

## TUNNEL NUMBERS OF SMALL KNOTS DO NOT GO DOWN UNDER CONNECTED SUM

KANJI MORIMOTO AND JENNIFER SCHULTENS

(Communicated by Ronald A. Fintushel)

ABSTRACT. Let  $K_1$  and  $K_2$  be two knots in  $S^3$  and  $t(K_1)$ ,  $t(K_2)$  the tunnel numbers of them. In this paper, we show that if both  $K_1$  and  $K_2$  are small, then  $t(K_1\#K_2) \geq t(K_1) + t(K_2)$ . Moreover we show that  $t(K_1\#K_2\#\cdots\#K_n) \geq t(K_1) + t(K_2) + \cdots + t(K_n)$  for any small knots  $K_1, K_2, \dots, K_n$ .

### 1. INTRODUCTION

Let  $K$  be a knot in the 3-sphere  $S^3$  and  $t(K)$  the tunnel number of  $K$ . Here,  $t(K)$  is the minimum number of mutually disjoint arcs properly embedded in  $E(K)$  whose exterior is a handlebody, where  $E(K) = cl(S^3 - N(K))$  and  $N(K)$  is a regular neighborhood of  $K$  in  $S^3$ . For two knots  $K_1$  and  $K_2$ , we denote the connected sum of them by  $K_1\#K_2$ .

Concerning the problem if tunnel numbers of knots go down or not under connected sum, in 1992 the first author showed the existence of those knots whose tunnel numbers go down. In fact he got:

**Theorem 1** ([Mo1, Theorem]). *There are infinitely many knots  $K$  such that  $t(K) = 2$  and  $t(K\#K') = 2$  for any 2-bridge knot  $K'$ .*

After then, by taking the connected sum of knots obtained by modifying those knots in Theorem 1, Kobayashi showed:

**Theorem 2** ([Ko, Theorem]). *For any positive integer  $n$ , there are infinitely many pairs of knots  $K_1$  and  $K_2$  such that  $t(K_1\#K_2) < t(K_1) + t(K_2) - n$ .*

Theorem 2 says that tunnel numbers of knots can arbitrarily highly degenerate. Moreover, we see that those knots in Theorem 1 and Theorem 2 have the property that the exteriors contain closed essential surfaces.

Now, we say that a knot  $K$  is small if  $E(K)$  contains no closed essential surfaces. Then the second author showed:

**Theorem 3** ([St2, Corollary 13]). *If both  $K_1$  and  $K_2$  are small, then*

$$t(K_1\#K_2) \geq t(K_1) + t(K_2) - 1.$$

---

Received by the editors March 1, 1998.

1991 *Mathematics Subject Classification*. Primary 57M25, 57N10.

*Key words and phrases*. Knots, connected sum, tunnel number.

This theorem says that tunnel numbers of small knots either do not go down or go down by one under connected sum. In this paper, we show that we can get rid of the term “ $-1$ ”. Namely we prove:

**Theorem 4.** *If both  $K_1$  and  $K_2$  are small, then*

$$t(K_1 \# K_2) \geq t(K_1) + t(K_2).$$

More generally, we will prove the following. We note that  $K \# K'$  is no longer small even if both  $K$  and  $K'$  are small.

**Theorem 5.** *For any small knots  $K_1, K_2, \dots, K_n$ ,*

$$t(K_1 \# K_2 \# \dots \# K_n) \geq t(K_1) + t(K_2) + \dots + t(K_n).$$

Throughout this paper, for an  $m$ -manifold  $M$  ( $m = 2$  or  $3$  resp.) and an  $n$ -manifold  $N$  ( $n = 1$  or  $2$  resp.) properly embedded in  $M$ , a component of  $M - N$  means the closure of a component of  $M - N$ . And for a manifold  $X$  and a subcomplex  $Y$  of  $X$ ,  $N(Y)$  denotes a regular neighborhood of  $Y$  in  $X$ .

## 2. PRELIMINARIES

Let  $K_1$  and  $K_2$  be two knots in  $S^3$ , and put  $K = K_1 \# K_2$  in  $S^3$ . Let  $N(K)$  be a regular neighborhood of  $K$  in  $S^3$ , and put  $E(K) = cl(S^3 - N(K))$ . Let  $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_t\}$  be an unknotting tunnel system for  $K$ , i.e.  $\gamma_i$  ( $i = 1, 2, \dots, t$ ) is an arc properly embedded in  $E(K)$  and  $cl(E(K) - N(\gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_t))$  is a genus  $t + 1$  handlebody, where  $N(\gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_t)$  is a regular neighborhood of  $\gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_t$  in  $E(K)$ .

Put  $V_1 = N(\partial E(K) \cup \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_t)$  in  $E(K)$ ,  $V_2 = cl(E(K) - V_1)$  and  $V_1 \cap V_2 = F$ . Then  $V_1$  is a genus  $t + 1$  compressionbody with  $\partial V_1 - F = \partial E(K)$  and  $V_2$  is a genus  $t + 1$  handlebody with  $\partial V_2 = F$ . Hence  $(V_1, V_2)$  is a genus  $t + 1$  Heegaard splitting of  $E(K)$ .

