l(r_n)}{(p - 1)r_n^{p-1}}.
\]

Hence
\begin{equation}
E \sin(t \max(|X| - r_n, 0) \text{sgn}(X)) \sim (c_2 - c_1)\text{sgn}(t) \frac{l(r_n)}{(p - 1)r_n^{p-1}}.
\end{equation}

From (2.8) and (2.9) it follows that
\begin{equation}
(EE^{itY_n})^n = \left( 1 - \frac{l(r_n)}{(p - 1)r_n^{p-1}}(c_1 + c_2 + isgn(t)(c_1 - c_2) + o(1)) \right)^n.
\end{equation}
Consider four different cases:

**Case 1.** When \( \lim \inf_{n \to \infty} \frac{n l(r_n)}{r_n^{p-1}} < \lim \sup_{n \to \infty} \frac{n l(r_n)}{r_n^{p-1}} \), (2.7) contradicts (2.10).

**Case 2.** When \( \lim_{n \to \infty} \frac{n l(r_n)}{r_n^{p-1}} = \infty \), (2.10) implies that for every \( t \neq 0 \),

\[ (E e^{i t Y_n})^n \to 0, \]

which is also in contradiction with (2.7).

**Case 3.** When \( \lim_{n \to \infty} \frac{n l(r_n)}{r_n^{p-1}} = a \), where \( 0 < a < \infty \), we obtain from (2.10) that

\[ \forall \ t \neq 0, \quad (E e^{i t Y_n})^n \to \exp \left( -\frac{a}{p-1}(c_1 + c_2 + \text{sgn}(t)(c_1 - c_2)) \right). \]

Since \( (E e^{i0 Y_n})^n \equiv 1 \), the limiting function of \( (E e^{i t Y_n})^n \) is not continuous, which contradicts the assumption that \( g(t) \) is a characteristic function.

**Case 4.** When \( \lim_{n \to \infty} \frac{n l(r_n)}{r_n^{p-1}} = 0 \), then we have

\[ (E e^{i t Y_n})^n \to 1 \]

and hence the limiting distribution is degenerate.

This completes the proof of the theorem.

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