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THE ROGERS-RAMANUJAN IDENTITIES, THE FINITE GENERAL LINEAR GROUPS, AND THE HALL-LITTLEWOOD POLYNOMIALS

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ABSTRACT. We connect Gordon's generalization of the Rogers-Ramanujan identities with the Hall-Littlewood polynomials and with generating functions which arise in a probabilistic setting in the finite general linear groups. This yields a Rogers-Ramanujan type product formula for the $n \to \infty$ probability that an element of $GL(n,q)$ or $Mat(n,q)$ is semisimple.

1. BACKGROUND AND NOTATION

The Rogers-Ramanujan identities are among the most remarkable partition identities in number theory and combinatorics. This paper will be concerned with the following generalization of the Rogers-Ramanujan identities, due to Gordon. Let $(x)_n$ denote $(1-x)(1-x^2)\cdots(1-x^n)$.

Theorem 1 ([A, page 111]). *For $1 \leq i \leq k, k \geq 2$, and complex x with $|x| < 1$,*

$$\sum_{n_1,\cdots,n_{k-1} \geq 0} \frac{x^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(x)_{n_1}\cdots(x)_{n_{k-1}}} = \prod_{\substack{r=1 \\ r \neq 0, \pm i (mod\ 2k+1)}}^{\infty} \frac{1}{1-x^r}$$

where $N_j = n_j + \cdots + n_{k-1}$.

Gordon's generalization of the Rogers-Ramanujan identities has been widely studied and appears in many places in mathematics and physics. Andrews [A] discusses combinatorial aspects of these identities. In an important series of papers, Lepowsky and Wilson [LW1], [LW2], [LW3] connect the Gordon identities with affine Lie algebras and structures that they called Z-algebras (later interpreted as parafermion algebras in conformal field theory). Meurman and Primc [MP] solve a problem left open in [LW3], proving the independence of a Z-algebra basis and obtaining a Z-algebra proof of the Gordon identities. Feigin and Frenkel [FF] interpret the Gordon identities as a character formula for the Virasoro algebra. Andrews, Baxter, and Forrester [ABF] and Warnaar [W] relate the Gordon identities with statistical mechanics. For some number theoretic connections see the conference proceedings [AABRR].

We use the following standard notation from the theory of partitions. Call $\lambda = (\lambda_1, \lambda_2, \cdots)$ a partition of $n = |\lambda|$ if $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ where the λ_i are

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a sequence of positive integers stabilizing to 0 such that $\sum_i \lambda_i = n$. The λ_i are referred to as the parts of λ. Let $m_i(\lambda)$ be the number of parts of λ of size i, and set $\lambda_i' = m_i(\lambda) + m_{i+1}(\lambda) + \cdots$. Define $n(\lambda)$ by $\sum_{i \geq 1}(i-1)\lambda_i$. Let $[u^n]f(u)$ denote the coefficient of u^n in a power series $f(u)$.

2. Main results

To begin we recall the Hall-Littlewood polynomials associated to a partition λ (page 208 of [Mac]). Let n be any integer such that $n \geq \lambda_1$. The permutation $w \in S_n$ acts on the variables $x_1, \cdots, x_n$ by sending x_i to $x_{w(i)}$. Letting t be a complex number, the Hall-Littlewood polynomials are defined as

$$P_\lambda(x_1, \cdots, x_n; t) = \frac{1}{\prod_{i \geq 0} \prod_{r=1}^{m_i(\lambda)} \frac{1-t^r}{1-t}} \sum_{w \in S_n} w(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}).$$

At first glance it is not obvious that these are polynomials, but the denominators cancel out after the symmetrization. The Hall-Littlewood polynomials interpolate between the Schur functions ($t = 0$) and the monomial symmetric functions ($t = 1$).

