PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 2, Pages 547–556
S 0002-9939(99)04999-0
Article electronically published on July 6, 1999

A DISTRIBUTIONAL CONVOLUTION FOR A GENERALIZED FINITE FOURIER TRANSFORMATION

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. In this paper we define a generalized finite Fourier transformation in distribution spaces. Also we investigate a distributional convolution for this finite integral transformation.

1. INTRODUCTION

K. Trimèche [T1] introduced the generalized Fourier transformation

$$(\mathcal{F}f)(\lambda) = \int_0^\infty \varphi_\lambda(x) f(x) A(x) dx$$

where (i) $A(t)$ is a real function defined on $[0, \infty)$ such that $A(t) = t^{2\alpha+1} C(t)$ with $a > -1/2$ and $C(t)$ is an even, infinitely differentiable and strictly positive function on $\mathbb{R}$, and (ii) the kernel φ_λ is the solution of the initial value problem

$$\begin{aligned} \Delta\varphi &= -\lambda^2 \varphi, \\ \varphi(0) &= 1, \\ D\varphi(0) &= 0 \end{aligned} \tag{1}$$

for every $\lambda \in \mathbb{C}$. In (1) Δ represents the operator $\frac{1}{A(t)} D(A(t)D) - q(t)$, $D = \frac{d}{dt}$, and the function $q(t)$ is even and infinitely differentiable on $\mathbb{R}$. Besides, he defined and investigated a convolution for the $\mathcal{F}$-transformation.

Later, K. Trimèche [T2] studied generalized Fourier series expansions associated to the operator Δ when $q(t) = -\gamma^2$, $\gamma \geq 0$. We now recall some of his definitions and results that will be useful in the sequel. Thus, $\mathcal{E}_*(\mathbb{R})$ is the space of all even infinitely differentiable functions on $\mathbb{R}$. We assign to $\mathcal{E}_*(\mathbb{R})$ the topology generated by the family of seminorms $\{p_{n,m}\}_{m\in\mathbb{N}}$ where

$$p_{n,m}(\phi) = \sup_{|x|\leq n} |D^m \phi(x)|, \qquad \phi \in \mathcal{E}_*(\mathbb{R}),$$

for every $n, m \in \mathbb{N}$.

The generalized translation operator associated to Δ is defined for every $\phi \in \mathcal{E}_*(\mathbb{R})$ by

$$(\tau_x \phi)(y) = \chi_x \chi_y [\sigma_x \chi^{-1} \phi(y)], \quad x, y \in \mathbb{R},$$

Received by the editors July 12, 1996 and, in revised form, April 7, 1998.
1991 *Mathematics Subject Classification.* Primary 46F12.
Key words and phrases. Convolution, distribution, Fourier transformation.

where $(\sigma_x\phi)(y) = \frac{\phi(x+y)+\phi(x-y)}{2}$, $x, y \in \mathbb{R}$, and χ represents a generalized Riemann-Liouville integral that transmutes the operator Δ into the operator D^2 (Definition 5.1 in [T1]).

Now let $a > 0$. Denote by $\mathcal{M}_a(\mathbb{R})$ the space of functions $\phi \in \mathcal{E}_*(\mathbb{R})$ such that $(\tau_x\phi)(a) = 0$, for each $x \in \mathbb{R}$. The space $\mathcal{E}_*(\mathbb{R})$ induces on $\mathcal{M}_a(\mathbb{R})$ the topology generated by the family $\{\alpha_k\}_{k\in\mathbb{N}}$ of seminorms where for every $k \in \mathbb{N}$

$$\alpha_k(\phi) = \sup_{x\in(0,a)} |\Delta^k\phi(x)|, \qquad \phi \in \mathcal{M}_a(\mathbb{R}).$$

Thus $\mathcal{M}_a(\mathbb{R})$ is a Fréchet Montel space. The dual space of $\mathcal{M}_a(\mathbb{R})$ will be denoted, as usual, by $\mathcal{M}_a(\mathbb{R})'$.