Let  $\Delta_i$  ( $i = 1, 2$ ) be a disk properly embedded in  $V_i$  with  $\partial \Delta_i \subset F$ . Then we say that  $\Delta_i$  is essential if  $\partial \Delta_i$  is an essential loop in  $F$ . We say that the Heegaard splitting  $(V_1, V_2)$  is reducible if there is an essential disk  $\Delta_i$  in  $V_i$  ( $i = 1, 2$ ) with  $\partial \Delta_1 = \partial \Delta_2$ , and that the Heegaard splitting  $(V_1, V_2)$  is irreducible if it is not reducible. Moreover according to [CG], we say that  $(V_1, V_2)$  is weakly reducible if there is an essential disk  $\Delta_i$  in  $V_i$  ( $i = 1, 2$ ) with  $\partial \Delta_1 \cap \partial \Delta_2 = \emptyset$ , that  $(V_1, V_2)$  is strongly irreducible if it is not weakly reducible.

Now, suppose that the unknotting tunnel system  $\Gamma$  for  $K = K_1 \# K_2$  realizes the tunnel number of  $K$ . Then the corresponding Heegaard splitting  $(V_1, V_2)$  of  $E(K)$  is irreducible. Thus hereafter we assume that the Heegaard splitting is irreducible. Then the second author showed:

**Lemma 6** ([St2, Theorem 9]). *Let both  $K_1$  and  $K_2$  be small, and suppose the corresponding Heegaard splitting  $(V_1, V_2)$  is weakly reducible. Then  $t(K_1 \# K_2) \geq t(K_1) + t(K_2)$ .*

Let  $S$  be the 2-sphere giving the connected sum of  $K = K_1 \# K_2$ . Then we can put  $S \cap N(K) = D_1^* \cup D_2^*$ , where  $D_i^*$  ( $i = 1, 2$ ) is a meridian disk of  $N(K)$ . Put  $A = cl(S - (D_1^* \cup D_2^*))$ ; then  $A$  is a separating essential annulus properly embedded in  $E(K)$ . Then the second author showed:

**Lemma 7** ([St2, Lemma 6]). *If the Heegaard splitting  $(V_1, V_2)$  is strongly irreducible, then after some ambient isotopy for  $A$ ,  $A \cap F$  consists of essential loops in both  $A$  and  $F$ .*

Moreover, she showed:

**Lemma 8** ([St2, Lemma 11]). *If both  $K_1$  and  $K_2$  are small, then  $(V_1, V_2)$  is weakly reducible, or we can choose the Heegaard splitting  $(V_1, V_2)$  so that  $A \cap F$  consists of two essential loops or of four essential loops in both  $A$  and  $F$ .*

*Remark 9.* The Heegaard splitting  $(V_1, V_2)$  in Lemma 8 may not be isotopic to the Heegaard splitting we take first because the argument in the proof of [St2, Lemma 11] exchanges the Heegaard splittings. In fact, the argument in the proof of [St2, Lemma 11] has been done under the two assumptions stated in [St2, section 4], but we have no those assumptions in this paper. Therefore the statement in the above lemma is somewhat different from that in [St2, Lemma 11].

### 3. PROOF OF THEOREM 4

In this section, we show the following, which is a refinement of Lemma 8.

**Lemma 10.** *If both  $K_1$  and  $K_2$  are small, then  $(V_1, V_2)$  is weakly reducible, or we can choose the Heegaard splitting  $(V_1, V_2)$  so that  $A \cap F$  consists of two essential loops in both  $A$  and  $F$ .*

*Remark 11.* If  $A \cap F$  consists of two essential loops in both  $A$  and  $F$ , then by a more detailed argument we can show that  $(V_1, V_2)$  is weakly reducible. Hence this lemma says that we can always take the Heegaard splitting of  $E(K_1 \# K_2)$  corresponding to the tunnel number  $t(K_1 \# K_2)$  to be weakly reducible.

*Proof.* We denote the number of components of  $A \cap F$  by  $|A \cap F|$ . Then by Lemma 8, we may assume that  $|A \cap F| = 4$  and each component of  $A \cap F$  is an essential loop in both  $A$  and  $F$ . Then since  $A$  is a separating essential annulus in  $E(K)$ , we can put  $A \cap V_1 = E_1 \cup E_2 \cup E_0$  and  $A \cap V_2 = G_1 \cup G_2$ , where  $E_i$  ( $i = 1, 2$ ) is an annulus in  $V_1$  connecting  $F$  and  $\partial E(K)$ ,  $E_0$  is an essential annulus in  $V_1$  with  $\partial E_0 \subset F$  and  $G_i$  ( $i = 1, 2$ ) is an essential annulus in  $V_2$ . Then we can regard  $E_0$  as a union of an essential disk  $D_0$  in  $V_1$  and a band  $b_0$ ,  $G_i$  ( $i = 1, 2$ ) as a union of an essential disk  $D_i$  in  $V_2$  and a band  $b_i$ . Since the annulus  $E_i$  ( $i = 1, 2$ ) extends to a non-separating disk  $E_i \cup D_i^*$  in the handlebody  $V_1 \cup N(K)$ ,  $E_i$  is a non-separating annulus in  $V_1$ . Moreover, since  $A$  is a separating annulus in  $E(K)$ ,  $E_1 \cup E_2 \cup E_0$  is a separating 2-manifold in  $V_1$ . Hence according to whether  $E_1 \cup E_2$  is a separating 2-manifold or not in  $V_1$ , we have the following two cases.