Theorem 2. *For $q > 1$ and an integer $k \geq 2$,*

$$\sum_{\lambda:\lambda_1<k} \frac{P_\lambda(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \cdots; \frac{1}{q})}{q^{n(\lambda)}} = \prod_{\substack{r=1 \\ r \neq 0, \pm k (mod \ 2k+1)}}^{\infty} (\frac{1}{1 - \frac{1}{q^r}}),$$

$$\sum_{\lambda:\lambda_1<k} \frac{P_\lambda(\frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \frac{1}{q^5}, \cdots; \frac{1}{q})}{q^{n(\lambda)}} = \prod_{\substack{r=1 \\ r \neq 0, \pm 1 (mod \ 2k+1)}}^{\infty} (\frac{1}{1 - \frac{1}{q^r}}).$$

Proof. Macdonald's principal specialization formula (page 337 of [Mac]) states that

$$P_\lambda(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \cdots; \frac{1}{q}) = \frac{1}{q^{|\lambda|+n(\lambda)}} \prod_i \frac{1}{(\frac{1}{q})_{m_i(\lambda)}}.$$

Combining this with the elementary fact that $n(\lambda) = \sum_i \binom{\lambda_i'}{2}$ shows that

$$\frac{P_\lambda(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \cdots; \frac{1}{q})}{q^{n(\lambda)}} = \frac{1}{q^{\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q})_{m_i(\lambda)}}.$$

The first equality of the theorem follows by applying Theorem 1 with $i = k, x = \frac{1}{q}, n_j = \lambda_j$ and $N_j = \lambda_j'$. For the second equality, observe that for u complex,

$$\begin{aligned} \frac{P_\lambda(\frac{u}{q}, \frac{u}{q^2}, \frac{u}{q^3}, \frac{u}{q^4}, \cdots; \frac{1}{q})}{q^{n(\lambda)}} &= u^{|\lambda|} \frac{P_\lambda(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \cdots; \frac{1}{q})}{q^{n(\lambda)}} \\ &= \frac{u^{|\lambda|}}{q^{\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q})_{m_i(\lambda)}}. \end{aligned}$$

Now set $u = \frac{1}{q}$ and apply Theorem 1 with $i = 1, x = \frac{1}{q}, n_j = \lambda_j$ and $N_j = \lambda_j'$. ☐

Remark. Stembridge [Ste] used properties of the Hall-Littlewood polynomials as a tool in giving proofs of the Rogers-Ramanujan identities. The statement of Theorem 2 gives a direct connection.

Recall that the conjugacy classes of $GL(n,q)$ are parameterized by rational canonical form (Chapter 6 of Herstein [H]). This form corresponds to the following combinatorial data. To each monic non-constant irreducible polynomial ϕ over a field of q elements, associate a partition λ_ϕ of some non-negative integer $|\lambda_\phi|$. Let $deg(\phi)$ denote the degree of ϕ. This data represents a conjugacy class of $GL(n,q)$ if and only if $|\lambda_z| = 0$ and $\sum_\phi |\lambda_\phi| deg(\phi) = n$.

Definition. For $\alpha \in GL(n,q)$ and ϕ a monic, irreducible polynomial over F_q, a field of q elements, define $\lambda_\phi(\alpha)$ to be the partition corresponding to the polynomial ϕ in the rational canonical form of α.

The following elementary lemmas will be of use in studying the partitions λ_ϕ.

Lemma 1. *If the Taylor series of $f(u)$ around 0 converges at $u = 1$, then*

$$lim_{n\to\infty}[u^n]\frac{f(u)}{1-u} = f(1).$$

Proof. Write the Taylor expansion $f(u) = \sum_{n=0}^{\infty} a_n u^n$. Then observe that $[u^n]\frac{f(u)}{1-u}$ $= \sum_{i=0}^{n} a_i$. □

Lemma 2. *For $t \geq 1$, q a prime power, and u a formal variable,*

$$\prod_{\phi \ irred.} (1 - \frac{u^{deg(\phi)}}{q^{t\cdot deg(\phi)}}) = 1 - \frac{u}{q^{t-1}}.$$

