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A LOCAL APPROACH TO FUNCTIONALS ON $L^\infty(\mu, X)$

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ABSTRACT. Let (Ω, Σ, μ) be a probability space and X a Banach space. We show that the dual of $L^\infty(\mu, X)$ can be "locally" identified with $L^1(\mu, X^*)$.

1. Introduction

It is a classical result that the dual of $L^\infty(\mu)$ can be isometrically described through finitely additive measures of bounded variation. Moreover, thanks to the Lebesgue and Yosida-Hewitt decomposition theorems, we know that the dual of $L^\infty(\mu)$ is the sum of a "nice" part and a "singular" part [DU, Chapter I, Section 5]. In the vector case, a related decomposition still holds but an isometric description of $L^\infty(\mu, X)^*$ seems to be unknown, in spite of many papers that have been published on this question (see [CV, Chapter VIII] and the references therein). In this paper, we show that the elements of $L^\infty(\mu, X)^*$ can be "locally" (i.e., acting over finite dimensional subspaces) identified with elements of $L^1(\mu, X^*)$ so, at least in this situation, the "singular" part can be partially forgotten. Moreover, comparing our main theorems with [DJT, Theorem 8.16], it is reasonable to say that we have proved for $L^\infty(\mu, X)^*$ two vector extensions of the principle of local reflexivity for $L^1(\mu)^{**}$. At this point, it is worth recalling that the measure approach works in the scalar case because simple functions are dense in $L^\infty(\mu)$, a situation far from being true in the general case. In the end of the paper, we also derived a vector version of the well-known theorem of Grothendieck-Lindenstrauss-Pełczynski saying that every operator from $L^\infty(\mu)$ into a $\mathcal{L}_1$-space is 2-summing.

Throughout this paper (Ω, Σ, μ) denotes a probability space, X a Banach space and $L^1(\mu, X)$ (resp. $L^\infty(\mu, X)$) the space of (classes of) of Bochner integrable (resp. essentially bounded and measurable) functions defined over Ω. For some of our arguments, we also consider the Banach space $L^1_w(\mu, X^*)$ (resp. $L^\infty_w(\mu, X^*)$) of (equivalence classes of) w^*-measurable functions $f : \Omega \to X^*$ for which there is a μ-measurable integrable (resp. essentially bounded) function g with $\|f(\omega)\| \leq g(\omega)$, μ-almost everywhere. The norm is the infimum of those $\|g\|_1$ (resp. $\|g\|_\infty$) when f runs through the corresponding class and two w^*-measurable functions f_1, f_2 are identified if, for each $x \in X$, we have that $\langle x, f_1(\omega)\rangle = \langle x, f_1(\omega)\rangle$, μ-almost everywhere. We recall that $L^\infty_w(\mu, X^*)$ is isometrically isomorphic to $L_1(\mu, X)^*$.

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We also denote by $S(\mu, X)$ the subspace of $L^1(\mu, X)$ formed by the simple functions and by $S_\omega(\mu, X)$ the subspace of $L^\infty(\mu, X)$ formed by the "functions" with μ-essentially countable range. The canonical embedding of X into its bidual will be written as π. This allows us to consider the following linear isometry

$$\Pi : \ell^\infty(X) \to \ell^\infty(X^{**}), \qquad (x_n) \mapsto (\pi(x_n)).$$

The isometric identifications given in the paper will sometimes be shortened with the symbol "$\equiv$". The rest of the terminology and notations are quite standard and we refer the reader to the monographs of Diestel [Di] and Diestel-Uhl [DU].

2. Results

Every element $f \in L^1(\mu, X^*)$ defines an element of $L^\infty(\mu, X)^*$ in the following natural way:

$$\widetilde{f} : L^\infty(\mu, X) \to \mathbb{K}, \qquad g \mapsto \widetilde{f}(g) = \int \langle f(\omega), g(\omega) \rangle \, d\mu(\omega)$$

and $\left\| \widetilde{f} \right\| = \|f\|$. In what follows, and for the sake of clearness, we do not distinguish between these two elements. We also point out that an operator between Banach spaces $T : X_0 \to X_1$ is an ε-isometry if

$$(1 - \varepsilon) \|x\| \leq \|T(x)\| \leq (1 + \varepsilon) \|x\|, \text{ for all } x \in X_0.$$

