AN NOTE ON $\lambda$-OPERATIONS IN ORTHOGONAL K-THEORY

MOHAMED ELHAMDADI

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1. Introduction

Let $A$ be a commutative ring in which 2 is invertible, and let $\mathcal{C}_A$ be the category of symmetric bilinear $A$-modules. Denote by $\phi_{p,q}$ the diagonal bilinear form $\langle 1, \ldots, 1, -1, \ldots, -1 \rangle$ over $A^{p+q}$ and let $Q_A$ be the full subcategory of $\mathcal{C}_A$ with objects $(M, \phi)$ isomorphic to $(A^{p+q}, \phi_{p,q})$ for a certain pair $(p, q)$ (we were led to consider this category because the exterior powers of hyperbolic forms are not generally hyperbolic). The isometry group of $(A^{p+q}, \phi_{p,q})$ will be denoted by $O_{p,q}(A)$. Let $G$ be a group and let $RQ_A(G)$ be the ring of isometry classes of representation of $G$ in $Q_A$. Recall [3, definition 3.1] that a $\lambda$-ring is a commutative ring with unit equipped with a sequence of operations $\{\lambda^n\}_{n \geq 0}$ satisfying the following properties:

(a) $\lambda^0(x) = 1$ and $\lambda^1(x) = x$, (b) $\lambda^n(x + y) = \sum_{k=0}^{n} \lambda^k(x) \lambda^{n-k}(y)$. The following observation is straightforward to check and can be found as proposition 1.3.2 on page 13 of [2].

Proposition 1.1. The exterior powers induce a $\lambda$-ring structure on $RQ_A(G)$.

In fact this proposition comes from the known formula:

$$\Lambda^k[(M, \phi) \perp (N, \psi)] \cong \sum_{0 \leq i \leq k} \Lambda^i(M, \phi) \otimes \Lambda^{k-i}(N, \psi)$$

where $(M, \phi)$ and $(N, \psi)$ are symmetric bilinear modules (proposition 1.3.2 on page 11 in [2]). We denote by $O(A)$ the infinite orthogonal group, by $BO(A)^+$ the plus construction of Quillen applied to the classifying space $BO(A)$ [4], and by $[X, Y]$ the based homotopy class of maps from a based space $X$ to another $Y$. The key to the result is the following construction of a morphism from $RQ_A(G)$ to $[BG, BO(A)^+]$. 

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2. A morphism from $RQ_A(G)$ to $[BG, BO(A)^+]$

Let $\rho$ be a representation of the group $G$ in an object $(E, \phi)$ of $Q_A$. Then we have a commutative diagram:

$$
\begin{array}{ccc}
(E, \phi) & \xrightarrow{\Psi} & (A^{p+q}, \phi_{p,q}) \\
& & \\
& & (A^{p'+q'}, \phi_{p',q'})
\end{array}
$$

The action of $G$ on $(E, \phi)$ gives a mapping $R : G \to Isom(E)$ and we obtain the commutative diagram:

$$
\begin{array}{ccc}
O_{p,q}(A) & \xrightarrow{\Upsilon} & O_{n,n}(A) \\
G & \xrightarrow{\Upsilon} & G \\
O_{p',q'}(A) & \xrightarrow{\Upsilon} & O_{n,n}(A)
\end{array}
$$

where the homomorphism $\Upsilon$ is the conjugate by $\Psi$. By using a theorem of Witt of extension of isometries in the case of a commutative ring in which 2 is invertible, which comes from [8, theorem 7.2], we can extend $\Psi$ to an isometry $\Psi$ on $(A^{2n}, \phi_{n,n})$ where $n = p + q = p' + q'$ and then obtain the commutative diagram:

$$
\begin{array}{ccc}
O_{p,q}(A) & \xrightarrow{\Upsilon} & O_{n,n}(A) & \xrightarrow{} & O(A) \\
G & \xrightarrow{} & O_{n,n}(A) & \xrightarrow{} & O(A) \\
O_{p',q'}(A) & \xrightarrow{} & O_{n,n}(A) & \xrightarrow{} & O(A)
\end{array}
$$