Proof. Assume that $t = 1$, the general case following by replacing u with $\frac{u}{q^{t-1}}$. Expanding $\frac{1}{1-\frac{u^{deg(\phi)}}{q^{deg(\phi)}}}$ as a geometric series, unique factorization in $F_q[x]$ implies that the coefficient of u^d in the reciprocal of the left hand side is $\frac{1}{q^d}$ times the number of monic polynomials of degree d, hence 1. Comparing with the reciprocal of the right hand side proves the lemma. □

Lemma 3. *For $t \geq 1$, q a prime power, and u a formal variable,*

$$1 - u = \prod_{\substack{\phi \neq z \\ \phi \ irred.}} \prod_{r=1}^{\infty} (1 - \frac{u^{deg(\phi)}}{q^{r\cdot deg(\phi)}}).$$

Proof. By Lemma 2,

$$\prod_{r=1}^{\infty}(1 - \frac{u}{q^{r-1}}) = \prod_{\phi \ irred.} \prod_{r=1}^{\infty} (1 - \frac{u^{deg(\phi)}}{q^{r\cdot deg(\phi)}}).$$

The lemma follows by cancelling the terms corresponding to $\phi = z$. □

Lemma 4. *For q a prime power and u complex with $|u| \leq 1$,*

$$\sum_\lambda \frac{u^{|\lambda|}}{q^{\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q})_{m_i(\lambda)}} = \prod_{r=1}^{\infty} (\frac{1}{1-\frac{u}{q^r}}).$$

Proof. On page 225 of [Mac] it is proved that

$$\prod_{r=1}^{\infty}(1-x_i)\sum_\lambda t^{n(\lambda)} P_\lambda(x_1, x_2, x_3, \cdots; t) = 1.$$

Applying Macdonald's principal specialization formula with $t = \frac{1}{q}, x_i = \frac{u}{q^i}$ as in Theorem 2 proves the result. □

Lemma 5 (Euler)**.** *For $|q| > 1$ and u complex with $|u| \leq 1$,*

1. $\prod_{r=1}^{\infty}(1-\frac{u}{q^r}) = \sum_{n=0}^{\infty} \frac{(-u)^n}{(q^n-1)\cdots(q-1)}$,
2. $\prod_{r=1}^{\infty}(\frac{1}{1-\frac{u}{q^r}}) = \sum_{n=0}^{\infty} \frac{u^n q^{\binom{n}{2}}}{(q^n-1)\cdots(q-1)}$.

Furthermore, these Taylor series converge at $u = 1$.

Theorem 3 relates the Gordon identities with probability in the finite general linear groups.

Theorem 3. *Let ϕ be a monic, irreducible polynomial over F_q. Let $k \geq 2$ be an integer. Then the $n \to \infty$ limit of the chance that a uniformly chosen element α of $GL(n,q)$ has the largest part of the partition $\lambda_{\phi}(\alpha)$ less than k is equal to*

$$\prod_{\substack{r=1 \\ r=0,\pm k (mod \;\; 2k+1)}}^{\infty} (1 - \frac{1}{q^{r \cdot deg(\phi)}}).$$

Proof. Assume for simplicity that $\phi = z - 1$. From the proof it will be clear that the general case follows. Stong [Sto] used Kung's [Ku] formula for the sizes of the conjugacy classes of $GL(n,q)$ to find a "cycle index" for the general linear groups. Using the notation

$$d_i(\lambda) = 1m_1(\lambda) + 2m_2(\lambda) + \cdots + (i-1)m_{i-1}(\lambda) + i(m_i(\lambda) + m_{i+1}(\lambda) + \cdots + m_j(\lambda)),$$

he obtained the equality

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n,q)|} \sum_{\alpha \in GL(n,q)} \prod_{\substack{\phi \neq z \\ \phi \;\; irred.}} x_{\phi,\lambda_{\phi}(\alpha)} \\ = \prod_{\substack{\phi \neq z \\ \phi \;\; irred.}} \left[\sum_{\lambda} x_{\phi,\lambda} \frac{u^{|\lambda| deg(\phi)}}{\prod_i \prod_{k=1}^{m_i(\lambda)} (q^{deg(\phi) d_i} - q^{deg(\phi)(d_i - k)})} \right]. \end{aligned}$$

