EXTENSIONS OF HEINZ-KATO-FURUTA INEQUALITY

MASATOSHI FUJII AND RITSUO NAKAMOTO

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ABSTRACT. We give an extension of Lin’s recent improvement of a generalized Schwarz inequality, which is based on the Heinz-Kato-Furuta inequality. As a consequence, we can sharpen the Heinz-Kato-Furuta inequality.

1. Introduction

First of all, we cite a generalized Schwarz inequality which is a base of Lin’s recent paper [9]. For a (bounded linear) operator $T$ acting on a Hilbert space $H$,

$$|(Tx, y)|^2 \leq (|T|^{2\alpha} x, x)(|T^*|^{2(1-\alpha)} y, y)$$

for all $\alpha \in [0, 1]$ and $x, y \in H$, where $|X|$ is the square root of $X^*X$ for an operator $X$ on $H$. It implies the Heinz-Kato inequality via the Löwner-Heinz inequality, cf. [3],[10]. On the other hand, Furuta [7] extended the Heinz-Kato inequality, the so-called Heinz-Kato-Furuta inequality. Rephrasing it parallel to (1), we have

$$|(T|^{\alpha+\beta-1} x, y)|^2 \leq (|T|^{2\alpha} x, x)(|T^*|^{2\beta} y, y)$$

for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \geq 1$ and $x, y \in H$.

Very recently, Lin [9] sharpened (1) as follows:

**Theorem L.** Let $T$ be an operator on $H$ and $0 \neq y \in H$. For $z \in H$ satisfying $Tz \neq 0$ and $(Tz, y) = 0$,

$$|(Tx, y)|^2 + \frac{(|T|^{2\alpha} x, z)^2 (|T^*|^{2(1-\alpha)} y, y)}{(|T|^{2\alpha} z, z)} \leq (|T|^{2\alpha} x, x)(|T^*|^{2(1-\alpha)} y, y)$$

for all $\alpha \in [0, 1]$ and $x, y \in H$. The equality holds if and only if $Tz \neq 0$ and $(Tz, y) = 0$, and $T^*y$ are proportional, or equivalently, $Tx - \frac{(|T|^{2\alpha} x, z)}{|T|^{2\alpha} z, z} Tz$ and $|T^*|^{2(1-\alpha)} y$ are proportional.

In this note, we extend Theorem L, which is based on the Heinz-Kato-Furuta inequality (2). Our proof is quite simple, in which we clarify the meaning of the assumption in Theorem L that $Tz \neq 0$ and $(Tz, y) = 0$. As a consequence, we can sharpen the Heinz-Kato-Furuta inequality, and Furuta’s further generalization [6, Theorem 3] of the Heinz-Kato inequality via the Furuta inequality [4]. Incidentally we discuss Bernstein type inequality along the lines of our result.

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2. Heinz-Kato-Furuta inequality

For the sake of convenience, we first cite the Heinz-Kato-Furuta inequality [7]:

Let $T$ be an operator on $H$. If $A$ and $B$ are positive operators on $H$ such that $T^*T \leq A^2$ and $TT^* \leq B^2$, then

$$\|(T|T|^{\alpha+\beta-1}x, y)\| \leq \|A^\alpha x\|\|B^\beta y\|$$

for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \geq 1$ and $x, y \in H$.

We here remark that the Heinz-Kato inequality is just the case $\alpha = \beta = 1$ in above and that it corresponds to (1). Thus we have the following extension of Theorem L. Throughout this paper, let $T = U|T|$ be the polar decomposition of an operator $T$ on $H$.

