PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 1, Pages 31–38
S 0002-9939(99)05412-X
Article electronically published on July 27, 1999

FREE G_a ACTIONS ON C^3

JAMES K. DEVENEY AND DAVID R. FINSTON

(Communicated by Ron Donagi)

ABSTRACT. It has been conjectured that every free algebraic action of the additive group of complex numbers on complex affine three space is conjugate to a global translation. The main result lends support to this conjecture by showing that the morphism to the variety defined by the ring of invariants of the associated action on the coordinate ring is smooth. As a consequence, the graph morphism is an open immersion, and simple proofs of certain cases of the conjecture are obtained.

1. INTRODUCTION

Let G_a denote the additive group of complex numbers, and X a complex affine variety. A fixed point free algebraic action of G_a on X will be referred to as free, although this terminology may not agree with notions of free already in the literature. Given such an action $\sigma : G_a \times X \to X$ and $\hat{\sigma} : \mathbf{C}[X] \to \mathbf{C}[X,t]$ the induced map on coordinate rings, differentiating $\hat{\sigma}$ yields a locally nilpotent derivation δ of $\mathbf{C}[X]$:

$$\delta(P) = \frac{\hat{\sigma}(P) - P}{t} \mid_{t=0}, \quad \hat{\sigma} = exp(t\delta).$$

Every $\hat{\sigma}$, hence every G_a action, arises as the exponential of a locally nilpotent derivation.

A central question is whether the action is "conjugate to a translation", i.e. whether X is (G_a) equivariantly isomorphic to $Y \times \mathbf{C}$ for some affine variety Y, where the action fixes the first coordinate and is addition on the second coordinate. In this case, the action is conjugate in the full group of automorphisms of X to a translation. The cancellation problem asks for $X = \mathbf{C}^n$ whether a G_a action which is conjugate to a translation has $Y \cong \mathbf{C}^{n-1}$. The cancellation problem is known to have a positive solution only for $n \leq 3$ (see [16] for instance).

It is well known that the ring C_0 of G_a invariants in $\mathbf{C}[X]$ is equal to the kernel of the associated derivation δ so that, in light of [18, Proposition 2.1], an action is conjugate to a translation if and only if there is a function $s \in \mathbf{C}[X]$ satisfying $\delta(s) = 1$. Such an s is called a slice. The action is said to be locally trivial (with respect to the Zariski topology) if there is a cover of X by G_a stable affine open subsets, on each of which the action is conjugate to a translation.

Received by the editors March 24, 1996 and, in revised form, July 3, 1997, November 26, 1997, and March 9, 1998.
1991 *Mathematics Subject Classification.* Primary 14L30; Secondary 20G20.
The second author was supported in part by NSA Grant MDA904-96-1-0069.

It has been conjectured that if U is a unipotent algebraic group acting freely as algebraic automorphisms of a smooth, factorial affine variety X and the ring of U invariants in $\mathbf{C}[X]$ is regular of dimension 2, then the action is locally trivial. In the special case where $X = \mathbf{C}^n$ and the dimension of U is at least $n-2$, a famous result of Miyanishi [10, Theorem 4] gives the requisite condition on the ring of invariants. For $n = 3$, $U = G_a$, and the action proper, an Euler characteristic argument shows that the action is not merely locally trivial, but in fact conjugate to a translation [4]. Fauntleroy and Magid prove in [6] that a free G_a action on any factorial quasiaffine surface is conjugate to a translation. If the surface is the affine plane, then the triangulability of all G_a actions on $\mathbf{C}^2$ (e.g. [1]) can also be used to prove the result.

The primary concern here is the conjecture for $U = G_a$ and $X = \mathbf{C}^3$. The main contribution is the result that the morphism $X \to Y$ is smooth, where Y denotes the variety associated to the ring of invariants (the aforementioned result of Miyanishi, or a well known theorem of Zariski, does indeed show that the ring of invariants is affine in this case). As a consequence we can realize the derivation generating the G_a action as a "cross product" in a nice way. Moreover, some known cases of the conjecture can be strengthened and proved in an appealing manner.