Observe that

$$\begin{aligned} \prod_i \prod_{k=1}^{m_i} (q^{d_i} - q^{(d_i-k)}) &= q^{\sum_i m_i(\lambda) d_i(\lambda)} \prod_i (\frac{1}{q})_{m_i(\lambda)} \\ &= q^{\sum_i m_i(\lambda)[(\sum_{h<i} h m_h(\lambda)) + i m_i(\lambda) + \sum_{i<k} i m_k(\lambda)]} \prod_i (\frac{1}{q})_{m_i(\lambda)} \\ &= q^{\sum_i [i m_i(\lambda)^2 + 2 m_i(\lambda) \sum_{h<i} h m_h(\lambda)]} \prod_i (\frac{1}{q})_{m_i(\lambda)} \\ &= q^{\sum_i [\sum_{h<i} m_h(\lambda)]^2} \prod_i (\frac{1}{q})_{m_i(\lambda)} \\ &= q^{\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q})_{m_i(\lambda)}. \end{aligned}$$

Combining this observation with Lemma 3 shows that

$$1+\sum_{n=1}^{\infty}\frac{u^n}{|GL(n,q)|}\sum_{\alpha\in GL(n,q)}\prod_{\substack{\phi\neq z\\ \phi\ irred.}} x_{\phi,\lambda_\phi(\alpha)}$$
$$= \frac{1}{1-u}\prod_{\substack{\phi\neq z\\ \phi\ irred.}}\left[\prod_{r=1}^{\infty}(1-\frac{u^{deg(\phi)}}{q^{r\cdot deg(\phi)}})\right]\left[\sum_{\lambda} x_{\phi,\lambda}\frac{u^{|\lambda| deg(\phi)}}{q^{deg(\phi)\sum_i(\lambda_i')^2}\prod_i(\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)}}\right].$$

Setting $x_{z-1,\lambda}=0$ if the largest part of λ is greater than or equal to k, and all $x_{\phi,\lambda}=1$ otherwise shows by Lemma 4 that the sought probability is

$$\lim_{n\to\infty}[u^n]\frac{1}{1-u}\left(\prod_{r=1}^{\infty}(1-\frac{u}{q^r})\right)\left(\sum_{\lambda:\lambda_1<k}\frac{u^{|\lambda|}}{q^{\sum_i(\lambda_i')^2}\prod_i(\frac{1}{q})_{m_i(\lambda)}}\right).$$

By Lemmas 4 and 5, Lemma 1 applies and

$$\lim_{n\to\infty}[u^n]\frac{1}{1-u}\left(\prod_{r=1}^{\infty}(1-\frac{u}{q^r})\right)\left(\sum_{\lambda:\lambda_1<k}\frac{u^{|\lambda|}}{q^{\sum_i(\lambda_i')^2}\prod_i(\frac{1}{q})_{m_i(\lambda)}}\right)$$
$$= \left(\prod_{r=1}^{\infty}(1-\frac{1}{q^r})\right)\left(\sum_{\lambda:\lambda_1<k}\frac{1}{q^{\sum_i(\lambda_i')^2}\prod_i(\frac{1}{q})_{m_i(\lambda)}}\right).$$

The theorem follows from the Gordon identities with $i=k, x=\frac{1}{q}, n_j=m_j(\lambda)$, and $N_j=\lambda_j'$. □

Let $Mat(n,q)$ denote the set of $n\times n$ matrices with entries in the finite field F_q. Recall that $\alpha\in Mat(n,q)$ is said to be semisimple if it is diagonalizable over the algebraic closure $\bar{F}_q$.