**Theorem 1.** Let $T$ be an operator on $H$ and $0 \neq y \in H$. For $z \in H$ satisfying $T^*T|T|^{\alpha+\beta-1}z \neq 0$ and $(T|T|^{\alpha+\beta-1}z, y) = 0$,

$$\|(T|T|^{\alpha+\beta-1}x, y)\|^2 \leq \frac{\|(T|T|^{\alpha+\beta-1}x, z)\|^2 (T|T|^2z, y)}{(T|T|^{2\alpha}z, z)}$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$ and $x, y \in H$. In the case $\alpha, \beta > 0$, the equality in (5) holds if and only if $|T|^{\alpha+\beta-1}T^*y$ and $|T|^{2\alpha}(x - \frac{(|T|^{2\alpha}x, z)}{(|T|^{2\alpha}z, z)})$ are proportional, or equivalently, $|T|^{2\beta}y$ and $T|T|^{\alpha+\beta-1}(x - \frac{(|T|^{2\alpha}x, z)}{(|T|^{2\alpha}z, z)})$ are proportional.

It is easily seen that Theorem L is the case $\alpha + \beta = 1$ in Theorem 1. As a consequence, we have the following improvement of the Löwner-Heinz inequality, i.e., $A \geq B \geq 0$ implies $A^\alpha \geq B^\alpha$ for $\alpha \in [0, 1]$:

**Theorem 2.** Let $T$ be an operator on $H$. If $A$ and $B$ are positive operators on $H$ such that $T^*T \leq A^2$ and $TT^* \leq B^2$, then

$$\|(T|T|^{\alpha+\beta-1}x, y)\|^2 \leq \frac{\|(T|T|^{\alpha+\beta-1}x, z)\|^2 (T|T|^2z, y)}{(T|T|^{2\alpha}z, z)}$$

for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \geq 1$ and $x, y, z \in H$ such that $T|T|^{\alpha+\beta-1}z \neq 0$ and $(T|T|^{\alpha+\beta-1}z, y) = 0$. In the case $\alpha, \beta > 0$, the equality in (6) holds if and only if $A^{2\alpha}x = |T|^{2\alpha}x$, $B^{2\beta}y = |T|^{2\beta}y$ and $|T|^{\alpha+\beta-1}T^*y$ and $|T|^{2\alpha}(x - \frac{(|T|^{2\alpha}x, z)}{(|T|^{2\alpha}z, z)})$ are proportional; the third condition is equivalent to the condition that $|T|^{2\beta}y$ and $T|T|^{\alpha+\beta-1}(x - \frac{(|T|^{2\alpha}x, z)}{(|T|^{2\alpha}z, z)})$ are proportional.

**Proof of Theorem 1.** We only use the positivity of the Gram matrix

$$G = G(U|T|^{\alpha}x, |T|^{\beta}y, U|T|^{\alpha}z).$$

Noting that

$$(|T|^\beta y, U|T|^{\alpha}z) = (y, |T|^\beta U|T|^{\alpha}z) = (y, T|T|^\alpha z) = 0$$

by the assumption, we have

$$G = \begin{pmatrix}
\|T|^{\alpha}x\|^2 & (U|T|^{\alpha}x, |T|^{\beta}y) & (U|T|^{\alpha}x, U|T|^{\alpha}z) \\
(U|T|^{\alpha}x, |T|^{\beta}y)^* & \|T|^{\beta}y\|^2 & 0 \\
(U|T|^{\alpha}x, U|T|^{\alpha}z)^* & 0 & \|T|^{\alpha}z\|^2 \\
\end{pmatrix}.$$

Since $|T|^\alpha z \neq 0$, we have

$$\|(T|T|^{\alpha+\beta-1}x, y)\|^2 \leq \frac{\|(T|T|^{\alpha+\beta-1}x, z)\|^2 (T|T|^2z, y)}{(T|T|^{2\alpha}z, z)}.$$
To prove the equality condition, we set up the following lemma, which is applied to the vectors \( u = U|T|^\alpha x, v = U|T|^\alpha z \) and \( w = |T^*|^\beta y \).