Theorem 2.1. *Let $\varepsilon > 0$, and let E and F be finite dimensional subspaces of $L^\infty(\mu, X)^*$ and $L^\infty(\mu, X)$, respectively. Then there is an operator $T : E \to L^1(\mu, X^*)$ such that*

1. *T is an ε-isometry on E.*
2. *$T(e) = e$, for all $e \in E \cap S(\mu, X^*)$.*
3. *$\|T(e) - e\| \leq \varepsilon \|e\|$, for all $e \in E \cap L^1(\mu, X^*)$.*
4. *$\langle T(e), f \rangle = \langle e, f \rangle$, for all $e \in E$ and for all $f \in F$.*

Proof. We are going to prove the theorem in several steps.

First step: Let $\varepsilon > 0$ and let E and F be finite dimensional subspaces of $\ell^\infty(X)^*$ and $\ell^\infty(X)$, respectively. Then there is an operator $T : E \to \ell^1(X^*)$ such that

(1.a) T is an ε-isometry on E.
(1.b) $T(e) = e$, for all $e \in E \cap \ell^1(X^*)$.
(1.c) $\langle T(e), f \rangle = \langle e, f \rangle$, for all $e \in E$ and $f \in F$.

Take $\delta > 0$ satisfying that $(1 + \delta)^2 \leq 1 + \varepsilon$. Since E is finite dimensional, we may enlarge F (if necessary), so that, for all $e \in E$,

$$(*) \qquad (1 - \varepsilon) \|e\| \leq \sup\{|\langle e, f \rangle| : f \in F \text{ and } \|f\| = 1\}.$$

In other words, F is an ε-norming subset for E.

We define the index set I to be the collection of all finite dimensional subspaces of $\ell^\infty(X^{**})$ and consider the partial order on I generated by the inclusion. Let $\mathcal{U}$ be an ultrafilter dominating the canonical order filter associated to that order.

Now, take an arbitrary element $\mathbf{H} \in I$. It is clear that we can find a sequence (H_n) of finite dimensional subspaces of X^{**} such that

$$\mathbf{H} \subset \left(\bigoplus_n H_n \right)_\infty .$$

Moreover, by the principle of local reflexivity and, for every $n \in \mathbb{N}$, we can obtain an operator $S_n^{\mathbf{H}} : H_n \to X$ such that

$$\begin{aligned} \left\| S_n^{\mathbf{H}}(x) \right\| &\leq (1+\delta) \|x\|, \text{ for all } x \in H_n, \\ S_n^{\mathbf{H}}(x) &= x, \text{ for all } x \in X \cap H_n. \end{aligned}$$

Then, we define the (non-linear) map

$$\begin{aligned} S_{\mathbf{H}} &: \ell^{\infty}(X^{**}) \to \ell^{\infty}(X), \\ (x_n^{**}) &\mapsto S_{\mathbf{H}}[(x_n^{**})] = \begin{cases} 0, & (x_n^{**}) \notin \mathbf{H}, \\ (S_n^{\mathbf{H}}(x_n^{**})), & (x_n^{**}) \in \mathbf{H}. \end{cases} \end{aligned}$$

Since, for all $(x_n^{**}) \in \ell^{\infty}(X^{**})$,

$$\sup \{ S_{\mathbf{H}}[(x_n^{**})] : \mathbf{H} \in I \} \leq (1+\delta) \sup \{ \|x_n^{**}\| : n \in \mathbb{N} \} < \infty,$$

we can assure the existence of the following family of limits under the ultrafilter $\mathcal{U}$:

$$\lim_{\mathcal{U}} \langle \Phi, S_{\mathbf{H}}[(x_n^{**})] \rangle, \quad \Phi \in \ell^{\infty}(X)^*, (x_n^{**}) \in \ell^{\infty}(X^{**}).$$

Therefore, we can consider the map

$$\begin{aligned} J &: \ell^{\infty}(X)^* \to \ell^{\infty}(X^{**})^*, \\ \Phi &\mapsto J(\Phi), \quad \langle J(\Phi), (x_n^{**}) \rangle = \lim_{\mathcal{U}} \langle \Phi, S_{\mathbf{H}}[(x_n^{**})] \rangle. \end{aligned}$$

It is not difficult to check that J is a linear continuous map with $\|J\| \leq (1+\delta)$.