The conjecture has important implications for the structure of the automorphism group of affine three space. The "Generation Gap" problem [1] asks whether the groups of invertible linear maps and triangular maps generate the full automorphism group of $\mathbf{C}[x_1, \ldots , x_n]$. It is known that the problem has a positive solution for $n \leq 2$, but remains open otherwise, although a candidate for an automorphism not in the subgroup generated by the linear and triangular subgroups for $n = 3$ was given by Nagata. Bass, in [1], was able to embed the Nagata automorphism in a G_a action which was shown not to be conjugate to a subgroup of the triangular group. Other classes of nontriangulable G_a actions have been discovered, yet in each case the nontriangulability is attributable to the existence of fixed points for the action, so a natural question is whether the presence of fixed points is the only obstruction to triangulability. The conjecture would certainly confirm that in dimension three a G_a action without fixed points can be conjugated into the triangular group.

Most of these results were presented, and some obtained, at the Curaçao conference on Automorphisms of Affine Spaces, July, 1994, organized by Professor Arno van den Essen. The authors wish to thank Prof. van den Essen for providing the stimulating environment at that conference, and also Prof. Hanspeter Kraft for many enlightening conversations there. The referee of an earlier version of this paper took great pains to correct certain inaccuracies, simplify some of our arguments, and show us how to generalize others. The paper was tremendously improved by these changes and we wish to express our sincere gratitude to the referee.

2. Smoothness of the quotient morphism

Throughout this section, X will denote a complex affine variety and $\mathbf{C}[X]$ its coordinate ring. The ring of G_a invariants for an action as automorphisms of $\mathbf{C}[X]$ will be denoted by C_0 and $\Omega = \Omega_{\mathbf{C}[X]/C_0}$ will denote the module of differentials of $\mathbf{C}[X]$ over C_0. The $\mathbf{C}[X]$ module of C_0 derivations of $\mathbf{C}[X]$, denoted $\mathfrak{D}$, is isomorphic to the dual of Ω. As usual, δ is the locally nilpotent derivation of $\mathbf{C}[X]$ associated to the action. It is readily verified that the zero set of $\{\delta f \mid f \in \mathbf{C}[X]\}$ is the fixed point set for the corresponding G_a action; thus the action is free if and only if the image of δ generates the unit ideal in $\mathbf{C}[X]$.

Proposition 2.1. *If G_a acts on the affine variety X, then the module of derivations of $\mathbf{C}[X]$ over C_0 is free of rank one, generated as a $\mathbf{C}[X]$ module by the derivation δ defining the action in each of the following cases:*

1. *The action is free and X arbitrary.*
2. *The fixed point set of the action has codimension at least two and X is normal.*

Proof. If the action admits a slice, then the result is clear. Otherwise, let $\delta g = h \neq 0 = \delta h$, and set $s = \frac{g}{h}$. Any $D \in \mathfrak{D}$ extends uniquely to a $C_0[\frac{1}{h}]$ derivation of $\mathbf{C}[X, \frac{1}{h}]$ which will also be denoted by D. Since the action δ induces on $\mathbf{C}[X, \frac{1}{h}]$ has a slice, it is clear that $D = \frac{H}{h^n}\delta$ for some $H \in \mathbf{C}[X]$. Write $\mathbf{C}[X] = \mathbf{C}[u_1, \dots, u_m]$ and evaluate $D(u_i) = \frac{H}{h^n}\delta(u_i) \in \mathbf{C}[X]$. In case 1 $\{\delta u_i : 1 \leq i \leq m\}$ generates the unit ideal in $\mathbf{C}[X]$, and it follows easily that $\frac{H}{h^n} \in \mathbf{C}[X]$. In case 2, note that the image of δ is contained in no height one prime ideal of $\mathbf{C}[X]$. For each height one prime ideal p therefore, at least one of δu_i is a unit in the localization $\mathbf{C}[X]_p$, and thus $\frac{H}{h^n} \in \mathbf{C}[X]_p$. By normality we conclude that $\frac{H}{h^n} \in \mathbf{C}[X]$. □

