PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 1, Pages 223–228
S 0002-9939(99)05242-9
Article electronically published on June 30, 1999

EXTENSIONS OF HEINZ-KATO-FURUTA INEQUALITY

MASATOSHI FUJII AND RITSUO NAKAMOTO

(Communicated by David R. Larson)

ABSTRACT. We give an extension of Lin's recent improvement of a generalized Schwarz inequality, which is based on the Heinz-Kato-Furuta inequality. As a consequence, we can sharpen the Heinz-Kato-Furuta inequality.

1. INTRODUCTION

First of all, we cite a generalized Schwarz inequality which is a base of Lin's recent paper [9]. For a (bounded linear) operator T acting on a Hilbert space H,

$$|(Tx,y)|^2 \le (|T|^{2\alpha}x,x)(|T^*|^{2(1-\alpha)}y,y) \tag{1}$$

for all $\alpha \in [0,1]$ and $x,\ y \in H$, where $|X|$ is the square root of X^*X for an operator X on H. It implies the Heinz-Kato inequality via the Löwner-Heinz inequality, cf. [3],[10]. On the other hand, Furuta [7] extended the Heinz-Kato inequality, the so-called Heinz-Kato-Furuta inequality. Rephrasing it parallel to (1), we have

$$|(T|T|^{\alpha+\beta-1}x,y)|^2 \le (|T|^{2\alpha}x,x)(|T^*|^{2\beta}y,y) \tag{2}$$

for all $\alpha,\ \beta \in [0,1]$ with $\alpha+\beta \ge 1$ and $x,\ y \in H$.

Very recently, Lin [9] sharpened (1) as follows:

Theorem L. *Let T be an operator on H and $0 \ne y \in H$. For $z \in H$ satisfying $Tz \ne 0$ and $(Tz,y)=0$,*

$$|(Tx,y)|^2 + \frac{|(|T|^{2\alpha}x,z)|^2(|T^*|^{2(1-\alpha)}y,y)}{(|T|^{2\alpha}z,z)} \le (|T|^{2\alpha}x,x)(|T^*|^{2(1-\alpha)}y,y) \tag{3}$$

for all $\alpha \in [0,1]$ and $x,\ y \in H$. The equality holds if and only if $|T|^{2\alpha}(x-\frac{(|T|^{2\alpha}x,z)}{(|T|^{2\alpha}z,z)}z)$ and T^*y are proportional, or equivalently, $Tx-\frac{(|T|^{2\alpha}x,z)}{(|T|^{2\alpha}z,z)}Tz$ and $|T^*|^{2(1-\alpha)}y$ are proportional.

In this note, we extend Theorem L, which is based on the Heinz-Kato-Furuta inequality (2). Our proof is quite simple, in which we clarify the meaning of the assumption in Theorem L that $Tz \ne 0$ and $(Tz,y)=0$. As a consequence, we can sharpen the Heinz-Kato-Furuta inequality, and Furuta's further generalization [6, Theorem 3] of the Heinz-Kato inequality via the Furuta inequality [4]. Incidentally we discuss Bernstein type inequality along the lines of our result.

Received by the editors November 3, 1997 and, in revised form, March 23, 1998.
1991 *Mathematics Subject Classification.* Primary 47A30, 47A63.
Key words and phrases. Heinz inequality, Heinz-Kato-Furuta inequality, Furuta inequality.

2. Heinz-Kato-Furuta inequality

For the sake of convenience, we first cite the Heinz-Kato-Furuta inequality [7]:

Let T be an operator on H. If A and B are positive operators on H such that $T^*T \le A^2$ and $TT^* \le B^2$, then

$$|(T|T|^{\alpha+\beta-1}x, y)| \le \|A^\alpha x\|\|B^\beta y\| \tag{4}$$

for all $\alpha, \beta \in [0,1]$ with $\alpha+\beta \ge 1$ and $x, y \in H$.