Let $\phi, \psi \in \mathcal{M}_a(\mathbb{R})$. Then, with the aid of the τ-translation, the convolution $\phi\#\psi$ is defined by

$$(\phi\#\psi)(x) = \int_0^a (\tau_x\phi)(y)\psi(y)A(y)dy, \qquad x \in \mathbb{R}.$$

Next, $\{\pm\lambda_k\}_{k\in\mathbb{N}}$ denote the zeros of the entire function $\lambda \to \varphi_\lambda(a)$ with $\lambda_0 = 0$. From now on φ_{λ_n} will always be represented by φ_n, where $n \in \mathbb{N}$. Note also that φ_n is in $\mathcal{M}_a(\mathbb{R})$, for every $n \in \mathbb{N}$. Then for every $\phi \in \mathcal{M}_a(\mathbb{R})$ one has

$$\phi(x) = \sum_{n=0}^{\infty} \rho_n \mathcal{F}(\phi)(n)\varphi_n(x) \tag{2}$$

where the series converges to ϕ in $\mathcal{E}_*(\mathbb{R})$. Moreover for every $n \in \mathbb{N}$

$$\rho_n = \frac{1}{\int_0^a \varphi_n^2(x)A(x)dx}$$

and

$$\mathcal{F}(\phi)(n) = \int_0^a \varphi_n(x)\phi(x)A(x)dx.$$

Let $\phi \in \mathcal{M}_a(\mathbb{R})$. The sequence $(\mathcal{F}(\phi)(n))_{n\in\mathbb{N}}$ will be called the generalized finite Fourier transform $\mathcal{F}(\phi)$.

Now some properties are listed:

i) The expression (2) can be seen as an inversion formula for the $\mathcal{F}$-transformation.

ii) Let $\phi, \psi \in \mathcal{M}_a(\mathbb{R})$. The convolution takes the form

$$(\phi\#\psi)(x) = \sum_{n=0}^{\infty} \rho_n \mathcal{F}(\phi)(n)\mathcal{F}(\psi)(n)\varphi_n(x), \qquad x \in \mathbb{R}.$$

iii) The #-convolution defines a continuous mapping from $\mathcal{M}_a(\mathbb{R})X\mathcal{M}_a(\mathbb{R})$ into $\mathcal{M}_a(\mathbb{R})$.

We now consider the space $\mathcal{V}$ of all complex sequences $(a_n)_{n\in\mathbb{N}}$ satisfying

$$\beta_k((a_n)_{n\in\mathbb{N}}) = \sum_{n=0}^{\infty} \rho_n|\lambda_n|^{2k}|a_n| < \infty, \quad \text{for every } k \in \mathbb{N}.$$

The space $\mathcal{V}$ is endowed with the topology associated to the family $\{\beta_k\}_{k\in\mathbb{N}}$ of seminorms.

Proposition 1. *The generalized finite Fourier transform is a homeomorphism from* $\mathcal{M}_a(\mathbb{R})$ *onto the space* $\mathcal{V}$.

Proof. It can be proved making use of Theorem 3.I.1, Lemma 3.I.2 and Proposition 3.I.2 (1,2) of [T2]. □

Motivated by the works of A. H. Zemanian [Z1] and R. S. Pathak [P], in this paper we investigate the generalized finite Fourier transformation $\mathcal{F}$ of distributions. In Section 2 we define the generalized finite Fourier transformation on $\mathcal{M}_a(\mathbb{R})'$. The distributional #-convolution is studied in Section 3.

Remark 1. Our study includes two important cases. On the one hand, when $A(t) = t^{2\alpha+1}$, $\gamma = 0$ and $\alpha > -1/2$, Δ reduces to the Bessel operator. On the other hand, if we choose $A(t) = 2^{2(\alpha+\beta+1)}(\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1}$, $\gamma = \alpha+\beta+1$ and $\alpha, \beta > -1/2$, then Δ turns out to be the Jacobi operator.

Throughout this paper C will always represent a positive constant not necessarily the same in each occurrence.

2. The generalized finite Fourier transformation

In this section we define the finite generalized Fourier transformation $\mathcal{F}$ on the space $\mathcal{M}_a(\mathbb{R})'$ and we establish its main properties.

The finite generalized Fourier transform $\mathcal{F}'(F)$ of $F \in \mathcal{M}_a(\mathbb{R})'$ is defined by

$$\langle \mathcal{F}'(F), (a_n)_{n\in\mathbb{N}} \rangle = \left\langle F(x), \sum_{n=0}^{\infty} \rho_n \varphi_n(x) a_n \right\rangle, \qquad (a_n)_{n\in\mathbb{N}} \in \mathcal{V}.$$

Using well-known results of duality we infer that $\mathcal{F}'$ is a homeomorphism from $\mathcal{M}_a(\mathbb{R})'$ onto $\mathcal{V}'$, whenever $\mathcal{M}_a(\mathbb{R})'$ and $\mathcal{V}'$ are endowed either with the strong topology or the weak* topology.