Remark. If the normal variety X admits a locally trivial G_a action and the geometric quotient $Y = X/G_a$ exists and is separated, then Ω is free of rank one as a $\mathbf{C}[X]$ module. Indeed, $G_a \times X \cong X \times_Y X \cong \mathbf{Spec}\ \mathbf{C}[X] \otimes_{C_0} \mathbf{C}[X]$, where $Y = \mathbf{Spec}\ C_0$ [5, Theorem 1.8]. The kernel I of the "multiplication" homomorphism $\mathbf{C}[X] \otimes_{C_0} \mathbf{C}[X] \to \mathbf{C}[X]$ is then identified with the ideal generated by t in the ring $\mathbf{C}[G_a \times X] = \mathbf{C}[X, t]$. Clearly $I/I^2 \cong (t)/(t)^2$ as $\mathbf{C}[X]$ modules. The latter is obviously free of rank one. For a locally trivial action on a factorial affine variety, the geometric quotient exists and is quasiaffine, hence separated [3].

The following example shows that $\mathfrak{D}$ may be free even if the action has fixed points and Ω is not free. It is conjectured that Ω is free of rank one when the extension $C_0 \hookrightarrow \mathbf{C}[X]$ is flat and the action is free.

Example 1. Let $X = \mathbf{C}^3$ and define δ on $\mathbf{C}[X] = \mathbf{C}[x_1, x_2, x_3]$ by

$$\delta(x_1) = 0,\ \delta(x_2) = x_1,\ \delta(x_3) = x_2.$$

The $\mathbf{C}[X]$ module of C_0 derivations of $\mathbf{C}[X]$ is free of rank one, but Ω is not free.

Proof. That the module of derivations is free of rank one follows from Proposition 2.1 (2), since the fixed point set for the G_a action is the line $x_1 = x_2 = 0$.

The kernel of δ is well known to be $\mathbf{C}[x_1, x_2^2 - 2x_1x_3] = C_0$. As a $\mathbf{C}[X]$ module, Ω is the quotient of the free module with basis $\{dx_1, dx_2, dx_3\}$ by the submodule generated by dx_1 and $d(x_2^2 - 2x_1x_3)$. Thus $\Omega \cong \mathbf{C}[X] \oplus \mathbf{C}[X]/\mathbf{C}[X](x_2, -x_1)$ which is easily seen not to be free ($(x_2, -x_1)$ is not unimodular). □

For a G_a action on the affine variety X with defining derivation δ, call the action primitive if $\delta = q\rho$ for $q \in \mathbf{C}[X]$ and $\rho \in \mathfrak{D}$ implies $q \in \mathbf{C}[X]^*$. If X is factorial, then every G_a action has the form $\exp(tq\delta)$ where $\exp(t\delta)$ is primitive and $\delta(q) = 0$.

Also, for X affine with G_a action, and $f \in \mathbf{C}[X]$, denote by $o(f)$ the degree in t of $\hat{\sigma}(f)$ (this is one less than the least power of δ annihilating f). Observe that $o(fg) = o(f) + o(g)$ for all $f, g \in \mathbf{C}[X]$. It follows easily for G_a, indeed for any unipotent group, that the action is trivial on $\mathbf{C}[X]^*$. If $o(f) = 1$, then $\frac{f}{\delta f}$ is a slice for the G_a action on the principal open subset $X_{\delta f}$.

Let d denote the universal derivation from $\mathbf{C}[X]$ to Ω.

Theorem 2.2. *A primitive G_a action on X is conjugate to a translation if and only if Ω is freely generated over $\mathbf{C}[X]$ by ds for some $s \in \mathbf{C}[X]$. In this case there is a unit λ for which λs is a slice.*

Proof. ($\Rightarrow$) If s is a slice, then $\mathbf{C}[X] \cong C_0[s]$, and ds clearly generates Ω.

($\Leftarrow$) Assume that Ω is freely generated by ds, and let $f \in Hom_{\mathbf{C}[X]}(\Omega, \mathbf{C}[X])$ determine δ. Since the image of δ is contained in the ideal generated by $f(ds)$ and the action is primitive, $f(ds)$ is a unit, say $\frac{1}{\lambda}$. As remarked above, $\mathbf{C}[X]^* \subset C_0$; thus $f(d(\lambda s)) = \delta(\lambda s) = 1$. □

The following is Winkelmann's example of a non-affine quotient [17, p.595].