It is well-known that $E = E_1 \oplus_1 E_2$, where $E_1 = E \cap \ell^1(X^*)$ [CM, Chapter 5]. To avoid trivial cases, we suppose that $E_2 \neq \{0\}$. Applying again the principle of local reflexivity to $J(E_2) \subset \ell^{\infty}(X^{**})^* \equiv \ell^1(X^*)^{**}$ and $\Pi(F) \subset \ell^{\infty}(X^{**})$, we obtain an operator $R : J(E_2) \to \ell^1(X^*)$ such that

(i) $\|RJ(e)\| \leq (1+\delta) \|J(e)\|$, for all $e \in E_2$.
(ii) $\langle RJ(e), \Pi(f) \rangle = \langle J(e), \Pi(f) \rangle$, for all $e \in E_2$ and for all $f \in F$.

Finally, we define the operator

$$\begin{aligned} T &: E \to \ell^1(X^*), \\ e &= e_1 + e_2 \in E_1 \oplus_1 E_2 \mapsto T(e) = e_1 + RJ(e_2). \end{aligned}$$

It is trivial that T is a well-defined linear operator with

$$T(e) = e, \text{ for all } e \in E \cap \ell^1(X^*).$$

Moreover, on the one hand, given $e = e_1 + e_2 \in E_1 \oplus_1 E_2$ and $f = (f_n) \in F$,

$$\begin{aligned} \langle T(e), f \rangle &= \langle e_1, f \rangle + \langle RJ(e_2), \Pi(f) \rangle = \langle e_1, f \rangle + \langle J(e_2), \Pi(f) \rangle \\ &= \langle e_1, f \rangle + \lim_{\mathcal{U}} \langle e_2, S_{\mathbf{H}}(\Pi(f)) \rangle = \langle e_1, f \rangle + \lim_{\mathcal{U}} \left\langle e_2, (S_n^{\mathbf{H}}[(\pi(f_n))]) \right\rangle \\ &= \langle e_1, f \rangle + \lim_{\mathcal{U}} \langle e_2, (f_n) \rangle = \langle e_1, f \rangle + \langle e_2, (f_n) \rangle = \langle e, f \rangle. \end{aligned}$$

On the other hand, given $e = e_1 + e_2 \in E_1 \oplus_1 E_2$ and using the above equality and condition $(*)$, we have that

$$(1-\varepsilon) \|e\| \leq \sup\{ |\langle T(e), f \rangle| : f \in F \text{ and } \|f\| \leq 1 \} \leq \|T(e)\|.$$

Likewise,

$$\begin{aligned} \|T(e)\| &\leq \|e_1\| + \|RJ(e_2)\| \leq \|e_1\| + (1+\delta)^2 \|e_2\| \\ &\leq (1+\varepsilon)(\|e_1\| + \|e_2\|) = (1+\varepsilon) \|e\|. \end{aligned}$$

Second step: Let $\varepsilon > 0$ and let E and F be finite dimensional subspaces of $L^\infty(\mu, X)^*$ and $S_\omega(\mu, X)$, respectively. Then, there is an operator $T : E \to L^1(\mu, X^*)$ such that

(2.a) T is an ε-isometry on E.
(2.b) $T(e) = e$, for all $e \in E \cap S(\mu, X^*)$.
(2.c) $\langle T(e), f\rangle = \langle e, f\rangle$, for all $e \in E$ and for all $f \in F$.

Since E is finite dimensional, we may obtain a finite family F_0 in the closed unit ball of $L^\infty(\mu, X)$ such that, for all e in the unit sphere of E,

$$1 - \varepsilon/2 \leq \sup\{|\langle e, f\rangle| : f \in F_0\}.$$

Bearing in mind that $S_\omega(\mu, X)$ is dense in $L^\infty(\mu, X)$ (see the proof of [DU, Chapter II, Section 1, Theorem 2]), we know that there is also a finite family F_1 in the closed unit ball of $S_\omega(\mu, X)$ such that

$$\inf\{\|f_0 - f_1\| : f_1 \in F_1\} \leq \varepsilon/2, \text{ for all } f_0 \in F_0.$$

Therefore, we may assume that F is an ε-norming subset for E.