Let $F \in \mathcal{M}_a(\mathbb{R})'$. There exist $C > 0$ and $r \in \mathbb{N}$ such that

$$|\langle F, \phi \rangle| \leq C \max_{0\leq k\leq r} \sup_{x\in(0,a)} |\Delta^k \phi(x)|, \quad \phi \in \mathcal{M}_a(\mathbb{R}). \tag{3}$$

Hence, according to Theorem 1.1 in [T1], we may write

$$\left| \left\langle F(x), \sum_{n=p}^{\infty} a_n \rho_n \varphi_n(x) \right\rangle \right| \leq C \max_{0\leq k\leq r} \sum_{n=p}^{\infty} |a_n| \rho_n |\lambda_n|^{2k}$$

for every $(a_n)_{n\in\mathbb{N}} \in \mathcal{V}$ and $p \in \mathbb{N}$.

Now, by invoking again Proposition 3.I.2 (2) of [T2] we obtain

$$\left| \left\langle F(x), \sum_{n=p}^{\infty} a_n \rho_n \varphi_n(x) \right\rangle \right| \leq C \sum_{n=p}^{\infty} |a_n|\, |\lambda_n|^{2r} (|\lambda_n|^2 + \gamma^2)^{\alpha+1/2},$$

provided that $p \geq 1$.

Therefore, given an $\varepsilon > 0$, there exists a $p_0 \in \mathbb{N}$ in such a way that

$$\left| \left\langle F(x), \sum_{n=p}^{\infty} a_n \rho_n \varphi_n(x) \right\rangle \right| < \varepsilon, \quad \text{for every } p \geq p_0. \tag{4}$$

Next, keeping (3) in mind, we can write

$$\left| \sum_{n=p}^{\infty} a_n \rho_n \langle F, \varphi_n \rangle \right| \leq C \sum_{n=p}^{\infty} |a_n| (|\lambda_n|^2 + \gamma^2)^{\alpha+1/2} |\lambda_n|^{2r} < \varepsilon, \tag{5}$$

provided that p is sufficiently large.

From (4) and (5) we conclude that

$$\left\langle F(x), \sum_{n=0}^{\infty} a_n \rho_n \varphi_n(x) \right\rangle = \sum_{n=0}^{\infty} a_n \rho_n \langle F(x), \varphi_n(x) \rangle, \qquad (a_n)_{n\in\mathbb{N}} \in \mathcal{V}.$$

Thus we have established the following.

Proposition 2. *Let $F \in \mathcal{M}_a(\mathbb{R})'$. Then the sequence $(\langle F(x), \varphi_n(x)\rangle)_{n\in\mathbb{N}}$ defines a member of $\mathcal{V}'$ by*

$$\langle (\langle F(x), \varphi_n(x)\rangle)_{n\in\mathbb{N}}, (a_n)_{n\in\mathbb{N}} \rangle = \sum_{n=0}^{\infty} \rho_n \langle F(x), \varphi_n(x)\rangle a_n, \qquad (a_n)_{n\in\mathbb{N}} \in \mathcal{V},$$

and $\mathcal{F}'(F) = (\langle F(x), \varphi_n(x)\rangle)_{n\in\mathbb{N}}$ in the sense of equality in $\mathcal{V}'$. □

In the sequel we set $\mathcal{F}'(F)(n) = \langle F(x), \varphi_n(x)\rangle$, $n \in \mathbb{N}$ and $F \in \mathcal{M}_a(\mathbb{R})'$.

Remark 2. If $\phi \in \mathcal{M}_a(\mathbb{R})$, we get $\mathcal{F}'(\phi) = (\langle \phi(x), \varphi_n(x)\rangle)_{n\in\mathbb{N}} = (\mathcal{F}(\phi)(n))_{n\in\mathbb{N}}$. Consequently, the classical $\mathcal{F}$-transformation on $\mathcal{M}_a(\mathbb{R})$ is a particular case of the $\mathcal{F}'$-transformation.

We now show a representation for the elements of $\mathcal{M}_a(\mathbb{R})'$ that can be seen as an inversion formula for the generalized $\mathcal{F}'$-transformation.

Proposition 3. *Let $F \in \mathcal{M}_a(\mathbb{R})'$. Then*

$$F = \lim_{n\to\infty} \sum_{p=0}^{n} \rho_p \langle F, \varphi_p \rangle \varphi_p$$

where the convergence is understood in the strong topology of $\mathcal{M}_a(\mathbb{R})'$.

Proof. Let $\in \mathcal{M}_a(\mathbb{R})'$. Note that it is sufficient to prove $\{\sum_{p=0}^{n} \rho_p \langle F, \varphi_p\rangle \varphi_p\}_{n\in\mathbb{N}}$ is a weak* convergent sequence to F, because $\mathcal{M}_a(\mathbb{R})$ is a Montel space.