**Lemma.** (1) If \( v \neq 0 \) and \( (v, w) = 0 \), then \( \{u, v, w\} \) is linearly dependent if and only if \( w \) and \( u - \frac{(u,v)}{\|v\|^2} v \) are proportional.

(2) Let \( T = U|T| \) be the polar decomposition of an operator \( T \) on \( H \), (namely \( \ker(U) = \ker(T) \)). For \( \alpha, \beta > 0 \) with \( \alpha + \beta \geq 1 \) and \( y, w \in H \), the following conditions are mutually equivalent:

(i) \( |T|^\beta y \) and \( U|T|^\alpha w \) are proportional.

(ii) \( |T|^{\alpha+\beta-1} T^* y \) and \( |T|^{2\alpha} w \) are proportional.

(iii) \( |T|^\beta y \) and \( T|T|^{\alpha-1} w \) are proportional.

**Proof.** (1) Suppose that \( au + bv + cw = 0 \) for some \( (a, b, c) \neq 0 \). Then \( a(u, v) + b\|v\|^2 = 0 \) and so \( b = -\frac{a(u,v)}{\|v\|^2} \). Hence we have

\[
0 = au + bv + cw = a(u - \frac{(u,v)}{\|v\|^2} v) + cw.
\]

Since \( a = c = 0 \) does not occur by \( v \neq 0 \), vectors \( u - \frac{(u,v)}{\|v\|^2} v \) and \( w \) are proportional. The converse is easily checked.

(2) (i) is equivalent to the statement that \( U|T|^\beta U^* y \) and \( U|T|^\alpha w \) are proportional. Noting that \( \alpha, \beta > 0 \) and \( \ker(U) = \ker(T) \), it is equivalent to (ii). Similarly we have the equivalence between (i) and (iii).

\[\qed\]

3. Furuta inequality

In [6], the Heinz-Kato-Furuta inequality is extended by the use of the Furuta inequality; Theorem 1 also gives us an improvement of the extension due to Furuta. For the sake of convenience, we cite the Furuta inequality [4]; see also [2],[5],[8].

**The Furuta inequality.** If \( A \geq B \geq 0 \), then for each \( r \geq 0 \),

\[
(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q}
\]

holds for \( p \geq 0 \) and \( q \geq 1 \) with

(*) \begin{align*}
(1+2r)q &\geq p + 2r.
\end{align*}

The domain representing (*) is drawn in Figure 1 and it is shown in [11] that this domain is the best possible one for the Furuta inequality.

**Theorem 3.** Let \( T \) be an operator on \( H \). If \( A \) and \( B \) are positive operators on \( H \) such that \( T^* T \leq A^2 \) and \( TT^* \leq B^2 \). Then for each \( r, s \geq 0 \)

\[
|\langle |T|^{(1+2r)^\alpha+(1+2s)^\beta-1} x, y \rangle|^2 + |\langle |T|^{2(1+2r)\alpha} x, z \rangle|^2 |\langle |T^*|^{2(1+2s)\beta} y, y \rangle|^2

\leq \frac{((|T|^{2(1+2r)\alpha} x, z))}{(|T|^{2(1+2r)\alpha} z, z) |\langle |T^*|^{2(1+2s)\beta} y, y \rangle|^2}
\]

for all \( p, q \geq 1 \), \( \alpha, \beta \in [0, 1] \) with \( (1+2r)\alpha + (1+2s)^\beta \geq 1 \) and \( x, y, z \in H \) such that \( T|T|^{(1+2r)\alpha+(1+2s)^\beta-1} z \neq 0 \) and \( T|T|^{(1+2r)^\alpha+(1+2s)^\beta-1} y, z \neq 0 \). In the case \( \alpha, \beta > 0 \), the equality in (7) holds if and only if

\[
|T|^{2(1+2r)\alpha} x = |T|^{2(1+2s)\beta} y \text{ and } |T^*|^{2(1+2s)\beta} y = |T^*|^{2(1+2r)^\alpha} x.
\]

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$|T|^2(1+2r)^\alpha + 2(1+2s)^\beta - 1 T^* y$ are proportional; the latter is equivalent to the statement that $T|T|^{(1+2r)^\alpha + (1+2s)^\beta - 1} (x - \frac{(T|^2(1+2r)^\alpha x, z)}{(T|^2(1+2r)^\alpha z, z)} z)$ and $|T^*|^2(1+2s)^\beta y$ are proportional.

