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SETS OF p-POWERS AS CONJUGACY CLASS SIZES

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Abstract. We show that any finite set of powers of a fixed prime p which includes 1 can be the set of conjugacy class sizes of a p-group of nilpotency class 2. This corresponds to a result of Isaacs for degrees of irreducible characters.

Introduction

If G is a finite group, we denote by $cd(G)$ and $cs(G)$ the sets of numbers which occur as the degrees of the irreducible characters of G and as the sizes of the conjugacy classes of G respectively. Results about the set of irreducible character degrees sometimes correspond to similar results about the set of conjugacy class sizes. Here we give an instance of such a correspondence. Let p be a prime, and let $\mathcal{S}$ be a set of powers of p containing $p^0 = 1$. Isaacs proves in [1] that there is a p-group P of class 2 for which $cd(P) = \mathcal{S}$. We will prove an analogous result for the set of conjugacy class sizes.

Theorem. *Let p be a prime and $\mathcal{S}$ a finite set of p-powers containing* 1. *Then there exists a p-group P of class 2 with the property that $cs(P) = \mathcal{S}$.*

Proof of the Theorem

The set of G-conjugates of an element x of a group G will be denoted by x^G and the minimal number of generators of G by $d(G)$.

Denote the given set $\mathcal{S}$ of powers of the prime p by

$$\mathcal{S} = \{p^{\alpha_0}, p^{\alpha_1}, ..., p^{\alpha_n}\}$$

with the convention that $0 = \alpha_0 < \alpha_1 < \alpha_2 < ... < \alpha_n$. We will construct a group $P_{\mathcal{S}}$ satisfying the following conditions:

- $cs(P_{\mathcal{S}}) = \mathcal{S}$,
- $P_{\mathcal{S}}$ has class 2, exponent p if p is odd, and exponent 4 if $p = 2$, and
- $d(P_{\mathcal{S}}) = \alpha_n + 1$.

The construction requires some ideas from the theory of varieties of groups. We refer the reader to Hanna Neumann's book [2] for the meaning of our notation and for any unexplained ideas and results.

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When p is odd, $\mathfrak{V}_p$ will denote the variety of p-groups of class at most 2 and exponent p (note that since p is odd, $\mathfrak{V}_p$ contains nonabelian groups); $\mathfrak{V}_2$ will denote the variety generated by the dihedral group of order 8. For a positive integer n, we denote by F_n the free group of rank n in the variety $\mathfrak{V}_p$ and by A_n the elementary abelian p-group of rank n. Note that $F_n/\Phi(F_n)$ is isomorphic to A_n, where $\Phi(F_n)$ is the Frattini subgroup of F_n. Note also that $F_n' \leq \zeta(F_n)$ (since F_n has class 2) and that pth powers are central (since they are trivial if p is odd and squares are central in the dihedral group of order 8). Thus $\Phi(F_n) \leq \zeta(F_n)$. We will need the following crucial fact about F_n $(n \geq 2)$:

Elements x and y of F_n that are independent modulo $\Phi(F_n)$ do not commute.
To see this, observe that $\langle x, y\rangle$ is a free group of rank 2 in $\mathfrak{V}_p$ with x and y as free generators by [2], Theorem 42.31. If x commuted with y, then $[x, y] = 1$ would be a law in $\mathfrak{V}_p$ by [2], Corollary 13.25, and then $\mathfrak{V}_p$ would be abelian, a contradiction. It follows, in particular, that $\zeta(F_n) = \Phi(F_n)$.

We will also need the following special case of the varietal product (see [2], Section 1.8, for a detailed account of this construction). Suppose that X, Y are groups in $\mathfrak{V}_p$. Then the $\mathfrak{V}_p$-product $X *_{\mathfrak{V}_p} Y$ of X and Y is defined by

$$X *_{\mathfrak{V}_p} Y = (X * Y)/V,$$

where $X * Y$ is the free product of X and Y and V is the verbal subgroup of $X * Y$ corresponding to $\mathfrak{V}_p$. We note that $V \leq [X, Y]$ since $[X, Y]$ is the kernel of the natural homomorphism of $X * Y$ onto $X \times Y$, and so by definition