The composite mappings $u : G \to O_{p,q}(A) \to O_{n,n}(A)$ and $v : G \to O_{p',q'}(A) \to O_{n,n}(A)$ are conjugate by the element $\overline{\Psi}$ of the orthogonal group $O_{n,n}(A)$. By including $O_{n,n}(A)$ in $O_{2n,2n}(A)$ and replacing $\overline{\Psi}$ by $\overline{\Psi} \oplus \overline{\Psi}^{-1}$, we can suppose that $\overline{\Psi}$ is an element of the elementary group $E_{2n,2n}(A) \subset O_{2n,2n}(A)$. By Proposition 1.1.9 of [7], we conclude that $u$ and $v$ induce homotopic maps from $BG$ to $BO(A)^+$. The homotopy class of these maps will be denoted by $r(E)$. We define the mapping from $RQ_A(G)$ to $[BG, BO(A)^+]$ on generators by:

$$
\begin{array}{ccc}
r : RQ_A(G) & \xrightarrow{} & [BG, BO(A)^+] \\
[\rho] & \mapsto & r(E).
\end{array}
$$

**Theorem 2.1.** There exist mappings $\lambda^k$, up to weak homotopy, from $BO(A)^+$ to $BO(A)^+$, inducing $\lambda$-operations on orthogonal $K$-theory.
A note on \( \lambda \)-operations in orthogonal K-theory

**Proof.** This was inspired by the first paragraph in section 5 of [6]. Let \( G \) be the group \( O_{n,n}(A) \) and \([A_{id}^{n,n}]\) the class of the identity representation of \( G \) in the object \((A^{2n}, \phi_{n,n})\) of \( Q_A \). Denote by \( 2n.1 : O_{n,n}(A) \to O_{n,n}(A) \) the trivial representation which sends each element to the identity matrix. The difference \([A_{id}^{n,n}] - [2n.1]\) will be denoted by \([A_{id}^{n,n}]^{-}\). Since the structure of a \( \lambda \)-ring in \( RQ_A \) is given by exterior powers, by considering the exterior powers of \([A_{id}^{n,n}]\) and \([2n.1]\) we see that the following diagram commutes:

\[
\begin{array}{ccc}
RQ_A(O_{n+1,n+1}(A)) & \xrightarrow{i_n^*} & RQ_A(O_{n,n}(A)) \\
\lambda^k \downarrow & & \lambda^k \\
RQ_A(O_{n+1,n+1}(A)) & \xrightarrow{i_n^*} & RQ_A(O_{n,n}(A))
\end{array}
\]

Let us consider the following diagram:

\[
\begin{array}{ccc}
RQ_A(O_{n+1,n+1}(A)) & \xrightarrow{r} & [BO_{n+1,n+1}(A), BO(A)^+] \\
i_n^* \downarrow & & \downarrow \\
RQ_A(O_{n,n}(A)) & \xrightarrow{r} & [BO_{n,n}(A), BO(A)^+]
\end{array}
\]

From the construction of \( r \) and the fact that the restriction morphism sends \([A_{id}^{n+1,n+1}]^{-}\) to \([A_{id}^{n,n}]^{-}\), the restriction of \( r(\lambda^k[A_{id}^{n+1,n+1}]^{-}) \) to \( BO_{n,n}(A) \) is \( r(\lambda^k[A_{id}^{n,n}]^{-}) \). Let \( \lambda^k_{n,n} \) be a continuous map in the homotopy class \( r(\lambda^k[A_{id}^{n,n}]^{-}) \); then the following triangle is homotopy commutative:

\[
\begin{array}{ccc}
BO_{n,n}(A) & \xrightarrow{\lambda^k_{n,n}} & BO(A)^+ \\
\downarrow & & \downarrow \\
BO_{n+1,n+1}(A) & \xrightarrow{\lambda^k_{n+1,n+1}} & BO(A)^+
\end{array}
\]

We also obtain a map \( \lambda^k : BO(A) \to BO(A)^+ \) which passes to the Plus-construction and gives a map \( \lambda^k : BO(A)^+ \to BO(A)^+ \). For any compact subset \( K \) of \( BO(A)^+ \), there exists a positive integer \( m \) such that \( K \subset BO_{m,m}(A)^+ \), and we deduce that the precedent map is defined up to weak homotopy.

We define the operations on orthogonal K-theory by:

\[
\lambda^k : \pi_n(BO(A)^+) = [S^n, BO(A)^+] \to [S^n, BO(A)^+] \\
[f] \mapsto [\lambda \circ f].
\]

These operations produce a \( \lambda \)-ring structure on orthogonal K-theory ([2], page 21) analogous to the \( \lambda \)-ring structure in classical algebraic K-theory [6, theorem 5.1].

**Remark.** To get a special \( \lambda \)-ring, one needs a splitting principle [10]. We do not know if there exists an analogue for the study of rank two modules in this case.

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References


Department of Mathematics, University of South Florida, 4202 East Fowler Ave., PHY 114, Tampa, Florida 33620-5700
E-mail address: emohamed@math.usf.edu