Example 2. Let $X = \mathbf{C}^5$ and let δ be the derivation of $\mathbf{C}[X] = \mathbf{C}[x_1, \ldots, x_5]$ given by $x_1, x_2 \mapsto 0,\ x_3 \mapsto x_1,\ x_4 \mapsto x_2,\ x_5 \mapsto 1 + x_1x_4 - x_2x_3$. Then Ω is free of rank one, but is not generated by ds for any $s \in \mathbf{C}[X]$.

Proof. Winkelmann has shown that the associated G_a action is not conjugate to a global translation, so that Ω cannot be generated by any ds. However, it was also shown that the action is locally trivial and that a geometric quotient exists and is a quasiaffine variety. □

Proposition 2.3. *Let $X = \mathbf{C}^3$. The ring of invariants, C_0, is generated by two algebraically independent polynomials: $C_0 = \mathbf{C}[f, g]$. With Y denoting the affine variety with coordinate ring C_0, the corresponding morphism $\pi : X \to Y$ is flat and the image has finite complement.*

Proof. Such f, g exist by [10, Theorem 4]. Flatness follows from the regularity of C_0 and [9, Corollary p.179], once it is established that fibers of the morphism $\pi : X \to Y$ are one dimensional or empty.

It suffices to exclude the existence of two dimensional fibers. An irreducible two dimensional component Z of a fiber is the zero set of a polynomial h, which is necessarily an invariant and irreducible in both C_0 and $\mathbf{C}[X]$. These assertions follow since G_a can only act trivially on a finite set and has no multiplicative characters. In addition, C_0 is factorially closed in $\mathbf{C}[X]$. Thus h defines an irreducible curve L in Y which contains $\pi(Z)$ as a dense subset. But $\pi(Z)$ is a single point, a contradiction.

Since π is flat, it is open, and therefore the complement is closed. If L is an irreducible curve in the complement, then it is defined by an invariant h which is irreducible in both C_0 and $\mathbf{C}[X]$. Since $\pi(X)$ doesn't meet L, it follows that h has no zeros in X, a contradiction. □

For the remainder of this section X will denote $\mathbf{C}^3$ and we assume that G_a acts on X. Denote by $Y (\cong \mathbf{C}^2)$ the affine variety with coordinate ring C_0, and by $\pi : X \to Y$ the morphism induced by the inclusion of C_0 in $\mathbf{C}[X]$. We refer to π as the quotient morphism.

Remark. It is unknown whether π is surjective in general. The following corollary casts a somewhat different light on a special case of a result of Rentschler.

Corollary 2.4 ([13])**.** *Let G_a act on $\mathbf{C}^3$. Then the action has no isolated fixed points.*

Proof. Assuming a nontrivial action, let F denote the fixed point set and suppose that $x \in F$. Then $\pi^{-1}(\pi(x))$ is one dimensional, closed, and G_a stable. If Z is the irreducible component containing x, then $Z - F$ is either empty or an orbit. But all orbits are closed, so that $Z - F$ is empty. □

The following remark, suggested by the referee, is used in the next lemma. The relevant facts about smoothness can be found in [7, Chap. III, Sec. 10].

Remark. Let $\phi : X \to Y$ be a dominant morphism of smooth varieties, Y' a locally closed subset of Y, $X' = \phi^{-1}(Y')$ the schematic inverse image, and $\phi' : X' \to Y'$ the induced morphism. For $x \in X'$, we observe that ϕ is smooth at x if and only if ϕ' is smooth at x. The smoothness of ϕ' at x follows from that of ϕ by base change. For the converse, note that both X and Y are smooth, so that ϕ is smooth at x if and only if the schematic fiber $\phi^{-1}(\phi(x))$ is smooth at x. But the schematic fiber of ϕ is the same as that of ϕ'.