Now, we suppose that F is spanned by $f_1, ..., f_k$ and $E \cap S(\mu, X^*)$ is spanned by $\varphi_1, ..., \varphi_m$. It is clear that we may find a sequence (A_n) of pairwise disjoint subsets with positive measure of Σ such that

$$f_i = \sum_{n=1}^{\infty} x_{i,n} \chi_{A_n},$$

where $(x_{i,n})_n \in \ell^\infty(X)$ $(i = 1, ..., k)$ and,

$$\varphi_j = \sum_{n=1}^{\infty} x^*_{j,n} \chi_{A_n}$$

where $\operatorname{card}\{x^*_{j,n} : n \in \mathbb{N}\}$ is finite $(j = 1, ..., m)$. Consider the subspace of $L^\infty(\mu, X)$ defined as

$$Z := \{\sum_{n=1}^{\infty} x_n \chi_{A_n} : (x_n)_n \in \ell^\infty(X)\}.$$

It is clear that Z is isometrically isomorphic to $\ell^\infty(X)$. Let $J : Z \to \ell^\infty(X)$ be the canonical isometry and let F' be the finite dimensional subspace of $\ell^\infty(X)$ spanned by $J(F)$. Moreover, given $\Phi \in L^\infty(\mu, X)^*$, let $\Phi|_Z$ be the restriction of Φ to Z and let $R(\Phi)$ be the element of $\ell^\infty(X)^*$ defined as

$$\langle R(\Phi), (x_n^{**})\rangle = \Phi|_Z J^{-1}\left[(x_n^{**})\right], \qquad (x_n^{**}) \in \ell^\infty(X).$$

Applying the first step to the finite dimensional subspaces $F' \subset \ell^\infty(X)$ and

$$E' := \{R(e) : e \in E\} \subset \ell^\infty(X)^*,$$

we obtain an operator $S : E' \to \ell^1(X^*)$ such that

(i) $\|SR(e)\| \leq (1 + \delta)\|R(e)\|$, for all $e \in E$.
(ii) $S(e') = e'$, for all $e' \in E' \cap \ell^1(X^*)$.
(iii) $\langle SR(e), J(f)\rangle = \langle R(e), J(f)\rangle$, for all $e \in E$ and for all $f \in F$.

We shall also use the following linear isometry:

$$\begin{array}{rcl} U & : & \ell^1(X^*) \to L^1(\mu, X^*), \\ (x_n^*) & \mapsto & \langle U, (x_n^*)\rangle := \displaystyle\sum_{n=1}^{\infty} \frac{1}{\mu(A_n)} x_n^* \chi_{A_n}. \end{array}$$

Put

$$T : E \to L^1(\mu, X^*), \qquad e \mapsto T(e) = USR(e).$$

Let us check that T satisfies the conditions $(2.a)$, $(2.b)$ and $(2.c)$. We begin with the third condition, so we take $e \in E$ and $f \in F$. We assume that $SR(e) = (x_n^*) \in \ell^1(X^*)$ and $f = \sum_{n=1}^\infty x_n \chi_{A_n}$, where $(x_n) \in \ell^\infty(X)$. Then,

$$\begin{aligned} \langle T(e), f\rangle &= \left\langle \sum_{n=1}^\infty \frac{1}{\mu(A_n)} x_n^* \chi_{A_n}, \sum_{n=1}^\infty x_n \chi_{A_n} \right\rangle = \sum_{n=1}^\infty \langle x_n^*, x_n\rangle \\ &= \langle (x_n^*), J(f)\rangle = \langle SR(e), J(f)\rangle = \langle R(e), J(f)\rangle \\ &= \langle e, f\rangle. \end{aligned}$$

We pass to the first condition. Take $e \in E$. Then, on the one hand,

$$\begin{aligned} \|T(e)\| &= \|USR(e)\| = \|SR(e)\| \leq (1+\varepsilon)\|R(e)\| \\ &\leq (1+\varepsilon)\sup\{|\langle R(e), (x_n)\rangle| : \|(x_n)\|_{\ell^\infty(X)} \leq 1\} \\ &\leq (1+\varepsilon)\sup\left\{\left|\left\langle e, \sum_{n=1}^\infty x_n \chi_{A_n}\right\rangle\right| : \|(x_n)\|_{\ell^\infty(X)} \leq 1\right\} \\ &\leq (1+\varepsilon)\|e\|. \end{aligned}$$