Let $\phi \in \mathcal{M}_a(\mathbb{R})$. According to [T2, Theorem 3.I.1] we get

$$\phi(x) = \sum_{p=0}^{\infty} \rho_p \mathcal{F}(\phi)(p) \varphi_p(x), \qquad x \in \mathbb{R}.$$

Moreover the series is convergent in $\mathcal{M}_a(\mathbb{R})$. By taking into account that, for every $p \in \mathbb{N}$, φ_p defines an element of $\mathcal{M}_a(\mathbb{R})'$ by

$$\langle \varphi_p, \phi \rangle = \int_0^a \varphi_p(x)\phi(x)A(x)dx, \qquad \phi \in \mathcal{M}_a(\mathbb{R}),$$

it follows that

$$\begin{aligned}\langle F, \phi\rangle &= \lim_{n\to\infty} \left\langle F, \sum_{p=0}^{n} \rho_p \mathcal{F}(\phi)(p)\varphi_p \right\rangle \\ &= \lim_{n\to\infty} \sum_{p=0}^{n} \rho_p \mathcal{F}(\phi)(p) \langle F, \varphi_p\rangle = \lim_{n\to\infty} \left\langle \sum_{p=0}^{n} \rho_p \langle F, \varphi_p\rangle \varphi_p, \phi \right\rangle.\end{aligned}$$

□

Our next objective is to characterize the complex sequences that are $\mathcal{F}'$-transforms of an element of $\mathcal{M}_a(\mathbb{R})'$. First, we need to characterize the elements of $\mathcal{V}'$.

Proposition 4. *Let H be a linear functional on $\mathcal{V}$. Then $H \in \mathcal{V}'$ if, and only if, there exists a complex sequence $(b_n)_{n\in\mathbb{N}}$ such that*

$$|b_n| \leq C\lambda_n^{2k}, \qquad n \in \mathbb{N},\ n \geq 1, \tag{6}$$

for some $C > 0$ and some $k \in \mathbb{N}$, for which

$$\langle H, (a_n)_{n\in\mathbb{N}}\rangle = \sum_{n=0}^{\infty} b_n a_n \rho_n, \qquad (a_n)_{n\in\mathbb{N}} \in \mathcal{V}. \tag{7}$$

Proof. First, assume that H takes the form (7) where $(b_n)_{n\in\mathbb{N}}$ satisfies (6). Because of Proposition 3.I.2 (2) in [T2] it is not hard to see that $H \in \mathcal{V}'$.

Conversely, let $H \in \mathcal{V}'$. As is well-known there exists $k \in \mathbb{N}$ such that

$$|\langle H, (a_n)_{n\in\mathbb{N}}\rangle| \leq C \max_{0\leq p\leq k} \sum_{n=0}^{\infty} |\lambda_n|^{2p}|a_n|, \qquad (a_n)_{n\in\mathbb{N}} \in \mathcal{V}.$$

Even more, since $\lambda_n \neq 0$, $n \in \mathbb{N}$ and $n \geq 1$, and 0 is not an adherent point of the set $\{\lambda_n\}_{n\in\mathbb{N}}$ we can write

$$|\langle H, (a_n)_{n\in\mathbb{N}}\rangle| \leq C \sum_{n=1}^{\infty} |\lambda_n|^{2k}|a_n|, \qquad (a_n)_{n\in\mathbb{N}} \in \mathcal{V}. \tag{8}$$

We now introduce the subset of $\mathcal{V}$ defined by

$$W = \{(a_n)_{n\in\mathbb{N}} \in \mathcal{V} : a_0 = 0\}$$

and the mapping

$$\begin{aligned} J :& W \longrightarrow J(W) \subset \ell_1 \\ & (a_n)_{n\in\mathbb{N}} \longrightarrow (\lambda_n^{2k} a_n)_{n\in\mathbb{N}},\ n \geq 1. \end{aligned}$$

Here as usual ℓ_1 stands for the space of all those complex sequences $(a_n)_{n\in\mathbb{N},\ n\geq 1}$ such that $\sum_{n=1}^{\infty} |a_n| < \infty$. It is obvious that J is one to one. Moreover, by virtue of (8), the linear mapping

$$\begin{aligned} L :& J(W) \subset \ell \longrightarrow \mathbb{C} \\ & (\lambda_n^{2k} a_n)_{n=1}^{\infty} \longrightarrow \langle H, (a_n)_{n\in\mathbb{N}}\rangle \end{aligned}$$

is continuous when $J(W)$ is endowed with the topology induced on it by ℓ_1. Therefore, by invoking the Hahn-Banach theorem, L can be extended to ℓ_1 as a member of ℓ_1'. Then there exists $(\beta_n)_{n\in\mathbb{N}} \in \ell_\infty$ such that

$$L(\lambda_n^{2k} a_n)_{n=1}^{\infty} = \langle H, (a_n)_{n\in\mathbb{N}}\rangle = \sum_{n=1}^{\infty} \beta_n a_n \lambda_n^{2k}, \qquad (a_n)_{n\in\mathbb{N}} \in W. \tag{9}$$