Case I:  $E_1 \cup E_2$  splits  $V_1$  into two components and  $E_0$  is a separating annulus in one of the two components (Figure 1(I)).

Case II:  $E_1 \cup E_2$  does not split  $V_1$  and  $E_0$  is a non-separating annulus in  $V_1$  such that  $E_1 \cup E_2 \cup D_0$  splits  $V_1$  into two components (Figure 1(II)).

Suppose we are in Case I.

In this case, since  $E_1 \cup E_2$  splits  $V_1$  into two components and  $D_0$  splits one of the two components into two components,  $b_0$  is contained in one of the two components. Then we have the following claim. Recall the annulus  $A$  consists of five annuli  $E_1 \cup G_1 \cup E_0 \cup G_2 \cup E_2$ .

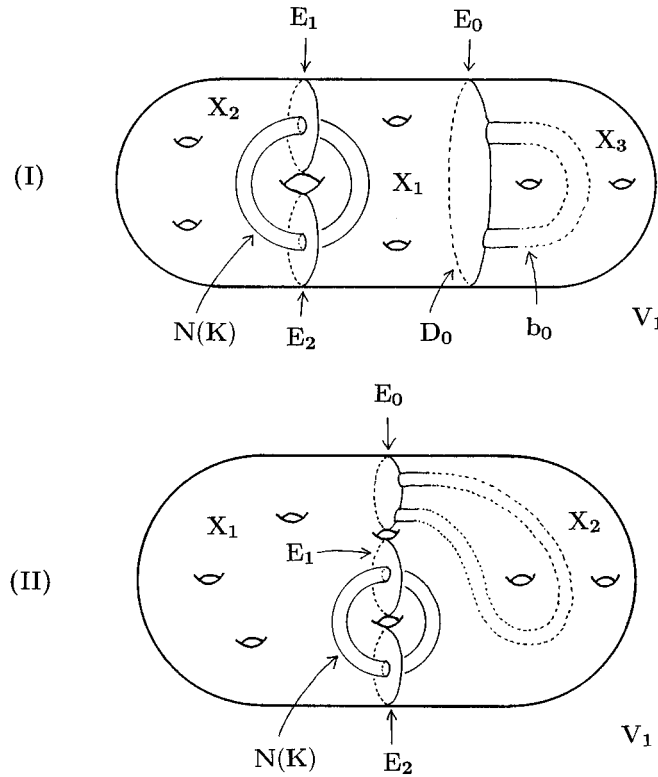


FIGURE 1

**Claim 12.** *We may assume that the band  $b_0$  is contained in the component of  $V_1 - (E_1 \cup E_2 \cup D_0)$  not meeting  $E_1 \cup E_2$  as illustrated in Figure 1(I).*

*Proof.* Suppose  $b_0$  is contained in the component of  $V_1 - (E_1 \cup E_2 \cup D_0)$  meeting  $E_1 \cup E_2$ . Push out the band  $b_1$  from  $V_2$  into  $V_1$ ; then a band in  $V_1$ , say  $b'_1$ , is produced. Since  $b_1$  is a part of  $G_1$ ,  $b'_1$  is a band connecting  $E_1$  and  $E_0$ . Then, since  $b_0$  is contained in the above component,  $b'_1$  does not run over  $b_0$ . Hence we can push out  $b_0$  from  $V_1$  into  $V_2$  leaving  $b'_1$  in  $V_1$ . Then after these ambient isotopies,  $A \cap F$  consists of two loops, and this completes the proof.  $\square$

Suppose one of  $G_1$  and  $G_2$  is a separating annulus in  $V_2$ ; then since  $A$  is a separating annulus in  $E(K)$ , the other is a separating annulus too. Then both  $D_1$  and  $D_2$  are separating disks in  $V_2$ , and  $D_1 \cup D_2$  splits  $V_2$  into three components  $R_0, R_1, R_2$ , where  $D_1 \subset \partial R_1$ ,  $D_2 \subset \partial R_2$  and  $D_1 \cup D_2 \subset \partial R_0$ .

Suppose the band  $b_1$  is not contained in  $R_1$ , and let  $H_1$  be the component of  $V_2 - G_1$  containing  $R_1$ . Then  $H_1 \cap F$  is a connected surface with two boundary components, and  $H_1 \cap F$  is identified with  $X_2 \cap F$  or  $X_3 \cap F$ . Hence  $G_1$  connects  $E_1$  and  $E_2$  or  $\partial G_1 = \partial E_0$ , a contradiction. If the band  $b_2$  is not contained in  $R_2$ , then we have the same contradiction. Thus  $b_i$  is contained in  $R_i$  ( $i = 1, 2$ ). Then  $G_1 \cup G_2$  splits  $V_2$  into three components  $H_1, H_2, H_0$ , where  $G_1 \subset \partial H_1$ ,  $G_2 \subset \partial H_2$  and  $G_1 \cup G_2 \subset \partial H_0$ . If one of  $H_1 \cap F$  and  $H_2 \cap F$  is connected, then we have a contradiction as above. Hence each of  $H_1 \cap F$  and  $H_2 \cap F$  has two components.