On the other hand, using the ε-norming property of F and looking at the end of the proof of the first step, we obtain that

$$(1-\varepsilon)\|e\| \leq \|T(e)\|, \text{ for all } e \in E.$$

To check the second condition, let φ be an element of $E \cap S(\mu, X^*)$. We can write

$$\varphi = \sum_{n=1}^\infty x_n^* \chi_{A_n},$$

where $(x_n^*) \in \ell^\infty(X)$. Bearing in mind the definition of R, we quickly have that

$$R(\varphi) = (x_n^* \mu(A_n))_n \in E' \cap \ell^1(X^*).$$

Therefore, using the first step, we deduce that

$$\begin{aligned} T(\varphi) &= U\left(S(x_n^* \mu(A_n))_n\right) = U\left((x_n^* \mu(A_n))_n\right) \\ &= \sum_{n=1}^\infty \frac{1}{\mu(A_n)} x_n^* \mu(A_n) \chi_{A_n} = \sum_{n=1}^\infty x_n^* \chi_{A_n} = \varphi. \end{aligned}$$

Third step: Let $\varepsilon > 0$ and let E and F be finite dimensional subspaces of $L^\infty(\mu, X)^*$ and $L^\infty(\mu, X)$, respectively. Then, there is an operator $T : E \to L^1(\mu, X^*)$ such that

(3.a) T is an ε-isometry on E.
(3.b) $T(e) = e$, for all $e \in E \cap S(\mu, X^*)$.
(3.c) $\langle T(e), f\rangle = \langle e, f\rangle$, for all $e \in E$ and for all $f \in F$.

Take $\delta > 0$ satisfying that $(1+\delta)^2 \leq 1+\varepsilon$. Once again, we may assume that F is an ε-norming subset for E.

Suppose that F is spanned by $f_1, ..., f_k$. Recalling again that $S_\omega(\mu, X)$ is dense in $L^\infty(\mu, X)$, we can obtain, for every $n \in \mathbb{N}$, a finite collection $g_1(n), ..., g_k(n)$ in

$S_\omega(\mu, X)$ with

$$\|f_i - g_i(n)\| \le \frac{1}{n}, \text{ for all } n \in \mathbb{N} \text{ and } i = 1, ..., k.$$

Applying the second step to the subspaces $E \subset L^\infty(\mu, X)^*$ and

$$G_n := \operatorname{span}\left\{\bigcup_{i=1}^{n}\{g_1(i), ..., g_k(i)\}\right\} \subset S_\omega(\mu, X),$$

we obtain an operator $T_n : E \to L^1(\mu, X^*)$ such that

(i) T_n is an δ-isometry on E.
(ii) $T_n(e) = e$, for all $e \in E \cap S(\mu, X^*)$.
(iii) $\langle T_n(e), g\rangle = \langle e, g\rangle$, for all $e \in E$ and for all $g \in G_n$.

Consider an ultrafilter $\mathcal{V}$ dominating the Fréchet filter in $\mathbb{N}$ of all subsets of natural numbers whose complement is finite. Bearing in mind the Alaoglu-Bourbaki theorem and denoting the weak*-topology on $L^1(\mu, X^*)^{**}$ by w^*, we see that the following map is well-defined:

$$S : E \to L^1(\mu, X^*)^{**}, \qquad e \mapsto S(e) := w^*\text{-}\lim_{\mathcal{V}} T_n(e) \in L^1(\mu, X^*)^{**}.$$

Moreover, given $e \in E$,

$$\begin{aligned} \|S(e)\| &\le \left\|w^*\text{-}\lim_{\mathcal{V}} T_n(e)\right\| \le \lim_{\mathcal{V}} \|T_n(e)\| \\ &\le \lim_{\mathcal{V}}(1+\delta)\|e\| = (1+\delta)\|e\|. \end{aligned}$$