Proof. We use Theorem 1 by replacing $\alpha$ (resp. $\beta$) by $\alpha_1 = (1 + 2r)^\alpha$ (resp. $\beta_1 = (1 + 2s)^\beta$). Then we have

$$|T|^2\alpha_1 + (T|^2(1+2r)^\alpha z, z) \leq \frac{|T|^2\alpha_1 x, x}{(T|^2(1+2r)^\alpha x, x)} (1+2r)^\alpha \alpha_1 + 2(1+2s)^\beta - 1 (x - \frac{(T|^2(1+2r)^\alpha x, z)}{(T|^2(1+2r)^\alpha z, z)} z)$$

Next we use the Furuta inequality for $|T|^2 \leq A^2$ and $|T^*|^2 \leq B^2$; namely (for the former) we replace $A$, $B$; $q$ in the Furuta inequality by $A^2$, $B^2$; $\frac{p+2r}{(1+2r)^\alpha}$ respectively. Then we have

$$|T|^2\alpha_1 = |T|^2(1+2r)^\alpha \leq \frac{|T|^2A^2B^2}{(|T|^2A^2)^{1+2r}}$$

and similarly

$$|T^*|^2\beta_1 = |T^*|^2(1+2s)^\beta \leq \frac{|T^*|^2B^2A^2}{(|T^*|^2B^2)^{1+2s}}.$$
that is, $A^2 \geq B^2$ ensures

$$(B^{2(1+2r)\alpha}x,x)^2 \leq ((B^{2r}A^{2p}B^{2r})^{\frac{1+2r\alpha}{1+2r}} x,x)$$

for all $p \geq 1$, $r \geq 0$ and $\alpha \in [0,1]$. This is nothing but the Furuta inequality.

4. A CONCLUDING REMARK

Lin also discussed Bernstein type inequalities independently of Theorem L [9, Theorem 3], see [1]. As an application of Theorem 1, we have a generalization of it:

**Theorem 4.** Let $T$ be an operator on $H$ having a nonzero normal eigenvalue $\lambda$ with an eigenvector $e$. If $y \in H$ satisfies $(e, y) = 0$ and $T^* y \neq 0$, then

$$|\lambda|^2 |(x, e)|^2 \leq \frac{\|Tx\|^2 \|T^*|\beta T^* y\|^2 - \|T|T|^\beta x, T^* y\|^2}{\|T^*|\beta T^* y\|^2}$$

for all $x \in H$ and $\beta \in [0,1]$.

**Proof.** We put $\alpha = 1$, $z = e$ and replace $y$ by $T^* y$ in Theorem 1. Since $(|T|^\beta e, T^* y) = 0$ by $(e, y) = 0$, it follows from Theorem 1 that

$$|(T|T|^\beta x, T^* y)|^2 + \|T^*|\beta T^* y\|^2 |\lambda|^2 |(x, e)|^2 \leq \|Tx\|^2 \|T^*|\beta T^* y\|^2,$$

so that we have the desired inequality. \qed

We obtain Lin’s inequality [9, Theorem 3] by taking $\beta = 0$ in Theorem 4.

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Department of Mathematics, Osaka Kyoiku University, Kashiwara, Osaka 582, Japan
E-mail address: mfujii@cc.osaka-kyoiku.ac.jp

Faculty of Engineering, Ibaraki University, Hitachi, Ibaraki 316, Japan
E-mail address: nakamoto@base.ibaraki.ac.jp