Theorem 4. *The $n\to\infty$ limiting probability that an element of $GL(n,q)$ is semisimple is*

$$\prod_{\substack{r=1\\ r=0,\pm 2(mod\ \ 5)}}^{\infty}\frac{(1-\frac{1}{q^{r-1}})}{(1-\frac{1}{q^r})}.$$

Proof. From the theory of Jordan canonical forms, an element α of $GL(n,q)$ is semisimple if and only if the largest part of $\lambda_\phi(\alpha)$ is less than two for all ϕ. Thus by the cycle index for $GL(n,q)$ (see the proof of Theorem 3) and Lemma 3, the sought probability is

$$lim_{n\to\infty}[u^n]\prod_{\substack{\phi\neq z\\ \phi\ irred.}}\sum_{\lambda:\lambda_1<2}\frac{u^{deg(\phi)|\lambda|}}{q^{deg(\phi)\sum_i(\lambda_i')^2}\prod_i(\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)}}$$
$$= lim_{n\to\infty}[u^n]\frac{1}{1-u}\prod_{\substack{\phi\neq z\\ \phi\ irred.}}\left(\prod_{r=1}^{\infty}(1-\frac{u^{deg(\phi)}}{q^{r\cdot deg(\phi)}})\right)$$
$$\cdot\left(\sum_{\lambda:\lambda_1<2}\frac{u^{deg(\phi)|\lambda|}}{q^{deg(\phi)\sum_i(\lambda_i')^2}\prod_i(\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)}}\right).$$

By Theorem 1 with $i = 1, k = 2, x = \frac{1}{q^{deg(\phi)}}, n_j = \lambda_j, N_j = \lambda'_j$ and Lemma 2, the desired probability becomes

$$\begin{aligned}
&\prod_{\substack{\phi \neq z \\ \phi\ irred.}} \prod_{\substack{r=1 \\ r=0,\pm 2 (mod\ 5)}}^{\infty} (1 - \frac{1}{q^{r \cdot deg(\phi)}}) \\
&= \prod_{\substack{r=1 \\ r=0,\pm 2 (mod\ 5)}}^{\infty} (\frac{1}{1 - \frac{1}{q^r}}) \prod_{\phi\ irred.} \prod_{\substack{r=1 \\ r=0,\pm 2 (mod\ 5)}}^{\infty} (1 - \frac{1}{q^{r \cdot deg(\phi)}}) \\
&= \prod_{\substack{r=1 \\ r=0,\pm 2 (mod\ 5)}}^{\infty} \frac{(1 - \frac{1}{q^{r-1}})}{(1 - \frac{1}{q^r})}.
\end{aligned}$$

□

Theorem 5. *The $n \to \infty$ limiting probability that an element of $Mat(n, q)$ is semisimple is*

$$\prod_{\substack{r=1 \\ r=0,\pm 2 (mod\ 5)}}^{\infty} (1 - \frac{1}{q^{r-1}}).$$

Proof. The orbits of $GL(n, q)$ on $Mat(n, q)$ under conjugation are also parameterized by the data λ_ϕ. However the polynomial z may appear with non-zero multiplicity, so the restriction that $|\lambda_z| = 0$ does not apply. Stong [Sto] obtained the generating function equality

$$\begin{aligned}
&1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n,q)|} \sum_{\alpha \in Mat(n,q)} \prod_{\phi\ irred.} x_{\phi, \lambda_\phi(\alpha)} \\
&= \prod_{\phi\ irred.} \left[\sum_{\lambda} x_{\phi,\lambda} \frac{u^{|\lambda| deg(\phi)}}{\prod_i \prod_{k=1}^{m_i(\lambda)} (q^{deg(\phi) d_i} - q^{deg(\phi)(d_i - k)})} \right],
\end{aligned}$$

where $d_i(\lambda)$ is defined by

$$\begin{aligned}
d_i(\lambda) =& 1 m_1(\lambda) + 2 m_2(\lambda) + \cdots + (i-1) m_{i-1}(\lambda) \\
&+ i(m_i(\lambda) + m_{i+1}(\lambda) + \cdots + m_j(\lambda)).
\end{aligned}$$