That is, S is a linear continuous map with $\|S\| \le (1+\delta)$. By the weak*-definition, we obviously have that

$$\text{(C1)} \qquad S(e) = e, \text{ for all } e \in E \cap S(\mu, X^*).$$

Fix $m \in \mathbb{N}$ and consider G_m as a subset of $L_w^\infty(\mu, X^{**}) \equiv L^1(\mu, X^*)^*$. If we take a natural number $k_0 > m$ and define $V := [k_0, \infty) \cap \mathbb{N} \in \mathcal{V}$, we see that

$$\langle T_k(e), g\rangle = \langle e, g\rangle, \text{ for all } k \in V.$$

Therefore, we have that $\lim_{\mathcal{V}} \langle T_n(e), g\rangle = \langle e, g\rangle$.

Now, take $e \in E$ and $f \in F$. For every $n \in \mathbb{N}$, we choose $g \in G_n$ with $\|f - g\| \le 1/n$. Then,

$$\begin{aligned} |\langle S(e), f\rangle - \langle e, f\rangle| &= |\langle S(e), f-g\rangle + \langle S(e), g\rangle - \langle e, g\rangle - \langle e, f-g\rangle| \\ &= |\langle S(e), f-g\rangle - \langle e, f-g\rangle| \le ((1+\delta)+1)\|f-g\|\|e\| \\ &\le ((1+\delta)+1)\frac{\|e\|}{n}. \end{aligned}$$

Since n was arbitrary, we have that

$$\text{(C2)} \qquad \langle S(e), f\rangle = \langle e, f\rangle, \text{ for all } e \in E \text{ and } f \in F.$$

If we apply the principle of local reflexivity to the finite dimensional subspaces $E' = S(E) \subset L^1(\mu, X^*)^{**}$ and $F \subset L^\infty(\mu, X) \subset L_w^\infty(\mu, X^{**}) \equiv L^1(\mu, X^*)^*$, we obtain an operator $R : S(E) \to L^1(\mu, X^*)$ such that

(i) R is an δ-isometry on E'.
(ii) $R(e') = e'$, for all $e' \in E' \cap L^1(\mu, X^*)$.
(iii) $\langle R(e'), f\rangle = \langle e', f\rangle$, for all $e' \in E'$ and for all $f \in F$.

Finally, we define $T : E \to L^1(\mu, X^*)$ as $T = RS$. Since S satisfies conditions (C2) and (C2), it is trivial that T satisfies (3.b) and (3.c). Now, combining the ε-norming property of F, condition (3.c) and bearing in mind that $\|R\|, \|S\| \le (1+\delta)$, we quickly deduce that T is an ε-isometry, that is, T satisfies condition (3.a).

Fourth and last step: We prove the theorem.

Again, we assume that F is an ε-norming subset for E. Let $e_1, ..., e_m$ be a basis of $E \cap L^1(\mu, X^*)$ and take $C > 0$ such that

$$\sum_{i=1}^m |\lambda_i| \le C \left\| \sum_{i=1}^m \lambda_i e_i \right\|,$$

for all $\lambda_1, ..., \lambda_m \in \mathbb{K}$. Also take $\delta > 0$ with $\delta C(2+\varepsilon) \le \varepsilon$. Since simple functions are dense in $L^1(\mu, X^*)$, we can find $\varphi_1, ..., \varphi_m$ with $\|e_i - \varphi_i\| \le \delta$. Now, we apply the third step to the finite dimensional subspaces $F \subset L^\infty(\mu, X)$ and

$$E' := \operatorname{span}(E \cup \{\varphi_1, ..., \varphi_m\}) \subset L^\infty(\mu, X)^*.$$

Therefore, we can find an operator $R : E' \to L^1(\mu, X^*)$ such that

(i) R is an ε-isometry on E'.
(ii) $R(e') = e'$, for all $e' \in E' \cap S(\mu, X^*)$.
(iii) $\langle R(e), f \rangle = \langle e, f \rangle$, for all $e' \in E'$ and for all $f \in F$.