From (9) we can conclude that

$$\langle H, (a_n)_{n\in\mathbb{N}}\rangle = a_n \langle H, e_0\rangle + \sum_{n=1}^{\infty} \beta_n a_n \lambda_n^{2k}, \qquad (a_n)_{n\in\mathbb{N}} \in \mathcal{V},$$

where $e_0 = (1, 0, \dots)$, and the proof is finished by taking $b_0 = \frac{\langle H, e_0\rangle}{\rho_0}$ and $b_n = \frac{\beta_n \lambda_n^{2k}}{\rho_n}$, $n \in \mathbb{N},\ n \geq 1$. □

Corollary 1. *Let* $(b_n)_{n\in\mathbb{N}}$ *be a complex sequence. There exists* $F \in \mathcal{M}_a(\mathbb{R})'$ *such that* $\mathcal{F}'(F)(n) = b_n$, $n \in \mathbb{N}$, *if, and only if, there exist* $C > 0$ *and* $k \in \mathbb{N}$ *such that*

$$|b_n| \le C|\lambda_n|^{2k}, \qquad n \ge 1,\ n \in \mathbb{N}. \tag{10}$$

Proof. Necessity is an immediate consequence of [Z2, Theorem 1.8.1]. In order to establish the sufficiency, we consider a sequence $(b_n)_{n\in\mathbb{N}}$ satisfying (10). Then the series $\sum_{n=0}^{\infty} b_n\rho_n\varphi_n$ converges in the strong topology of $\mathcal{M}_a(\mathbb{R})'$. By denoting said limit by F, from Proposition 3 we can infer that $\mathcal{F}'(F)(n) = b_n$, $n \in \mathbb{N}$. □

3. The generalized convolution

In this section we introduce and investigate a convolution operation in $\mathcal{M}_a(\mathbb{R})'$.

According to [T2] for every $\phi \in \mathcal{M}_a(\mathbb{R})$ we define the translation operator τ_x, $x \in \mathbb{R}$, on $\mathcal{M}_a(\mathbb{R})$ as

$$(\tau_x\phi)(y) = \sum_{n=0}^{\infty} \rho_n \mathcal{F}(\phi)(n)\varphi_n(x)\varphi_n(y), \qquad x, y \in \mathbb{R}.$$

It is clear that $\mathcal{F}(\tau_x\phi)(n) = \varphi_n(x)\mathcal{F}(\phi)(n)$, $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Proposition 5. *For every* $x \in \mathbb{R}$, τ_x *continuously maps* $\mathcal{M}_a(\mathbb{R})$ *into itself.*

Proof. Let $x \in \mathbb{R}$, $\phi \in \mathcal{M}_a(\mathbb{R})$ and $k \in \mathbb{N}$. We have

$$\Delta^k(\tau_x\phi)(y) = (-1)^k \sum_{n=0}^{\infty} \rho_n \mathcal{F}(\phi)(n)\varphi_n(x)\lambda_n^{2k}\varphi_n(y), \qquad x, y \in \mathbb{R}.$$

Then, from [T2, Proposition 3.I.2 (2)] it is deduced that

$$\sup_{x\in(0,a)} |\Delta^k(\tau_x\phi)(y)| \le C \sum_{n=0}^{\infty} (|\lambda_n|^2 + \gamma^2)^{\alpha+1/2}|\lambda_n|^{2k}|\mathcal{F}(\phi(n)|.$$

Hence, since $\mathcal{F}$ is a homeomorphism from $\mathcal{M}_a(\mathbb{R})$ onto $\mathcal{V}$, we conclude that τ_x is a continuous mapping from $\mathcal{M}_a(\mathbb{R})$ into itself. □

Proposition 5 allows us to define the #-convolution $F\#\phi$ of $F \in \mathcal{M}_a(\mathbb{R})'$ and $\phi \in \mathcal{M}_a(\mathbb{R})$ as follows:

$$(F\#\phi)(x) = \langle F(y), (\tau_x\phi)(y)\rangle, \qquad x \in \mathbb{R}. \tag{11}$$

Remark 3. Note that if $f \in \mathcal{M}_a(\mathbb{R})$, then $f \in \mathcal{M}_a(\mathbb{R})'$ according to Proposition 2. Besides, $f\#\phi$ given by (11) coincides with the classical #-convolution of f and ϕ.