$$[X, Y] \leq \zeta(X *_{\mathfrak{V}_p} Y).$$

Next, let $x \in X \setminus \Phi(X)$ and $y \in Y \setminus \Phi(Y)$, and let M, N be maximal subgroups of X and Y respectively with $x \notin M$, $y \notin N$. By [2], Theorem 18.42, there is an epimorphism of $X *_{\mathfrak{V}_p} Y$ onto $(X/M) *_{\mathfrak{V}_p} (Y/N)$. Since X/M and Y/N have order p, the product $(X/M) *_{\mathfrak{V}_p} (Y/N)$ is $\mathfrak{V}_p$-free of rank 2 if p is odd (by [2], Corollary 18.43), while if $p = 2$, it is easy to see that $(X/M) *_{\mathfrak{V}_p} (Y/N)$ is the dihedral group of order 8. In either case, the images of x and y in $(X/M) *_{\mathfrak{V}_p} (Y/N)$ do not commute, whence $[x, y] \neq 1$ in $X *_{\mathfrak{V}_p} Y$. We will need the following consequences of this fact. Let X and Y be nontrivial groups in $\mathfrak{V}_p$ and set $G = X *_{\mathfrak{V}_p} Y$; then $\zeta(G) = \Phi(X)\Phi(Y)[X, Y] = \Phi(G)$ and so $d(G/\zeta(G)) = d(X) + d(Y)$.

We will now show by induction on $|\mathcal{S}| = n + 1$ that, for $\mathcal{S} = \{p^{\alpha_0}, p^{\alpha_1}, ..., p^{\alpha_n}\}$ with $0 = \alpha_0 < \alpha_1 < \alpha_2 < ... < \alpha_n$, we can construct a group $P_{\mathcal{S}}$ in $\mathfrak{V}_p$ on $\alpha_n + 1$ generators satisfying $cs(P_{\mathcal{S}}) = \mathcal{S}$ and also, when $n > 0$, $\zeta(P_{\mathcal{S}}) = \Phi(P_{\mathcal{S}})$. If $n = 0$, we choose $P_{\mathcal{S}} = A_1$, for which we clearly have $cs(P_{\mathcal{S}}) = \mathcal{S}$ and $d(P_{\mathcal{S}}) = 1 = \alpha_0 + 1$. For $n = 1$ and $\mathcal{S} = \{1, p^{\alpha}\}$, we choose $P_{\mathcal{S}} = F_{\alpha+1}$. In this case, we have $d(P_{\mathcal{S}}) = \alpha + 1$ and $\zeta(P_{\mathcal{S}}) = \Phi(P_{\mathcal{S}})$ since $P_{\mathcal{S}}$ is free in $\mathfrak{V}_p$. To see that $cs(P_{\mathcal{S}}) = S$, suppose that x is chosen in $P_{\mathcal{S}}$ but not in $\Phi(P_{\mathcal{S}})$. From earlier observations, we have $C_{P_{\mathcal{S}}}(x) = \Phi(P_{\mathcal{S}})\langle x\rangle$ and therefore $|P_{\mathcal{S}}/C_{P_{\mathcal{S}}}(x)| = p^{\alpha}$. It follows immediately that $P_{\mathcal{S}}$ has only the two conjugacy class sizes 1 and p^{α}, as required.

Now let $\mathcal{S} = \{1, p^{\alpha_1}, ..., p^{\alpha_n}\}$ with $n \geq 2$, and set $\mathcal{S}^* = \{1, p^{\alpha_2-\alpha_1}, ..., p^{\alpha_{n-1}-\alpha_1}\}$. Since $|\mathcal{S}^*| = n-1$, our inductive hypothesis yields a group $P_{\mathcal{S}^*}$ in $\mathfrak{V}_p$ on $\alpha_{n-1} - \alpha_1 + 1$ generators with $cs(P_{\mathcal{S}^*}) = \mathcal{S}^*$ and $\zeta(P_{\mathcal{S}^*}) = \Phi(P_{\mathcal{S}^*})$. We now set

$$P_{\mathcal{S}} = F_{\alpha_1} *_{\mathfrak{V}_p} (P_{\mathcal{S}^*} \times A_{(\alpha_n - \alpha_{n-1})}),$$

the free $\mathfrak{V}_p$-product of F_{α_1} and $P_{\mathcal{S}^*} \times A_{(\alpha_n-\alpha_{n-1})}$. Observe that $\zeta(P_{\mathcal{S}}) = \Phi(P_{\mathcal{S}})$ and that $d(P_{\mathcal{S}}) = d(F_{\alpha_1}) + d(P_{\mathcal{S}^*} \times (A_{(\alpha_n-\alpha_{n-1})}))$ by our remark above. In particular,

$d(P_{\mathcal{S}}) = \alpha_1 + \alpha_{n-1} - \alpha_1 + 1 + \alpha_n - \alpha_{n-1} = \alpha_n + 1$. To complete the induction, it remains to show that $cs(P_{\mathcal{S}}) = \mathcal{S}$.