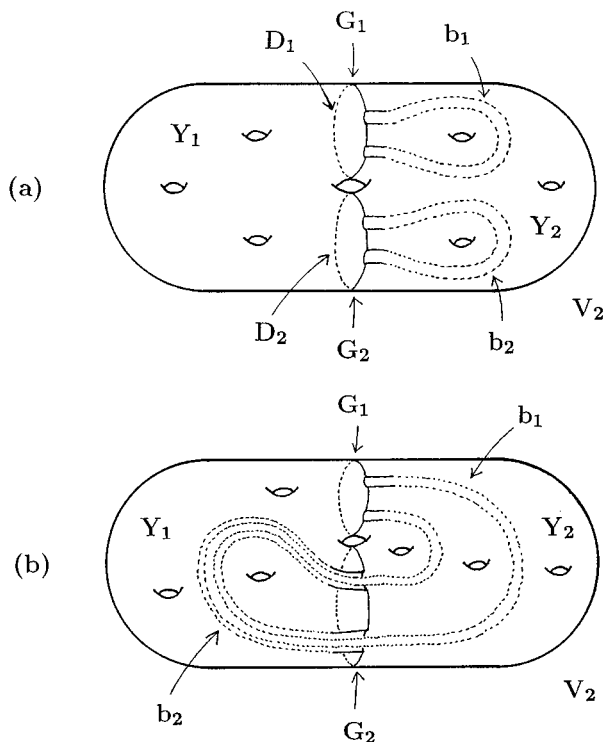


FIGURE 2

Then there is no component in  $H_1 \cap F$ ,  $H_2 \cap F$  and  $H_0 \cap F$  which has two boundary components. But  $X_2 \cap F$  has two boundary components, a contradiction.

Thus both  $G_1$  and  $G_2$  are non-separating annuli in  $V_2$  and  $G_1 \cup G_2$  splits  $V_2$  into two components (Figure 2). Figure 2(a) is the case when the bands  $b_1$  and  $b_2$  are contained in one of the two components  $V_2 - (D_1 \cup D_2)$ . Figure 2(b) is the case when  $b_2$  is contained in one of the two components  $V_2 - (D_1 \cup D_2)$  and  $b_1$  or a part of  $b_1$  is contained in the other component, and in the latter case  $b_1$  runs over the band  $b_2$ . We note that in the case of Figure 2(b) the band  $b_2$  does not run over the band  $b_1$ .

Let  $X_1$ ,  $X_2$  and  $X_3$  be the three components of  $V_1 - (E_1 \cup E_2 \cup E_0)$  indicated in Figure 1(I), and  $Y_1$  and  $Y_2$  the two components of  $V_2 - (G_1 \cup G_2)$  indicated in Figure 2.

**Claim 13.**  $X_1 \cap F$  is identified with  $Y_1 \cap F$ .

*Proof.* Suppose  $X_1 \cap F$  is identified with a part of  $Y_2 \cap F$ . Push out the band  $b_1$  from  $V_2$  into  $V_1$ , and let  $b'_1$  be the band in  $V_1$  produced by the ambient isotopy. Then  $b'_1$  is contained in  $X_2$  or  $X_3$ . This means that  $\partial G_1$  is contained in  $\partial(E_1 \cup E_2)$  or that  $\partial G_1 = \partial E_0$ . This is a contradiction because  $\partial G_1$  consists of a component of  $\partial E_1$  and a component of  $\partial E_0$ , and completes the proof.  $\square$

By the above claim,  $(X_2 \cup X_3) \cap F$  is identified with  $Y_2 \cap F$ . We denote the images of  $E_1$ ,  $E_2$ ,  $E_0$ ,  $G_1$  and  $G_2$  in  $\partial X_1$ ,  $\partial X_2$ ,  $\partial X_3$ ,  $\partial Y_1$  and  $\partial Y_2$  by the same

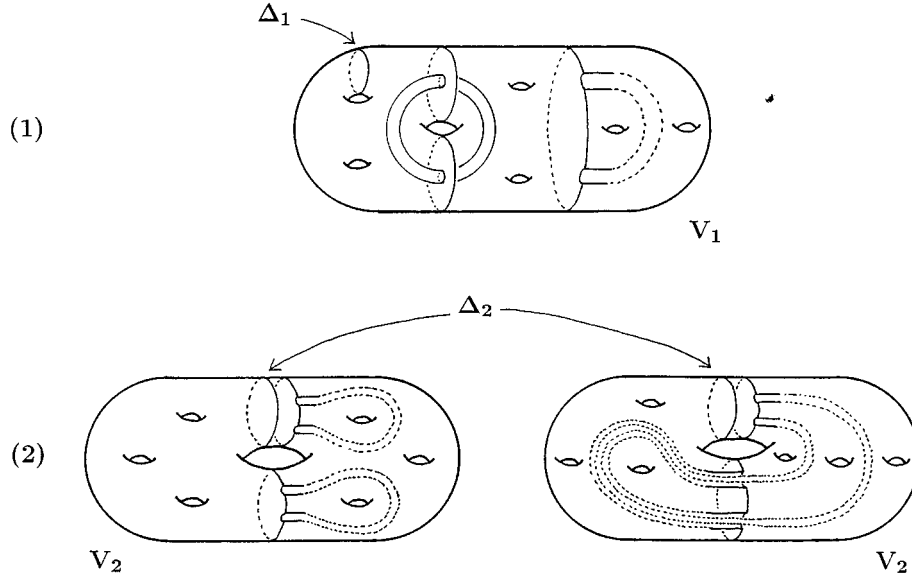


FIGURE 3

notations. According as  $X_2 \cap F$  is an annulus or not, we have the following two subcases.