We here remark that the Heinz-Kato inequality is just the case $\alpha + \beta = 1$ in above and that it corresponds to (1). Thus we have the following extension of Theorem L. Throughout this paper, let $T = U|T|$ be the polar decomposition of an operator T on H.

Theorem 1. *Let T be an operator on H and $0 \ne y \in H$. For $z \in H$ satisfying $T|T|^{\alpha+\beta-1}z \ne 0$ and $(T|T|^{\alpha+\beta-1}z, y) = 0$,*

$$|(T|T|^{\alpha+\beta-1}x, y)|^2 + \frac{|(|T|^{2\alpha}x, z)|^2(|T^*|^{2\beta}y, y)}{(|T|^{2\alpha}z, z)} \le (|T|^{2\alpha}x, x)(|T^*|^{2\beta}y, y) \tag{5}$$

for all $\alpha, \beta \ge 0$ with $\alpha+\beta \ge 1$ and $x, y \in H$. In the case $\alpha, \beta > 0$, the equality in (5) holds if and only if $|T|^{\alpha+\beta-1}T^*y$ and $|T|^{2\alpha}(x - \frac{(|T|^{2\alpha}x,z)}{(|T|^{2\alpha}z,z)}z)$ are proportional, or equivalently, $|T^*|^{2\beta}y$ and $T|T|^{\alpha+\beta-1}(x - \frac{(|T|^{2\alpha}x,z)}{(|T|^{2\alpha}z,z)}z)$ are proportional.

It is easily seen that Theorem L is the case $\alpha+\beta = 1$ in Theorem 1. As a consequence, we have the following improvement of the Heinz-Kato-Furuta inequality via the Löwner-Heinz inequality, i.e., $A \ge B \ge 0$ implies $A^\alpha \ge B^\alpha$ for $\alpha \in [0,1]$:

Theorem 2. *Let T be an operator on H. If A and B are positive operators on H such that $T^*T \le A^2$ and $TT^* \le B^2$, then*

$$|(T|T|^{\alpha+\beta-1}x, y)|^2 + \frac{|(|T|^{2\alpha}x, z)|^2(|T^*|^{2\beta}y, y)}{(|T|^{2\alpha}z, z)} \le \|A^\alpha x\|^2\|B^\beta y\|^2 \tag{6}$$

for all $\alpha, \beta \in [0,1]$ with $\alpha+\beta \ge 1$ and $x, y, z \in H$ such that $T|T|^{\alpha+\beta-1}z \ne 0$ and $(T|T|^{\alpha+\beta-1}z, y) = 0$. In the case $\alpha, \beta > 0$, the equality in (6) holds if and only if $A^{2\alpha}x = |T|^{2\alpha}x$, $B^{2\beta}y = |T^*|^{2\beta}y$ and $|T|^{\alpha+\beta-1}T^*y$ and $|T|^{2\alpha}(x - \frac{(|T|^{2\alpha}x,z)}{(|T|^{2\alpha}z,z)}z)$ are proportional; the third condition is equivalent to the condition that $|T^*|^{2\beta}y$ and $T|T|^{\alpha+\beta-1}(x - \frac{(|T|^{2\alpha}x,z)}{(|T|^{2\alpha}z,z)}z)$ are proportional.

Proof of Theorem 1. We only use the positivity of the Gram matrix

$$G = G(U|T|^\alpha x, |T^*|^\beta y, U|T|^\alpha z).$$

Noting that

$$(|T^*|^\beta y, U|T|^\alpha z) = (y, |T^*|^\beta U|T|^\alpha z) = (y, T|T|^{\alpha+\beta-1}z) = 0$$

by the assumption, we have

$$G = \begin{pmatrix} \||T|^\alpha x\|^2 & (U|T|^\alpha x, |T^*|^\beta y) & (U|T|^\alpha x, U|T|^\alpha z) \\ (U|T|^\alpha x, |T^*|^\beta y)^* & \||T^*|^\beta y\|^2 & 0 \\ (U|T|^\alpha x, U|T|^\alpha z)^* & 0 & \||T|^\alpha z\|^2 \end{pmatrix}.$$