Manipulations identical to those in Theorem 3 show that

$$\begin{aligned}
&1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n,q)|} \sum_{\alpha \in Mat(n,q)} \prod_{\phi\ irred.} x_{\phi, \lambda_\phi(\alpha)} \\
= \quad & \frac{1}{1-u} \prod_{\phi\ irred.} \left[\prod_{r=1}^{\infty} (1 - \frac{u^{deg(\phi)}}{q^{r \cdot deg(\phi)}}) \right] \left[\sum_{\lambda} x_{\phi,\lambda} \frac{u^{|\lambda| deg(\phi)}}{q^{deg(\phi) \sum_i (\lambda'_i)^2} \prod_i (\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)}} \right].
\end{aligned}$$

Arguing as in Theorem 4 shows that the $n \to \infty$ limit of the chance that an element of $Mat(n,q)$ is semisimple is

$$\begin{aligned} & lim_{n\to\infty} \frac{|GL(n,q)|}{q^{n^2}} [u^n] \prod_{\phi\ irred.} \sum_{\lambda:\lambda_1<2} \frac{u^{deg(\phi)|\lambda|}}{q^{deg(\phi)\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)}} \\ = \ & lim_{n\to\infty} \frac{|GL(n,q)|}{q^{n^2}} [u^n] \frac{1}{1-u} \frac{1}{\prod_{r=1}^{\infty}(1-\frac{u}{q^r})} \prod_{\phi\ irred.} \\ & \left[\prod_{r=1}^{\infty} (1-\frac{u^{deg(\phi)}}{q^{r\cdot deg(\phi)}}) \sum_{\lambda:\lambda_1<2} \frac{u^{deg(\phi)|\lambda|}}{q^{deg(\phi)\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)}} \right] \\ = \ & \prod_{\phi\ irred.} \left[\prod_{r=1}^{\infty} (1-\frac{1}{q^{r\cdot deg(\phi)}}) \right] \left[\sum_{\lambda:\lambda_1<2} \frac{1}{q^{deg(\phi)\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)}} \right] \\ = \ & \prod_{\phi\ irred.} \prod_{\substack{r=1 \\ r=0,\pm 2 (mod\ 5)}}^{\infty} (1-\frac{1}{q^{r\cdot deg(\phi)}}) = \prod_{\substack{r=1 \\ r=0,\pm 2 (mod\ 5)}}^{\infty} (1-\frac{1}{q^{r-1}}). \qquad \square \end{aligned}$$

Theorems 2 and 3 suggest a connection between the Hall-Littlewood polynomials and the finite general linear groups. Theorem 6 makes this connection precise.

Theorem 6. *For q a prime power and u a formal variable,*

$$\begin{aligned} & (1-u) \left[1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n,q)|} \sum_{\alpha\in GL(n,q)} \prod_{\substack{\phi\neq z \\ \phi\ irred.}} x_{\phi,\lambda_\phi(\alpha)} \right] \\ = \ & \prod_{\substack{\phi\neq z \\ \phi\ irred.}} \prod_{r=1}^{\infty} (1-\frac{u^{deg(\phi)}}{q^{r\cdot deg(\phi)}}) \left[\frac{P_\lambda\left((\frac{u}{q})^{deg(\phi)}, (\frac{u}{q})^{2deg(\phi)}, (\frac{u}{q})^{3deg(\phi)}, \cdots ; (\frac{1}{q})^{deg(\phi)} \right)}{q^{n(\lambda)\cdot deg(\phi)}} \right]. \end{aligned}$$