Lemma 2.5. *The quotient morphism $\pi : X \to Y$ is singular on a set of codimension at least two in X. If the action is primitive, then the singular set is contained in finitely many π fibers.*

Proof. Let X_s denote the closed set in X at which π is singular, and suppose that $W \subset X_s$ is a two dimensional component. Since X_s is stable under the group action and G_a can only act trivially on a finite set, we see that W is defined by an irreducible $h \in C_0$. If Z denotes the subset of Y defined by h, we have $Z = \pi(W)$, and W is equal to the schematic inverse image $\pi^{-1}(Z)$. By assumption π is singular on all of W, so that, by the remark above, $\pi |_W : W \to Y$ is singular everywhere on W, a contradiction [7, p. 271, Lemma 10.5].

Note that X_s is stable under the group action. If the action is free, it is clear that X_s consists of finitely many orbits, hence finitely many π fibers. Let F denote the fixed point set for the action. Primitivity implies that F has no two dimensional components, since such a component would be defined by an invariant dividing the image of δ. Since F is closed and stable under the group action, F consists of finitely many π fibers as there are no isolated fixed points. □

Theorem 2.6. *Suppose that the additive group G_a acts freely on $\mathbf{C}^3$ with quotient map $\pi : \mathbf{C}^3 \to Y$. Then $Y \cong \mathbf{C}^2$ and π is a smooth morphism.*

Proof. We show that the matrix

$$J(f,g) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} \end{bmatrix}$$

has rank two at all points of X.

From the lemma above we know that $J(f,g)$ is singular on only finitely many fibers in X. Define $D \in \mathfrak{D}$ by

$$D(h) = \begin{vmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \end{vmatrix}$$

Since $\mathfrak{D}$ is a rank one free $\mathbf{C}[X]$ module, $D = H\delta$ for some $H \in \mathbf{C}[X]$ (as usual, δ is the locally nilpotent derivation generating the action). Write $D = \sum_{i=1}^{3} J_i \frac{\partial}{\partial x_i}$, and $\delta = \sum_{i=1}^{3} \delta x_i \frac{\partial}{\partial x_i}$. Thus $J_i = H\delta x_i$ and, since the δx_i generate the unit ideal in

$\mathbf{C}[X]$, the ideal generated by the J_i is the principal ideal generated by H. Unless $H \in \mathbf{C}$, we obtain a two dimensional component of points at which π is singular, contradicting the smoothness in codimension one.

Thus $D = \lambda\delta$ for some nonzero complex number λ. Observe, however, that any point at which π is singular is then a common zero of the δx_i, hence a fixed point for the group action. □

Remark. We have shown that the vector field corresponding to the derivation δ can be realized as a "cross product". That is to say, $\delta = \mathbf{grad}\ f \times \mathbf{grad}\ g$.

Corollary 2.7. *The fiber product $X \times_Y X$ is smooth.*

Proof. By base extension, $\tilde{\pi} : X \times_Y X \to X$ is smooth and, therefore, so is $X \times_Y X$. □

Corollary 2.8. *The morphism $\tilde{\sigma} : G_a \times X \to X \times_Y X$ is an open immersion.*

Proof. Since, by the previous corollary, $X \times_Y X$ is normal, the Zariski Main Theorem applies once it is shown that the morphism has finite fibers and is birational. But since the action is free, $\tilde{\sigma}$ is injective, and since the image of $\bar{\sigma}$ contains $\mathbf{C}[X]$ and a polynomial linear in t, $\tilde{\sigma}$ is birational. □

Remark. A free G_a action on $\mathbf{C}^3$ is either proper, and therefore conjugate to a translation, or the complement of the image under $\tilde{\sigma}$ of $G_a \times X$ in $X \times_Y X$ is pure of codimension one. Indeed, $\tilde{\sigma}$ is an isomorphism if and only if the action is proper, and it was shown in [4, Theorem 2.4] that any proper G_a action on $\mathbf{C}^3$ is conjugate to a translation. Otherwise, the structure of the complement of the image is well known to be pure of codimension one (e.g. [12, Theorem 3′, p.51]).