Case I-(1):  $X_2 \cap F$  is not an annulus.

In this case, since  $X_2 \cap F$  has a positive genus, we can take an essential disk, say  $\Delta_1$ , properly embedded in  $X_2 \subset V_1$  indicated in Figure 3(1). Let  $\Delta_2$  be an essential disk properly embedded in  $Y_1 \subset V_2$  parallel to  $D_1$  indicated in Figure 3(2).

Then, since  $X_1 \cap F$  is identified with  $Y_1 \cap F$ ,  $\partial\Delta_1 \cap \partial\Delta_2 = \emptyset$ . This shows that  $(V_1, V_2)$  is weakly reducible.

Case I-(2):  $X_2 \cap F$  is an annulus.

Let  $N_1$  and  $N_2$  be the two components of  $N(K) - (D_1^* \cup D_2^*)$  so that  $N_i \cap X_i$  ( $i = 1, 2$ ) is an annulus. Put  $B_1 = X_1 \cup N_1 \cup Y_1$  and  $B_2 = X_2 \cup N_2$ . Then  $B_2$  is a 3-ball. And since  $B_1$  is a component of  $S^3 - S$ ,  $B_1$  is a 3-ball too. Moreover, since  $B_1 \cap X_3 = X_1 \cap X_3 = E_0$  is an annulus,  $B_1$  is a 2-handle for the handlebody  $X_3$  along  $E_0$ . And since  $B_2 \cap Y_2 = X_2 \cap F$  is an annulus,  $B_2$  is a 2-handle for the handlebody  $Y_2$ .

Put  $W_1 = B_1 \cup X_3$  and  $W_2 = B_2 \cup Y_2$ . Then  $W_1 \cup W_2 = S^3$  and  $W_1 \cap W_2 = \partial W_1 = \partial W_2$ . Hence at least one of  $W_1$  and  $W_2$  has a compressible boundary. Then we have the following two subcases.

(i):  $\partial W_1$  is compressible in  $W_1$ .

Suppose  $W_1$  is a 3-ball. Then  $X_3$  is a solid torus. If the annulus  $E_0$  winds around the handle of  $X_3$  more than once, then by [Mo2, Proposition 1.3]  $S^3$  has a lens space summand, a contradiction. Hence  $E_0$  winds around the handle of  $X_3$  exactly once, and we can push out  $E_0$  from  $V_1$  into  $V_2$ . This makes  $|A \cap F|$  to be 2. Hence we may assume that  $W_1$  is not a 3-ball. Then  $W_1$  has a compressible boundary with a positive genus. Then by [Ja, Theorem 2] (cf. [Sr1, Lemma 1.1]),  $\partial X_3 - E_0$  is compressible in  $X_3$ . Then there is a compressing disk, say  $\Delta_1$ , for  $\partial X_3 - E_0$  in  $X_3$ . And, since  $E_0$  is an incompressible annulus in  $\partial X_3$ ,  $\Delta_1$  is an essential disk in

$X_3$  and in  $V_1$ . Let  $\Delta_2$  be an essential disk properly embedded in  $Y_1 \subset V_2$  parallel to  $D_1$  indicated in Figure 3(2). Then  $\partial\Delta_1 \cap \partial\Delta_2 = \emptyset$ . This shows that  $(V_1, V_2)$  is weakly reducible.

(ii):  $\partial W_2$  is compressible in  $W_2$ .

Put  $B_2 \cap Y_2 = G_3$ . Then  $G_3$  is an annulus in  $\partial Y_2$ , and  $\partial G_3$  consists of a component of  $\partial G_1$  and a component of  $\partial G_2$ . In this case, the four components  $\partial G_1 \cup \partial G_2$  are all mutually parallel to each other in  $\partial Y_2$ .

If  $W_2$  is a 3-ball, then  $Y_2$  is a solid torus. Then  $\partial Y_2 - (G_1 \cup G_2 \cup G_3)$  is an annulus, and it is identified with  $\partial X_3 - E_0$ . Hence  $X_3$  is a solid torus, and by the argument in the case (i), we may assume that  $W_2$  is not a 3-ball. Then by the argument in the case (i) ([Ja, Theorem 2]), we have an essential disk, say  $\Delta_2$ , properly embedded in  $Y_2 \subset V_2$  with  $\partial\Delta_2 \subset \partial Y_2 - (G_1 \cup G_2 \cup G_3)$ . Let  $\Delta_1$  be an essential disk properly embedded in  $X_1 \subset V_1$  parallel to  $D_0$  (then  $\Delta_1$  is a separating disk in  $X_1 \subset V_1$ ). Then  $\partial\Delta_1 \cap \partial\Delta_2 = \emptyset$ , and this shows that  $(V_1, V_2)$  is weakly reducible. This completes the proof of Case I.

Suppose we are in Case II.