Then, if $e = \sum_{i=1}^m \lambda_i e_i \in E \cap L^1(\mu, X^*)$, we note that

$$\begin{aligned} \|R(e) - e\| &= \left\| R(e) - R(\sum_{i=1}^m \lambda_i \varphi_i) + \sum_{i=1}^m \lambda_i \varphi_i - e \right\| \\ &\le (1+\varepsilon+1) \left\| \sum_{i=1}^m \lambda_i (e_i - \varphi_i) \right\| \le (2+\varepsilon) \sum_{i=1}^m |\lambda_i| \, \|e_i - \varphi_i\| \\ &\le \delta(2+\varepsilon) \sum_{i=1}^m |\lambda_i| \le C\delta(2+\varepsilon) \left\| \sum_{i=1}^m \lambda_i e_i \right\| \le \varepsilon \|e\|. \end{aligned}$$

If we define T as the restriction of R to E and, bearing in mind the ε-norming property of F, it is straightforward to show that T satisfies the four conditions of the theorem. $\square$

Remark. Answering a question of Labuda (also implicitly posed by Saab and Saab), the author showed that, whenever X is a Banach lattice, the space $L^\infty(\mu, X)$ contains a complemented copy of ℓ_1 if and only if X contains all ℓ_1^n uniformly complemented [D]. This result was extended by Kalton to general Banach spaces [CM, Theorem 5.2.3]. The starting point of the proof of the above "first step" uses, in the context of ultrapowers, the new ideas introduced by Kalton.

Using the vector version of the Lebesgue decomposition theorem mentioned in the introduction, we can give the following variant of Theorem 2.1.

Theorem 2.2. *Let $\varepsilon > 0$ and let E and F be finite dimensional subspaces of $L^\infty(\mu, X)^*$ and $L^\infty(\mu, X)$, respectively. Then, there is an operator $T : E \to L^1(\mu, X^*)$ such that*

1. *T is an ε-isometry on E.*
2. *$T(e) = e$, for all $e \in L^1_w(\mu, X^*) \cap E$.*
3. *$\langle T(e), f \rangle = \langle e, f \rangle$, for all $e \in E$ and for all $f \in F$.*

Proof. According to [CV, Chapter VIII], the Banach space $L^\infty(\mu, X)^*$ is isometrically isomorphic to the sum $L^1_w(\mu, X^*) \oplus_1 L_S$, where L_S is the subspace of singular linear functionals on $L^\infty(\mu, X)$.

Therefore, we can write $E = E_1 \oplus_1 E_2$, where $E_1 = E \cap L^1_w(\mu, X^*)$ and $E_2 = E \cap L_S$. Now, applying Theorem 2.1 to $\varepsilon > 0$, $E_2 \subset L^\infty(\mu, X)^*$ and $F \subset L^\infty(\mu, X)$, we find a certain operator

$$R : E_2 \to L^1(\mu, X^*) \subset L^1_w(\mu, X^*).$$

Then, we define

$$T : E \to L^1_w(\mu, X^*),$$

$$e = e_1 + e_2 \in E_1 \oplus_1 E_2 \mapsto T(e) = e_1 + R(e_2).$$

We omit the rest of the proof since it is almost identical to the end part of the proof of the "first step" of Theorem 2.1. $\square$

As we quoted in the introduction, every operator from $L^\infty(\mu)$ into an $\mathcal{L}_1$-space is 2-summing, that is, maps weak ℓ^2-sequences in strong ℓ^2-sequences [Di, Chapter X]. Using the above theorem, it is possible to give the following vector-valued version of this result.

Corollary 2.3. *Assume that X^* has cotype 2. Then, every operator from $L^\infty(\mu, X)$ into an $\mathcal{L}_1$-space is 2-summing.*

Proof. Take an arbitrary operator T from $L^\infty(\mu, X)$ into an $\mathcal{L}_1$-space. Using Theorem 2.1, we see that $L^\infty(\mu, X)^*$ is finitely represented in $L^1(\mu, X^*)$, thus both spaces have the same cotype. By [DJT, Theorem 11.12], we find that this cotype is 2. Now, applying [DJT, Remark to Theorem 11.14], we know that T^* is 2-summing, as is T. $\square$

We note that a simple combination of the closed graph theorem and Theorem 2.1 shows that a sequence (f_n) in $L^\infty(\mu, X)$ is a weak ℓ^2-sequence if and only if, for every $g \in L^1(\mu, X^*)$,

$$\sum_{n=1}^{\infty} |\langle f_n, g \rangle|^2 < \infty.$$

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