The preceding results allow for a strengthening and simpler proof of [4, Theorem 2.4] (avoiding the use of [3, Theorem 2.8]). Assuming a proper action, the diagram

$$\begin{CD}
G_a \times X @>{\cong}>> X \times_Y X @>>> X \\
@V{pr_X}VV @V{\tilde{\pi}}VV @VV{\pi}V \\
X @= X @>{\pi}>> Y
\end{CD}$$

shows, by smooth base change, that all π fibers are single orbits and that $\pi : X \to \pi(X)$ is a principal G_a bundle. As noted in Proposition 2.3, the complement of $\pi(X)$ in Y is finite. If π is not surjective, the Euler characteristic of $\pi(X)$ (and consequently of X) is ≤ 0. Thus π is surjective and the bundle is trivial. Naturally, if it could be shown in some fashion that, for a free action, all nonempty π fibers are single orbits, then this argument would imply that the action is conjugate to a translation.

An action of the algebraic group G on the variety X is said to be separated if $\{(x, gx) : x \in X,\ g \in G\}$ is closed in $X \times X$. We note the following result of Kraft [8] which follows immediately from Corollary 2.8.

Corollary 2.9. *Every free separated G_a action on $\mathbf{C}^3$ is conjugate to a translation.*

Corollary 2.10. *If $X \times_Y X$ is factorial, then the action is conjugate to a translation.*

Proof. If not, then the complement of the image of $G_a \times X$ is pure of codimension one and therefore by factoriality defined by a single function f. But such an f would generate the unit ideal in $\mathbf{C}[X, t]$, which is impossible. □

Remark. If a free G_a action on $\mathbf{C}^3$ is not conjugate to a translation, then the obstruction is a curve in Y whose π fibers are disconnected. The complement of the image of $G_a \times X$ in $X \times_Y X$, call it Z, consists precisely of those pairs (a, b) which don't lie in the same orbit, but are indistinguishable by the ring of invariants. The ideal I defining Z is G_a stable; it is the radical of the ideal of denominators of t, so it contracts to an ideal in $\mathbf{C}[X]$ generated by invariants. In fact this ideal is the radical of the ideal generated by the intersection of the image of the derivation with C_0, since on the complement of the closed set in $\mathbf{C}^3$ defined by this ideal, the action is conjugate to a translation.

Remark. Two other cases in which it is known that a free G_a action on complex affine three space is conjugate to a translation are

1. The action is triangulable [15, Theorem 3.2] (Martha Smith and David Wright also proved this in an unpublished manuscript),

and the more general

2. The ring of invariants contains a variable of $\mathbf{C}[X]$ [2, Corollary 3.3].

The methods presented here give a simple proof of case 2:

Assume that the locally nilpotent derivation δ, for which $\delta x_1 = 0$, generates a free action. For each complex number λ let L_λ denote the curve $x_1 = \lambda$ in Y and $E_\lambda(\cong \mathbf{C}^2)$ the inverse image of L_λ in X. Observe that $\pi_\lambda \equiv \pi\mid_{E_\lambda}: E_\lambda \to L_\lambda$ is a smooth morphism by Theorem 2.6 and the fact that E_λ is the schematic inverse image of L_λ (note that $x_1 - \lambda$ is irreducible). It is well known, [6] for instance, that every free G_a action on $\mathbf{C}^2$ is conjugate to a translation. Thus $E_\lambda \cong G_a \times \mathbf{C}$ with quotient isomorphic to $\mathbf{C}$. Since π_λ is constant on G_a orbits, it factors through the quotient, yielding a smooth morphism $\rho : \mathbf{C} \to L_\lambda$. This forces ρ to be an isomorphism. Indeed, L_λ is a smooth rational affine curve, hence isomorphic to an open subset of $\mathbf{C}$. Since ρ defines a smooth morphism, the coordinate ring $\mathbf{C}[L_\lambda]$ is isomorphic to a subring of $\mathbf{C}[t]$, hence to $\mathbf{C}[t]$ by the absence of nonconstant units. Thus all π_λ fibers, consequently all π fibers, are single orbits and the result follows from the remarks after Corollary 2.8. □

Sathaye has observed in [14, p.160] that if, for $q \in \mathbf{C}[x_1, x_2, x_3] = \mathbf{C}[X]$, it happens that $\mathbf{C}[X] \otimes_{\mathbf{C}[q]} \mathbf{C}(q)$ is isomorphic to a two variable polynomial ring over $\mathbf{C}(q)$, and q defines a smooth morphism $\mathbf{C}^3 \to \mathbf{C}$, then q is a variable of $\mathbf{C}[x_1, x_2, x_3]$. The latter condition is clearly satisfied by either of the two generators f and g of C_0, and the former condition is satisfied for f (resp. g) if $\mathbf{C}[f]$ (resp. $\mathbf{C}[g]$) contains a non zero element of the image of the derivation. Thus to prove that every free action is conjugate to a translation it would suffice to prove that $im(\delta) \cap \mathbf{C}[f] \neq \{0\}$ or that $X \times_Y X$ is factorial (see Corollary 2.10).