Let  $X_1$  and  $X_2$  be the two components of  $V_1 - (E_1 \cup E_2 \cup E_0)$  indicated in Figure 1(II). Suppose one of  $G_1$  and  $G_2$  is a separating annulus in  $V_2$ ; then by the same reason as in Case I, the other is a separating annulus too. Then both  $D_1$  and  $D_2$  are separating disks in  $V_2$ , and  $D_1 \cup D_2$  splits  $V_2$  into three components  $R_0, R_1, R_2$ , where  $D_1 \subset \partial R_1$ ,  $D_2 \subset \partial R_2$  and  $D_1 \cup D_2 \subset \partial R_0$ .

Suppose the band  $b_1$  is not contained in  $R_1$ , and let  $H_1$  be the component of  $V_2 - G_1$  containing  $R_1$ . Then  $H_1 \cap F$  is a connected surface with two boundary components, and  $H_1 \cap F$  is not identified with  $X_1 \cap F$ . Then by the argument in the proof of Claim 12, we can change the bands  $b_0$  and  $b_1$  and reduce the number of  $|A \cap F|$ . Hence  $b_i$  is contained in  $R_i$  ( $i = 1, 2$ ). Then  $G_1 \cup G_2$  splits  $V_2$  into three components  $H_1, H_2$  and  $H_0$ , where  $G_1 \subset \partial H_1$ ,  $G_2 \subset \partial H_2$  and  $G_1 \cup G_2 \subset \partial H_0$ . Since  $H_0 \cap F$  is a connected surface with four boundary components,  $H_0 \cap F$  is identified with  $X_1 \cap F$  and  $(H_1 \cap F) \cup (H_2 \cap F)$  is identified with  $X_2 \cap F$ .

Consider the incompressibility of  $H_1 \cap F$ . If  $H_1 \cap F$  is compressible in  $H_1 \subset V_2$ , then we have a compressing disk, say  $\Delta_2$ , for  $H_1 \cap F$  properly embedded in  $H_1 \subset V_2$ , and let  $\Delta_1$  be the essential disk properly embedded in  $X_1 \subset V_1$  parallel to  $D_0$ . Then  $\partial\Delta_1 \cap \partial\Delta_2 = \emptyset$ . This shows that  $(V_1, V_2)$  is weakly reducible, a contradiction. If  $H_1 \cap F$  is compressible in  $X_2 \subset V_1$ , then we have a compressing disk, say  $\Delta_1$ , for  $H_1 \cap F$  properly embedded in  $X_2 \subset V_1$ , and let  $\Delta_2$  be the essential disk properly embedded in  $H_0 \subset V_2$  parallel to  $D_1$ . Then  $\partial\Delta_1 \cap \partial\Delta_2 = \emptyset$ , a contradiction. Hence  $H_1 \cap F$  is incompressible in both  $H_1$  and  $X_2$ .

Glue  $H_1 \cup H_2$  and  $X_2$  along  $(H_1 \cap F) \cup (H_2 \cap F) = X_2 \cap F$ , and get  $E(K_2) = H_1 \cup H_2 \cup X_2$ . By this identification,  $X_2 \cap F$  has two components each of which has two boundary components. Hence  $H_1 \cap F$  is a connected surface with two boundary components. Put  $Q = H_1 \cap F$ . Then  $Q$  is an incompressible surface properly embedded in  $E(K_2)$ , and  $\partial Q$  consists of two meridian loops of  $N(K_2)$ . If  $Q$  is not an annulus, then  $Q$  is essential and by [CGLS, Theorem 2.0.3(iii)],  $E(K_2)$  contains a closed essential surface, a contradiction. Thus  $Q$  is an annulus and hence  $H_1$  is a solid torus. Then by [Mo2, Proposition 1.3],  $G_1$  is parallel to an annulus in  $\partial V_2$ . Then we can reduce the number of  $|A \cap F|$ , a contradiction. Thus both  $G_1$  and  $G_2$  are non-separating annuli.

Let  $Y_1$  and  $Y_2$  be the two components of  $V_2 - (G_1 \cup G_2)$  indicated in Figure 2. Then by the argument similar to the proofs of Claims 12 and 13,  $X_i \cap F$  ( $i = 1, 2$ )

resp.) is identified with  $Y_i \cap F$  ( $i = 1, 2$  resp.) We denote the images of  $E_1, E_2, E_0, G_1$  and  $G_2$  in  $\partial X_1, \partial X_2, \partial Y_1$  and  $\partial Y_2$  by the same notations.

If  $X_2 \cap F$  is compressible in  $X_2$ , then we have a compressing disk, say  $\Delta_1$ , for  $X_2 \cap F$  properly embedded in  $X_2 \subset V_1$ . Let  $\Delta_2$  be an essential disk in  $Y_1 \subset V_2$  indicated in Figure 3(2); then  $\partial\Delta_1 \cap \partial\Delta_2 = \emptyset$ . This shows that  $(V_1, V_2)$  is weakly reducible. If  $Y_2 \cap F$  is compressible in  $Y_2$ , then we have a compressing disk, say  $\Delta_2$ , for  $Y_2 \cap F$  properly embedded in  $Y_2 \subset V_2$ . Let  $\Delta_1$  be an essential disk in  $X_1 \subset V_1$  parallel to  $D_0$ ; then  $\partial\Delta_1 \cap \partial\Delta_2 = \emptyset$ . This shows that  $(V_1, V_2)$  is weakly reducible. Hence we assume that  $X_2 \cap F$  ( $Y_2 \cap F$  resp.) is incompressible in  $X_2$  ( $Y_2$  resp.).