Since $|T|^\alpha z \ne 0$, we have

$$|(T|T|^{\alpha+\beta-1}x, y)|^2 + \frac{|(|T|^{2\alpha}x, z)|^2(|T^*|^{2\beta}y, y)}{(|T|^{2\alpha}z, z)} \le (|T|^{2\alpha}x, x)(|T^*|^{2\beta}y, y).$$

To prove the equality condition, we set up the following lemma, which is applied to the vectors $u = U|T|^\alpha x,\ v = U|T|^\alpha z$ and $w = |T^*|^\beta y$. □

Lemma. (1) *If $v \neq 0$ and $(v, w) = 0$, then $\{u, v, w\}$ is linearly dependent if and only if w and $u - \frac{(u,v)}{\|v\|^2} v$ are proportional.*

(2) *Let $T = U|T|$ be the polar decomposition of an operator T on H, (namely $\ker(U) = \ker(T)$). For $\alpha, \beta > 0$ with $\alpha + \beta \geq 1$ and $y, w \in H$, the following conditions are mutually equivalent*:

(i) *$|T^*|^\beta y$ and $U|T|^\alpha w$ are proportional.*
(ii) *$|T|^{\alpha+\beta-1} T^* y$ and $|T|^{2\alpha} w$ are proportional.*
(iii) *$|T^*|^\beta y$ and $T|T|^{\alpha-1} w$ are proportional.*

Proof. (1) Suppose that $au + bv + cw = 0$ for some $(a, b, c) \neq 0$. Then $a(u, v) + b\|v\|^2 = 0$ and so $b = -\frac{a(u,v)}{\|v\|^2}$. Hence we have

$$0 = au + bv + cw = a(u - \frac{(u, v)}{\|v\|^2} v) + cw.$$

Since $a = c = 0$ does not occur by $v \neq 0$, vectors $u - \frac{(u,v)}{\|v\|^2} v$ and w are proportional. The converse is easily checked.

(2) (i) is equivalent to the statement that $U|T|^\beta U^* y$ and $U|T|^\alpha w$ are proportional. Noting that $\alpha,\ \beta > 0$ and $\ker(U) = \ker(T)$, it is equivalent to (ii). Similarly we have the equivalence between (i) and (iii). □

3. Furuta inequality

In [6], the Heinz-Kato-Furuta inequality is extended by the use of the Furuta inequality; Theorem 1 also gives us an improvement of the extension due to Furuta. For the sake of convenience, we cite the Furuta inequality [4]; see also [2],[5],[8].

The Furuta inequality. *If $A \geq B \geq 0$, then for each $r \geq 0$,*

$$(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q}$$

holds for $p \geq 0$ and $q \geq 1$ with

$$(*) \qquad (1 + 2r)q \geq p + 2r.$$

The domain representing (*) is drawn in Figure 1 and it is shown in [11] that this domain is the *best possible one* for the Furuta inequality.

Theorem 3. *Let T be an operator on H. If A and B are positive operators on H such that $T^*T \leq A^2$ and $TT^* \leq B^2$. Then for each $r,\ s \geq 0$*

$$(7) \qquad |(T|T|^{(1+2r)\alpha+(1+2s)\beta-1} x, y)|^2 + \frac{|(|T|^{2(1+2r)\alpha} x, z)|^2 (|T^*|^{2(1+2s)\beta} y, y)}{(|T|^{2(1+2r)\alpha} z, z)}$$

$$\leq ((|T|^{2r} A^{2p} |T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}} x, x)((|T^*|^{2s} B^{2q} |T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}} y, y)$$

for all $p,\ q \geq 1,\ \alpha, \beta \in [0, 1]$ with $(1+2r)\alpha + (1+2s)\beta \geq 1$ and $x, y, z \in H$ such that $T|T|^{(1+2r)\alpha+(1+2s)\beta-1} z \neq 0$ and $(T|T|^{(1+2r)\alpha+(1+2s)\beta-1} z, y) = 0$. In the case $\alpha, \beta > 0$, the equality in (7) holds if and only if $|T|^{2(1+2r)\alpha} x = (|T|^{2r} A^{2p} |T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}} x$, $|T^*|^{2(1+2s)\beta} y = (|T^*|^{2s} B^{2q} |T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}} y$ and $|T|^{2(1+2r)\alpha}(x - \frac{(|T|^{2(1+2r)\alpha} x, z)}{(|T|^{2(1+2r)\alpha} z, z)} z)$ and