References

[1] H. Bass: *A non-triangular action of G_a on A^3*. J. Pure and Appl. Alg. 33 (1984) 1-5. MR **85j:**14086

[2] D. Daigle, G. Freudenburg: *Locally nilpotent derivations over a UFD and an application to rank two locally nilpotent derivations of $K[X_1, \ldots, X_n]$.*, J. Algebra 204 (1998), 353–371. MR **99c:**13011

[3] J.K. Deveney, D.R. Finston, M. Gehrke: *G_a actions on $\mathbf{C}^n$*. Comm. Alg. 22 (1994) 6295-6302. MR **95e:**14038

[4] J.K. Deveney, D.R. Finston: *G_a actions on $\mathbf{C}^3$ and $\mathbf{C}^7$*. Comm. Alg. 22 (1994) 4977-4988. MR **95j:**13004

[5] A. Fauntleroy: *Geometric invariant theory for general algebraic groups.* Compositio Math. 55 (1985) 63-87. MR **86m:**14008

[6] A. Fauntleroy, A. Magid: *Quasiaffine surfaces with G_a-actions.* Proc. AMS 68 (1978) 265-270. MR **57:**12529

[7] R. Hartshorne: Algebraic Geometry Springer, Berlin, 1977. MR **57:**3116

[8] H. Kraft: *Challenging problems on affine n-space.* Séminaire BOURBAKI, Juin 1995, 47ème, 1994-95, no. 802. MR **97m:**14042

[9] H. Matsumura: CommutativeRing Theory. Cambridge Studies in Advanced Mathematics 8. Cambridge University Press, Cambridge, 1989. MR **90i:**13001

[10] M. Miyanishi: *Normal affine subalgebras of a polynomial ring.* In Algebraic and Topological Theories - to the memory of Dr. Takehiko Miyata. Kinokuniya, Tokyo (1985) 37-51.

[11] D. Mumford: The Red Book of Varieties and Schemes. Lecture Notes in Mathematics No. 1358. Springer-Verlag, Berlin-Heidelberg-New York (1988). MR **89k:**14001

[12] M. Nagata: *Lectures on the Fourteenth Problem of Hilbert.* Tata Institute of Fundamental Research, Bombay, 1965. MR **35:**6663

[13] R. Rentschler: *Opérations du groupe additif sur le plan affine.* C. R. Acad. Sc. Paris, t. 267, Série A (1968) 384-387. MR **38:**1093

[14] A. Sathaye: *Polynomial ring in two variables over a D.V.R.: A criterion.* Invent. Math. 74 (1983) 159-168. MR **85j:**14098

[15] D. Snow: *Unipotent actions on affine space.* In Topological Methods in Algebraic Transformation Groups. Birkhäuser, Boston (1989). MR **91f:**14048

[16] T. Sugie: *Algebraic characterization of the affine plane and the affine three space.* In Topological Methods in Algebraic Transformation Groups. Birkhäuser, Boston (1989). MR **91a:**14022

[17] J. Winkelmann: *On free holomorphic $\mathbf{C}$-actions on $\mathbf{C}^\mathbf{n}$ and homogeneous Stein manifolds.* Math. Ann. 286 (1990) 593-612. MR **90k:**32094

[18] D. Wright: *On the Jacobian conjecture.* Ill. J. Math. 25 (1981) 423-440. MR **83a:**12032

Department of Mathematical Sciences, Virginia Commonwealth University, 1015 W. Main St., Richmond, Virginia 23284

E-mail address: `jdeveney@atlas.vcu.edu`

Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico 88003

E-mail address: `dfinston@nmsu.edu`