Two interesting properties of the #-convolution are shown in the following assertion.

Proposition 6. (i) *For every* $F \in \mathcal{M}_a(\mathbb{R})'$ *and* $\phi \in \mathcal{M}_a(\mathbb{R})$

$$(F\#\phi)(x) = \sum_{n=0}^{\infty} \rho_n \mathcal{F}(\phi)(n)\varphi_n(x)(\mathcal{F}'F)(n), \qquad x \in \mathbb{R}.$$

(iii) *For every* $F \in \mathcal{M}_a(\mathbb{R})'$, *the mapping* $\phi \to F\#\phi$ *is continuous from* $\mathcal{M}_a(\mathbb{R})$ *into itself.*

Proof. To see (i) we have to prove that

$$\left\langle F(y), \sum_{n=0}^{\infty} \rho_n \mathcal{F}(\phi)(n)\varphi_n(x)\varphi_n(y) \right\rangle = \sum_{n=0}^{\infty} \rho_n \mathcal{F}(\phi)(n)\langle F(y), \varphi_n(y)\rangle \varphi_n(x), \qquad x \in \mathbb{R}, \tag{12}$$

for every $F \in \mathcal{M}_a(\mathbb{R})'$ and $\phi \in \mathcal{M}_a(\mathbb{R})$.

Let $f \in \mathcal{M}_a(\mathbb{R})'$. As is known there exists $k \in \mathbb{N}$ such that

$$|\langle F(y), \phi(y)\rangle| \le C \max_{0\le r\le k} \sup_{y\in(0,a)} |\Delta^r \phi(y)|, \qquad \phi \in \mathcal{M}_a(\mathbb{R}).$$

Hence, for every $p \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$\left|\left\langle F(y), \sum_{n=p}^{\infty} \rho_n \mathcal{F}(\phi)(n)\varphi_n(x)\varphi_n(y) \right\rangle\right| \le \max_{0\le r\le k} \sup_{y\in(0,a)} \left|\Delta^r \left(\sum_{n=p}^{\infty} \rho_n \mathcal{F}(\phi)(n)\varphi_n(x)\varphi_n(y)\right)\right|. \tag{13}$$

By invoking again [T2, Proposition 3.I.2 (2)] one has for every $x \in \mathbb{R}$, $y \in (0,a)$, $k, p \in \mathbb{N}$,

$$\sum_{n=p}^{\infty} \rho_n \mathcal{F}(\phi)(n)|\lambda_n|^{2k}|\varphi_n(x)|\,|\varphi_n(y)| \le C \sum_{n=p}^{\infty} (|\lambda_n|^2+\gamma^2)^{\alpha+1/2}|\lambda_n|^{2k}|\mathcal{F}(\phi)(n)|. \tag{14}$$

Also by proceeding in a similar way we obtain for every $x \in \mathbb{R}$

$$\left|\sum_{n=p}^{\infty} \rho_n \mathcal{F}(\phi)(n)\langle F(y), \varphi_n(y)\rangle \varphi_n(x)\right| \le C \sum_{n=p}^{\infty} (|\lambda_n|^2+\gamma^2)^{\alpha+1/2}|\mathcal{F}(\phi)(n)|\,|\lambda_n|^{2r}. \tag{15}$$

Since $\mathcal{F}(\phi) \in \mathcal{V}$, by combining (13), (14) and (15) we can establish (12).

Part (ii) can be proved taking into account (i) and using [T2, Proposition 3.I.2 (2)]. □

We will denote by $\mathcal{L}(\mathcal{M}_a(\mathbb{R}))$ the space of the continuous linear mappings from $\mathcal{M}_a(\mathbb{R})$ into itself. We now characterize the elements of $\mathcal{L}(\mathcal{M}_a(\mathbb{R}))$ that commute with τ-translations.

Proposition 7. *Let $L \in \mathcal{L}(M_a(\mathbb{R}))$. The following two properties are equivalent.*

(i) $L(\tau_x\phi) = \tau_x L\phi$, $\phi \in \mathcal{M}_a(\mathbb{R})$ *and* $x \in \mathbb{R}$.
(ii) *There exists* $F \in \mathcal{M}_a(\mathbb{R})'$ *such that* $L\phi = F\#\phi$, $\phi \in \mathcal{M}_a(\mathbb{R})$.