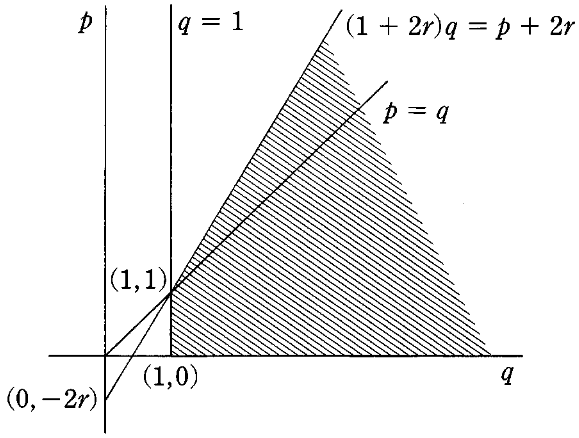

FIGURE 1

$|T|^{2(1+2r)\alpha+2(1+2s)\beta-1}T^*y$ are proportional; the latter is equivalent to the statement that $T|T|^{(1+2r)\alpha+(1+2s)\beta-1}(x-\frac{(|T|^{2(1+2r)\alpha}x,z)}{(|T|^{2(1+2r)\alpha}z,z)}z)$ and $|T^*|^{2(1+2s)\beta}y$ are proportional.

Proof. We use Theorem 1 by replacing α (resp. β) by $\alpha_1=(1+2r)\alpha$ (resp. $\beta_1=(1+2s)\beta$). Then we have

$$(8)\quad |(T|T|^{\alpha_1+\beta_1-1}x,y)|^2+\frac{|(|T|^{2\alpha_1}x,z)|^2(|T^*|^{2\beta_1}y,y)}{(|T|^{2\alpha_1}z,z)}\le(|T|^{2\alpha_1}x,x)(|T^*|^{2\beta_1}y,y).$$

Next we use the Furuta inequality for $|T|^2\le A^2$ and $|T^*|^2\le B^2$; namely (for the former) we replace $A,\ B;\ q$ in the Furuta inequality by $A^2,\ |T|^2;\ \frac{p+2r}{(1+2r)\alpha}$ respectively. Then we have

$$|T|^{2\alpha_1}=|T|^{2(1+2r)\alpha}\le(|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}}$$

and similarly

$$|T^*|^{2\beta_1}=|T^*|^{2(1+2s)\beta}\le(|T^*|^{2s}B^{2q}|T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}}.$$

Combining with (8), we obtain the inequality (7).

The equality condition is shown similarly to Theorem 2. □

Remark. (1) We remark that the condition $(1+2r)\alpha+(1+2s)\beta\ge1$ in Theorem 3 is unneccessary if T is either positive or invertible.

(2) Though Theorem 3 follows from the Furuta inequality, they are equivalent actually, that is, Theorem 3 is an alternative representation of the Furuta inequality. As a matter of fact, we put $T=B,\ \alpha=\beta,\ r=s$ and also $x=y$ in Theorem 3. Thus it follows from the above remark (1) that if $A^2\ge B^2$, then for $B^{2(1+2r)\alpha}z\ne0$ and $(B^{2(1+2r)\alpha}z,x)=0$

$$|(B^{2(1+2r)\alpha}x,x)|^2+\frac{|(B^{2(1+2r)\alpha}x,z)|^2(B^{2(1+2r)\alpha}x,x)}{(B^{2(1+2r)\alpha}z,z)}$$
$$\le((B^{2r}A^{2p}B^{2r})^{\frac{(1+2r)\alpha}{p+2r}}x,x)(B^{2(1+2)\alpha}x,x),$$

that is, $A^2 \geq B^2$ ensures

$$(B^{2(1+2r)\alpha}x, x)^2 \leq ((B^{2r}A^{2p}B^{2r})^{\frac{(1+2r)\alpha}{p+2r}}x, x)$$

for all $p \geq 1,\ r \geq 0$ and $\alpha \in [0,1]$. This is nothing but the Furuta inequality.