For notational convenience, set $X = F_{\alpha_1}$ and $Y = P_{\mathcal{S}^*} \times A_{(\alpha_n - \alpha_{n-1})}$; also write $P = P_{\mathcal{S}}$. We will now analyse the possible conjugacy class sizes for the noncentral elements of P. There are three cases.

(1) First, we consider an element $y \in A_{\alpha_n - \alpha_{n-1}}$. Then $C_P(y) = \zeta(P)Y = \Phi(X)[X,Y]Y$ and so $|y^P| = |X/\Phi(X)| = p^{\alpha_1}$.

(2) Next, we consider an element y in Y but not in $A_{\alpha_n - \alpha_{n-1}}$, writing $y = uv$ with $1 \neq u \in P_{\mathcal{S}^*}$ and $v \in A_{\alpha_n - \alpha_{n-1}}$. Since v is central in Y, we have $C_Y(y) = C_{P_{\mathcal{S}^*}}(u) \times A_{\alpha_n - \alpha_{n-1}}$ and $C_P(y) = \zeta(P)C_Y(y) = \Phi(X)[X,Y]C_Y(y)$. It now follows that for some $i \in \{2, ..., n-1\}$ we have

$$\begin{aligned} |y^P| &= |X/\Phi(X)|\,|Y/C_Y(y)| = |X/\Phi(X)|\,|P_{\mathcal{S}^*}/C_{P_{\mathcal{S}^*}}(u)| \\ &= |X/\Phi(X)|\,|Cl_{P_{\mathcal{S}^*}}(u)| = p^{\alpha_1}p^{\alpha_i - \alpha_1} = p^{\alpha_i}. \end{aligned}$$

For $i \in \{2, ..., n-1\}$ our induction hypothesis yields an element $w \in P_{\mathcal{S}^*}$ with $p^{\alpha_i - \alpha_1}$ conjugates in $P_{\mathcal{S}^*}$. The preceding calculation shows that $|w^P| = p^{\alpha_i}$, and therefore the conjugacy class sizes for the noncentral elements in $Y \setminus A_{\alpha_n - \alpha_{n-1}}$ are precisely p^{α_2}, p^{α_3}, $\dots$, and $p^{\alpha_{n-1}}$.

(3) Finally, we consider an element u of $P = P_{\mathcal{S}}$ not in $Y\zeta(P)$. Then $u = xyz$ with $x \in X \setminus \Phi(X) = X \setminus \zeta(X)$, $y \in Y$ and $z \in [X,Y]$. Suppose $u' = x'y'z' \in C_P(u)$, with $x' \in X$, $y' \in Y$ and $z' \in [X,Y]$. Now since $\Phi(P) = \Phi(X) \times \Phi(Y) \times [X,Y]$ and $[u,u'] = [x,x'][y,x'][x,y'][y,y'] = 1$ we have $[x,x'] = 1$ and $[x,y'] = [x',y]$. Then since X is free and $x \notin \Phi(X)$ we have $x' \in x^k\Phi(X)$ for some integer k and then $[x,y'] = [x^k,y] = [x,y^k]$. It now follows that $y'y^{-k} \in C_Y(x) = \Phi(Y)$. Thus $u' \in x^k y^k \zeta(P) = (xy)^k \zeta(P)$ and so $C_P(u) = \zeta(P)\langle u \rangle$ and therefore $|u^P| = p^{\alpha_1 - 1 + \alpha_{n-1} - \alpha_1 + 1 + \alpha_n - \alpha_{n-1}} = p^{\alpha_n}$.

Thus we have shown that $cs(P_{\mathcal{S}}) = \mathcal{S}$. This completes the induction step and with it the proof of the theorem.

References

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