Proof. Let $L \in \mathcal{L}(\mathcal{M}_a(\mathbb{R}))$ such that $L(\tau_x\phi) = \tau_x(L\phi)$ for every $x \in \mathbb{R}$ and $\phi \in \mathcal{M}_a(\mathbb{R})$. Define a functional F on $\mathcal{M}_a(\mathbb{R})$ by

$$\langle F, \phi\rangle = \langle \delta, L\phi\rangle, \qquad \phi \in \mathcal{M}_a(\mathbb{R}),$$

where δ represents the Dirac distribution.

It is clear that $F \in \mathcal{M}_a(\mathbb{R})'$. Moreover, for every $\phi \in \mathcal{M}_a(\mathbb{R})$

$$(F\#\phi)(x) = \langle F, \tau_x\phi\rangle = \langle \delta, L(\tau_x\phi)\rangle = \langle \delta, \tau_x L\phi\rangle = (L\phi)(x), \qquad x \in \mathbb{R},$$

because $\langle \delta, \tau_x\phi\rangle = \phi(x)$, for every $\phi \in \mathcal{M}_a(\mathbb{R})$ and $x \in \mathbb{R}$.

Conversely, if $L \in \mathcal{L}(\mathcal{M}_a(\mathbb{R}))$ is defined by

$$L\phi = F\#\phi, \qquad \phi \in \mathcal{M}_a(\mathbb{R}),$$

where $F \in \mathcal{M}_a(\mathbb{R})'$, then by invoking Proposition 6

$$\begin{aligned}\tau_x(L\phi) = \tau_x(F\#\phi) &= \tau_x\left(\sum_{n=0}^{\infty}\rho_n\mathcal{F}(\phi)(n)\mathcal{F}'(F)(n)\varphi_n(y)\right)\\ &= \sum_{n=0}^{\infty}\rho_n\mathcal{F}(\phi)(n)\mathcal{F}'(F)(n)\varphi_n(y)\varphi_n(x)\\ &= \left\langle F(z), \sum_{n=0}^{\infty}\rho_n\mathcal{F}(\phi)(n)\varphi_n(z)\varphi_n(y)\varphi_n(x)\right\rangle\\ &= F\#(\tau_x\phi) = L(\tau_x\phi), \qquad x \in \mathbb{R},\end{aligned}$$

because the series $\sum_{n=0}^{\infty}\rho_n\mathcal{F}(\phi)(n)\varphi_n(z)\varphi_n(y)\varphi_n(x)$ is convergent in $\mathcal{M}_a(\mathbb{R})$ for every $x, y \in \mathbb{R}$. □

Let F and G be two arbitrary elements of $\mathcal{M}_a(\mathbb{R})'$. In accordance with the preceding results, it is natural to try to define $F\#G$ by

$$\langle F\#G, \phi\rangle = \langle F, G\#\phi\rangle, \qquad \phi \in \mathcal{M}_a(\mathbb{R}). \tag{16}$$

From Proposition 6 if $F, G \in \mathcal{M}_a(\mathbb{R})'$ we have $F\#G \in \mathcal{M}_a(\mathbb{R})'$. Also Proposition 6 allows us to infer the following.

Proposition 8. *For every $F, G \in \mathcal{M}_a(\mathbb{R})'$*

$$F\#G = \sum_{n=0}^{\infty}\rho_n(\mathcal{F}'F)(n)(\mathcal{F}'G)(n)\varphi_n$$

where the convergence of the series is understood in the strong topology in $\mathcal{M}_a(\mathbb{R})'$.

Proof. By proceeding as in the proof of Proposition 6, for every $G \in \mathcal{M}_a(\mathbb{R})'$ it can be established that

$$(G\#\phi)(x) = \lim_{n\to\infty}\sum_{p=0}^{n}\rho_p\mathcal{F}(\phi)(p)(\mathcal{F}'G)(p)\varphi_p(x), \qquad \phi \in \mathcal{M}_a(\mathbb{R}),$$

in the sense of convergence in $\mathcal{M}_a(\mathbb{R})$.