Proof. The proof of Theorem 3 contained the equality

$$\begin{aligned} & (1-u) \left[1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n,q)|} \sum_{\alpha\in GL(n,q)} \prod_{\substack{\phi\neq z \\ \phi\ irred.}} x_{\phi,\lambda_\phi(\alpha)} \right] \\ = \ & \prod_{\substack{\phi\neq z \\ \phi\ irred.}} \left[\prod_{r=1}^{\infty} (1-\frac{u^{deg(\phi)}}{q^{r\cdot deg(\phi)}}) \right] \left[\sum_{\lambda} x_{\phi,\lambda} \frac{u^{|\lambda| deg(\phi)}}{q^{deg(\phi)\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)}} \right]. \end{aligned}$$

The theorem now follows from the use of Macdonald's principal specialization formula (as in Theorem 2) to conclude that

$$\frac{P_\lambda(\frac{u}{q}, \frac{u}{q^2}, \frac{u}{q^3}, \frac{u}{q^4}, \cdots ; \frac{1}{q})}{q^{n(\lambda)}} = \frac{u^{|\lambda|}}{q^{\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q})_{m_i(\lambda)}}. \qquad \square$$

Remark. Although Theorem 6 followed easily from techniques in Theorems 2 and 3, its formulation admits an attractive probabilistic interpretation. Fix u such that $0 < u < 1$ and fix an irreducible monic polynomial $\phi \neq z$. Then pick a natural number with probability of getting n equal to $(1-u)u^n$. Finally, choose α uniformly at random in $GL(n,q)$ and let $\lambda_\phi(\alpha)$ be the random partition so defined. This procedure induces a probability measure on the set of all partitions of all integers.

Theorem 6 implies that measures obtained for different ϕ are independent, and that the mass the measure for a given ϕ assigns to a partition λ is equal to

$$\prod_{r=1}^{\infty}(1-\frac{u^{deg(\phi)}}{q^{r\cdot deg(\phi)}})\left[\frac{P_{\lambda}\left((\frac{u}{q})^{deg(\phi)},(\frac{u}{q})^{2deg(\phi)},(\frac{u}{q})^{3deg(\phi)},\cdots;(\frac{1}{q})^{deg(\phi)}\right)}{q^{n(\lambda)\cdot deg(\phi)}}\right].$$

This connection with symmetric functions leads to a probabilistic algorithm for growing the random partitions λ_ϕ. This viewpoint is used in [Fu1] to give probabilistic proofs of some group theoretic results of Steinberg, Rudvalis/Shinoda, and Lusztig. Cycle indices for the finite unitary, symplectic, and orthogonal groups appear in [Fu2].

3. Conclusion

This paper has offered connections between Gordon's generalization of the Rogers-Ramanujan identities, the Hall-Littlewood polynomials, and generating functions which arise in the study of the finite general linear groups. The following questions seem natural.

- Are there analogs of the Gordon identities for the finite unitary, symplectic, and orthogonal groups? The Gordon identities do play a role in Misra's work on affine symplectic Lie algebras [Mi1] and in Mandia's work [Man] on the affine Lie algebras $B_l^{(1)}$, $F_4^{(1)}$, and $G_2^{(1)}$. Misra has also used Z-algebra methods to relate the Gordon identities to $\widetilde{A_n}$.
- Ian Macdonald has suggested that the results of this paper should carry over to the affine finite general linear groups. The Hall-Littlewood polynomials have affine analogs [EK].
- It is known (e.g. Kac [Ka]) that the product side of the Rogers-Ramanujan identities has interesting modular properties. Can this phenomenon be understood group theoretically in terms of the conjugacy classes of the general linear groups; i.e. is there "general linear group moonshine"? In this regard note that Jing [J1, J2] has connected the Hall-Littlewood polynomials with vertex operators.

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