4. A concluding remark

Lin also discussed Bernstein type inequalities independently of Theorem L [9, Theorem 3], see [1]. As an application of Theorem 1, we have a generalization of it:

Theorem 4. *Let T be an operator on H having a nonzero normal eigenvalue λ with an eigenvector e. If $y \in H$ satisfies $(e, y) = 0$ and $T^*y \neq 0$, then*

$$|\lambda|^2|(x,e)|^2 \leq \frac{\|Tx\|^2\||T^*|^\beta T^*y\|^2 - |(T|T|^\beta x, T^*y)|^2}{\||T^*|^\beta T^*y\|^2}$$

for all $x \in H$ and $\beta \in [0,1]$.

Proof. We put $\alpha = 1$, $z = e$ and replace y by T^*y in Theorem 1. Since $(|T|^\beta e, T^*y) = 0$ by $(e, y) = 0$, it follows from Theorem 1 that

$$|(T|T|^\beta x, T^*y)|^2 + \||T^*|^\beta T^*y\|^2|\lambda|^2|(x,e)|^2 \leq \|Tx\|^2\||T^*|^\beta T^*y\|^2,$$

so that we have the desired inequality. □

We obtain Lin's inequality [9, Theorem 3] by taking $\beta = 0$ in Theorem 4.

Acknowledgement

The authors would like to express their thanks to Prof. T.Furuta for his valuable comments and suggestions based on his own aesthetic sense.

References

1. H.J.Bernstein, *An inequality for selfajoint operators on a Hilbert space*, Proc. Amer. Math. Soc., **100** (1987), 319-321. MR **89b:**47013a
2. M.Fujii, *Furuta's inequality and its mean theoretic approach*, J. Operator theory, **23** (1990), 67-72. MR **91g:**47012
3. M.Fujii and T.Furuta, *Löwner-Heinz, Cordes and Heinz-Kato inequalities*, Math. Japon., **38** (1993), 73-78. MR **94d:**47005
4. T.Furuta, *$A \geq B \geq 0$ assures $(B^rA^pB^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$*, Proc. Amer. Math. Soc., **101** (1987), 85-88. MR **89b:**47028
5. T.Furuta, *Elementary proof of an order preserving inequality*, Proc. Japan Acad., **65** (1989), 126. MR **90g:**47029
6. T.Furuta, *Determinant type generalizations of the Heinz-Kato theorem via the Furuta inequality*, Proc. Amer. Math. Soc., **120** (1994), 223-231. MR **94b:**47025
7. T.Furuta, *An extension of the Heinz-Kato theorem*, Proc. Amer. Math. Soc., **120** (1994), 785-787. MR **94e:**47034
8. E.Kamei, *A satellite to Furuta's inequality*, Math. Japon., **33** (1988), 883-886. MR **89m:**47011
9. C.-S.Lin, *Heinz's inequality and Bernstein's inequality*, Proc. Amer. Math. Soc., **125** (1997), 2319-2325. MR **97j:**47028

10. G.K.Pedersen, *Some operator monotone functions*, Proc. Amer. Math. Soc., **36** (1972), 309-310. MR **46:**6078
11. K.Tanahashi, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc., **124** (1996), 141-146. MR **96d:**47025

Department of Mathematics, Osaka Kyoiku University, Kashiwara, Osaka 582, Japan
E-mail address: mfujii@cc.osaka-kyoiku.ac.jp

Faculty of Engineering, Ibaraki University, Hitachi, Ibaraki 316, Japan
E-mail address: nakamoto@base.ibaraki.ac.jp