Hence it follows for each $F, G \in \mathcal{M}_a(\mathbb{R})'$ that

$$\begin{aligned}\langle F\#G, \phi\rangle &= \lim_{n\to\infty}\sum_{p=0}^{n}\rho_p(\mathcal{F}\phi)(p)(\mathcal{F}'F)(p)(\mathcal{F}'G)(p)\\ &= \lim_{n\to\infty}\left\langle\sum_{p=0}^{n}(\mathcal{F}'F)(p)(\mathcal{F}'G)(p)\rho_p\varphi_p, \phi\right\rangle, \qquad \phi \in \mathcal{M}_a(\mathbb{R}).\end{aligned}$$

Thus we have proved that $\{\sum_{p=0}^{n}\rho_p(\mathcal{F}'F)(p)(\mathcal{F}'G)(p)\varphi_p\}_{n\in\mathbb{N}}$ is weak* convergent to $F\#G$, as $n \to \infty$. Since $\mathcal{M}_a(\mathbb{R})$ is a Montel space the proof is concluded. □

Remark 4. If $f, g \in \mathcal{M}_a(\mathbb{R})$, then the generalized convolution $f\#g$ defined by (16) coincides with the classical #-convolution.

It is not hard to prove the following properties for the #-convolution.

Proposition 9. *Let $F, G, H \in \mathcal{M}_a(\mathbb{R})'$. Then*:

(i) $F\#G = G\#F$.
(ii) $(F\#G)\#H = F\#(G\#H)$.
(iii) $F\#\delta = F$, *where δ as usual denotes the Dirac distribution.*
(iv) $\mathcal{F}'(F\#G)(n) = \mathcal{F}'(F)(n)\mathcal{F}'(G)(n)$, $n \in \mathbb{N}$. □

The τ-translation is defined on $\mathcal{M}_a(\mathbb{R})'$ in a usual way as the transpose of the τ-translation on $\mathcal{M}_a(\mathbb{R})$, i.e., for every $F \in \mathcal{M}_a(\mathbb{R})'$ we define

$$\langle \tau_x F, \phi \rangle = \langle F, \tau_x \phi \rangle, \qquad \phi \in \mathcal{M}_a(\mathbb{R}) \text{ and } x \in \mathbb{R}.$$

The space $\mathcal{L}(\mathcal{M}_a(\mathbb{R})')$ denotes the set of continuous linear mappings from $\mathcal{M}_a(\mathbb{R})'$ into itself when $\mathcal{M}_a(\mathbb{R})'$ is endowed with the weak* topology.

We now characterize the commuting elements with τ_x of $\mathcal{L}(\mathcal{M}_a(\mathbb{R})')$.

Proposition 10. *Let $L \in \mathcal{L}(\mathcal{M}_a(\mathbb{R})')$. The two following properties are equivalent.*

(i) $L(\tau_x G) = \tau_x LG$, *for every $G \in \mathcal{M}_a(\mathbb{R})'$ and $x \in \mathbb{R}$.*
(ii) *There exists $F \in \mathcal{M}_a(\mathbb{R})'$ such that $LG = F\#G$, for every $G \in \mathcal{M}_a(\mathbb{R})'$.*

Proof. To see that (i) implies (ii) we first note that the family $\{\tau_x\delta\}_{x\in\mathbb{R}}$ is a weakly* dense subset of $\mathcal{M}_a(\mathbb{R})'$ ([K, Problem W(b)]). Define the mapping $\Omega \in \mathcal{L}(\mathcal{M}_a(\mathbb{R})')$ by

$$\Omega G = L(G) - G\#(L\delta), \qquad G \in \mathcal{M}_a(\mathbb{R})'.$$

By invoking Proposition 9 (iii), for every $x \in \mathbb{R}$ we have

$$\begin{aligned}\Omega(\tau_x\delta) &= L(\tau_x\delta) - \tau_x\delta\#(L\delta) \\ &= \tau_x L\delta - \tau_x(\delta\#L\delta) = \tau_x L\delta - \tau_x L\delta = 0.\end{aligned}$$

Then $\{\tau_x\delta\}_{x\in\mathbb{R}}$ is contained in the kernel of Ω. Hence we have concluded that $\Omega = 0$ and $LG = L\delta\#G$, $G \in \mathcal{M}_a(\mathbb{R})'$.

Now let $G \in \mathcal{M}_a(\mathbb{R})'$ and $x \in \mathbb{R}$. If (ii) holds we can write

$$\begin{aligned}\langle \tau_x(LG), \phi\rangle &= \langle \tau_x(F\#G), \phi\rangle = \langle F\#G, \tau_x\phi\rangle \\ &= \langle F(y), \langle G(z), \tau_y(\tau_x\phi)(z)\rangle\rangle = \langle F(y), \langle G(z), \tau_x(\tau_y\phi)(z)\rangle\rangle \\ &= \langle F(y), \langle (\tau_x G)(z), (\tau_y\phi)(z)\rangle\rangle = \langle F\#\tau_x G, \phi\rangle, \qquad \phi \in \mathcal{M}_a(\mathbb{R}).\end{aligned}$$

This completes the proof. □

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