Glue  $X_2$  and  $Y_2$  along  $X_2 \cap F = Y_2 \cap F$ . Then we get  $E(K_2) = X_2 \cup Y_2$ . Put  $Q = X_2 \cap F = Y_2 \cap F$ . Then  $Q$  is an incompressible 2-manifold properly embedded in  $E(K_2)$ , and  $\partial Q$  consists of four meridian loops of  $N(K_2)$ . If  $Q$  has a component which is not an annulus, then the component is an essential surface in  $E(K_2)$  and by [CGLS, Theorem 2.0.3(iii)],  $E(K_2)$  contains a closed essential surface, a contradiction.

Suppose  $Q$  consists of two annuli. Then  $X_2$  is a solid torus homeomorphic to (an annulus, say  $R$ )  $\times [1, 4]$  so that  $R \times \{1\} = E_1$ ,  $R \times \{4\} = E_2$  and (a component of  $\partial R$ )  $\times [2, 3] = E_0$ . Put  $E_3 = R \times \{\frac{5}{2}\}$ . Then, since a component of  $\partial E_3$  is contained in  $E_0 \subset A$ , the component splits  $A$  into two annuli  $A_1$  and  $A_2$ . And, since small knots are prime, one of  $A_1 \cup E_3$  and  $A_2 \cup E_3$ , say  $A_1 \cup E_3$ , is a decomposing annulus of  $E(K_1 \# K_2)$  and  $A_2$  is isotopic rel.  $\partial E(K_1 \# K_2)$  to  $E_3$ . This reduces the number of  $|A \cap F|$  and completes the proof of Case II and Lemma 10.  $\square$

*Proof of Theorem 4.* Let  $(V_1, V_2)$  be a Heegaard splitting corresponding to an unknotting tunnel system for  $K = K_1 \# K_2$  which realizes the tunnel number of  $K$ . Then  $(V_1, V_2)$  is irreducible, and by Lemma 10 we may assume that  $(V_1, V_2)$  is weakly reducible or the decomposing 2-sphere of  $K_1 \# K_2$  intersects the Heegaard surface in two essential loops. In the former case, by Lemma 6 we have  $t(K_1 \# K_2) \geq t(K_1) + t(K_2)$ . In the latter case, by the argument in the proof of Case 1 of [St2, Theorem 12], we have  $t(K_1 \# K_2) \geq t(K_1) + t(K_2)$ . This completes the proof of Theorem 4.  $\square$

#### 4. PROOF OF THEOREM 5

Put  $K = K_1 \# K_2 \# \cdots \# K_n$ , and let  $(V_1, V_2)$  be the Heegaard splitting of  $E(K)$  corresponding to an unknotting tunnel system for  $K$  which realizes the tunnel number of  $K$ .

**Lemma 14.** *If  $(V_1, V_2)$  is weakly reducible, then*

$$t(K_1 \# K_2 \# \cdots \# K_n) \geq t(K_{i_1} \# \cdots \# K_{i_j}) + t(K_{i_{j+1}} \# \cdots \# K_{i_n})$$

for some  $j$ .

*Proof.* Since  $(V_1, V_2)$  is weakly reducible, by considering the untelescoping of  $(V_1, V_2)$  and by [CG, Theorem 3.1] and [St2, Remark 7] (cf. [LM, Theorem 1.3]), there is a closed incompressible surface  $S$  in  $E(K)$  such that  $S$  splits  $(V_1, V_2)$  into two Heegaard splittings  $(V_1^1, V_2^1)$  and  $(V_1^2, V_2^2)$ . Then  $(V_1, V_2)$  is an amalgamation of  $(V_1^1, V_2^1)$  and  $(V_1^2, V_2^2)$  along the surface  $S$ , and  $g(V_1, V_2) = g(V_1^1, V_2^1) + g(V_1^2, V_2^2) - g(S)$ , where  $g(\cdot)$  is the genus of the Heegaard splitting or is the genus of the surface. For the definition of amalgamation, see [St1]. And for the definition of the untelescoping, see [St2, Definition 12], [Sr2] or [ST].



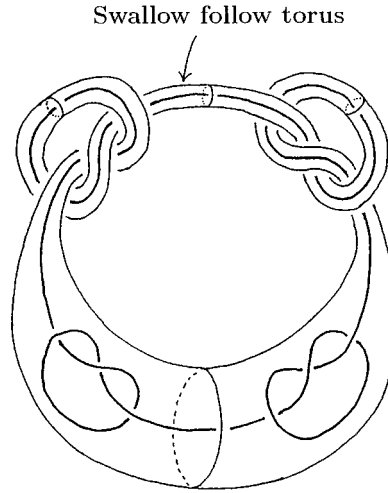


FIGURE 4

If  $S$  is boundary parallel in  $E(K)$ , then  $E(K)$  has a lower genus Heegaard splitting than  $g(V_1, V_2)$ , a contradiction. Hence  $S$  is essential. Then by [St2, Lemma 14],  $S$  is a swallow follow torus as illustrated in Figure 4, and  $S$  splits  $E(K)$  into  $E(K_{i_1} \# \cdots \# K_{i_j})$  (a solid torus) and  $E(K_{i_{j+1}} \# \cdots \# K_{i_n})$ . Then, since  $(V_1^1, V_2^1)$  ( $(V_1^2, V_2^2)$  resp.) is a Heegaard splitting of  $E(K_{i_1} \# \cdots \# K_{i_j})$  (a solid torus) ( $E(K_{i_{j+1}} \# \cdots \# K_{i_n})$  resp.),  $t(K_{i_1} \# \cdots \# K_{i_j}) \leq g(V_1^1, V_2^1) - 1$  and  $t(K_{i_{j+1}} \# \cdots \# K_{i_n}) \leq g(V_1^2, V_2^2) - 1$ . Thus by noting  $g(S) = 1$ , we have  $t(K_{i_1} \# \cdots \# K_{i_j}) + t(K_{i_{j+1}} \# \cdots \# K_{i_n}) \leq g(V_1^1, V_2^1) - 1 + g(V_1^2, V_2^2) - 1 = g(V_1, V_2) - 1 = t(K_1 \# K_2 \# \cdots \# K_n)$ . This completes the proof.  $\square$

Let  $S_1, S_2, \dots, S_{n-1}$  be the decomposing 2-spheres giving the connected sum of  $K = K_1 \# K_2 \# \cdots \# K_n$ . Put  $A_i = S_i \cap E(K)$  ( $i = 1, 2, \dots, n-1$ ). Then  $A_i$  is a separating essential annulus properly embedded in  $E(K)$ . By Claim 2 in the proof of [St2, Theorem 15], we have:

**Lemma 15.**  *$(V_1, V_2)$  is weakly reducible, or we can choose the Heegaard splitting  $(V_1, V_2)$  and the decomposing 2-spheres so that  $F$  intersects at most one of the decomposing annuli in 2 or 4 essential loops and intersects the others in 2 essential loops in both  $F$  and the annuli.*

By Lemmas 14 and 15, and by exchanging the decomposing 2-spheres if necessary, we can put  $|F \cap A_{i_1}| = 2$  or 4 and  $|F \cap A_{i_j}| = 2$  ( $j = 2, \dots, n-1$ ). Then by repeating the argument in the proof of Case 1 of [St2, Theorem 12], we have  $t(K_1 \# K_2 \# \cdots \# K_n) \geq t(K_{i_1} \# K_{i_2}) + t(K_{i_3}) + \cdots + t(K_{i_n})$ . Moreover by Theorem 4,  $t(K_{i_1} \# K_{i_2}) \geq t(K_{i_1}) + t(K_{i_2})$ . This completes the proof of Theorem 5.  $\square$

## REFERENCES

- [CG] A. J. Casson and C. McA. Gordon, *Reducing Heegaard splittings*, Topology Appl. **27**, (1987) 275-283. MR **89c**:57020
- [CGLS] M. Culler, C. McA. Gordon, J. Lueke and P. B. Shalen, *Dehn surgery on knots*, Ann. of Math. **125**, (1987) 237-300. MR **88a**:57026

- [Ja] W. Jaco, *Adding a 2-handle to a 3-manifold: an application to property R*, Proc. A. M. S. **92**, (1984) 288-292. MR **86b**:57006
- [Ko] T. Kobayashi, *A construction of arbitrarily high degeneration of tunnel numbers of knots under connected sum*, J. Knot. Rami. **3**, (1994) 179-186. MR **95g**:57011
- [LM] M. Lustig and Y. Moriah, *Closed incompressible surfaces in complements of wide knots and links*, Topology Appl. **92** (1999), no. 1, 1–13. CMP 99:07
- [Mo1] K. Morimoto, *There are knots whose tunnel numbers go down under connected sum*, Proc. A. M. S. **123**, (1995) 3527-3532. MR **96a**:57022
- [Mo2] ———, *Planar surfaces in a handlebody and a theorem of Gordon-Reid*, Proc. Knots 96 ed. by S.Suzuki (1997) 123-146. CMP 99:07
- [Sr1] M. Scharlemann, *Outermost forks and a theorem of Jaco*, Contemporary Math. **44**, (1985) 189-193. MR **87e**:57021
- [Sr2] ———, *Heegaard splittings of compact 3-manifolds*, Preprint.
- [ST] M. Scharlemann and A. Thompson, *Thin position for 3-manifolds*, A.M.S. Contemporary Math. **164** (1994) 231-238. MR **95e**:57032
- [St1] J. Schultens, *The classification of Heegaard splittings for (compact orientable surface)  $\times S^1$* , London Math. Soc. **67** (1993) 425-448. MR **94d**:57043
- [St2] ———, *Additivity of tunnel number for small knots*, Preprint.

DEPARTMENT OF MATHEMATICS, TAKUSHOKU UNIVERSITY TATEMACHI, HACHIOJI, TOKYO 193, JAPAN

*E-mail address:* morimoto@la.takushoku-u.ac.jp

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, EMORY UNIVERSITY, ATLANTA, GEORGIA 30322

*E-mail address:* jcs@mathcs.emory.edu