A DISTRIBUTIONAL CONVOLUTION FOR A GENERALIZED
FINITE FOURIER TRANSFORMATION

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Abstract. In this paper we define a generalized finite Fourier transformation in distribution spaces. Also we investigate a distributional convolution for this finite integral transformation.

1. Introduction

K. Trimèche [T1] introduced the generalized Fourier transformation

$$(Ff)(\lambda) = \int_0^\infty \varphi_\lambda(x)f(x)A(x)dx$$

where (i) $A(t)$ is a real function defined on $[0, \infty)$ such that $A(t) = t^{2a+1}C(t)$ with $a > -1/2$ and $C(t)$ is an even, infinitely differentiable and strictly positive function on $\mathbb{R}$, and (ii) the kernel $\varphi_\lambda$ is the solution of the initial value problem

$$\Delta \varphi = -\lambda^2 \varphi,$$

$$\varphi(0) = 1,$$

$$D\varphi(0) = 0$$

for every $\lambda \in \mathbb{C}$. In (1) $\Delta$ represents the operator $\frac{1}{A(t)}D(A(t)D) - q(t)$, $D = \frac{d}{dt}$, and the function $q(t)$ is even and infinitely differentiable on $\mathbb{R}$. Besides, he defined and investigated a convolution for the $F$-transformation.

Later, K. Trimèche [T2] studied generalized Fourier series expansions associated to the operator $\Delta$ when $q(t) = -\gamma^2$, $\gamma \geq 0$. We now recall some of his definitions and results that will be useful in the sequel. Thus, $\mathcal{E}_*(\mathbb{R})$ is the space of all even infinitely differentiable functions on $\mathbb{R}$. We assign to $\mathcal{E}_*(\mathbb{R})$ the topology generated by the family of seminorms $\{p_{n,m}\}_{m \in \mathbb{N}}$ where

$$p_{n,m}(\phi) = \sup_{|x| \leq n} |D^m\phi(x)|, \quad \phi \in \mathcal{E}_*(\mathbb{R}),$$

for every $n, m \in \mathbb{N}$.

The generalized translation operator associated to $\Delta$ is defined for every $\phi \in \mathcal{E}_*(\mathbb{R})$ by

$$(\tau_\sigma \phi)(y) = \chi_x\chi_y[\sigma_x\chi^{-1}\phi(y)], \quad x, y \in \mathbb{R},$$

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where \((\sigma_x \phi)(y) = \frac{\phi(x+y) + \phi(x-y)}{2}\), \(x, y \in \mathbb{R}\), and \(\chi\) represents a generalized Riemann-Liouville integral that transmutes the operator \(\Delta\) into the operator \(D^2\) (Definition 5.1 in [T1]).

Now let \(a > 0\). Denote by \(\mathcal{M}_a(\mathbb{R})\) the space of functions \(\phi \in \mathcal{E}_a(\mathbb{R})\) such that \((\tau_x \phi)(a) = 0\), for each \(x \in \mathbb{R}\). The space \(\mathcal{E}_a(\mathbb{R})\) induces on \(\mathcal{M}_a(\mathbb{R})\) the topology generated by the family \(\{\alpha_k\}_{k \in \mathbb{N}}\) of seminorms where for every \(k \in \mathbb{N}\)
\[
\alpha_k(\phi) = \sup_{x \in (0, a)} |\Delta^k \phi(x)|, \quad \phi \in \mathcal{M}_a(\mathbb{R}).
\]
Thus \(\mathcal{M}_a(\mathbb{R})\) is a Fréchet Montel space. The dual space of \(\mathcal{M}_a(\mathbb{R})\) will be denoted, as usual, by \(\mathcal{M}_a(\mathbb{R})^\prime\).

Let \(\phi, \psi \in \mathcal{M}_a(\mathbb{R})\). Then, with the aid of the \(\tau\)-translation, the convolution \(\phi \# \psi\) is defined by
\[
(\phi \# \psi)(x) = \int_0^a (\tau_x \phi)(y) \psi(y) A(y) dy, \quad x \in \mathbb{R}.
\]
Next, \(\{\pm \lambda_k\}_{k \in \mathbb{N}}\) denote the zeros of the entire function \(\lambda \to \varphi_\lambda(a)\) with \(\lambda_0 = 0\).
From now on \(\varphi_{\lambda_n}\) will always be represented by \(\varphi_n\), where \(n \in \mathbb{N}\). Note also that \(\varphi_n\) is in \(\mathcal{M}_a(\mathbb{R})\), for every \(n \in \mathbb{N}\). Then for every \(\phi \in \mathcal{M}_a(\mathbb{R})\) one has
\[
(\phi \# \psi)(x) = \sum_{n=0}^{\infty} \rho_n \mathcal{F}(\phi)(n) \varphi_n(x)
\]
where the series converges to \(\phi\) in \(\mathcal{E}_a(\mathbb{R})\). Moreover for every \(n \in \mathbb{N}\)
\[
\rho_n = \frac{1}{\int_0^a \varphi_n^2(x) A(x) dx}
\]
and
\[
\mathcal{F}(\phi)(n) = \int_0^a \varphi_n(x) \phi(x) A(x) dx.
\]
Let \(\phi \in \mathcal{M}_a(\mathbb{R})\). The sequence \((\mathcal{F}(\phi)(n))_{n \in \mathbb{N}}\) will be called the generalized finite Fourier transform \(\mathcal{F}(\phi)\).

Now some properties are listed:

i) The expression (2) can be seen as an inversion formula for the \(\mathcal{F}\)-transformation.

ii) Let \(\phi, \psi \in \mathcal{M}_a(\mathbb{R})\). The convolution takes the form
\[
(\phi \# \psi)(x) = \sum_{n=0}^{\infty} \rho_n \mathcal{F}(\phi)(n) \mathcal{F}(\psi)(n) \varphi_n(x), \quad x \in \mathbb{R}.
\]

iii) The \#-convolution defines a continuous mapping from \(\mathcal{M}_a(\mathbb{R}) \times \mathcal{M}_a(\mathbb{R})\) into \(\mathcal{M}_a(\mathbb{R})\).

We now consider the space \(\mathcal{V}\) of all complex sequences \((a_n)_{n \in \mathbb{N}}\) satisfying
\[
\beta_k((a_n)_{n \in \mathbb{N}}) = \sum_{n=0}^{\infty} \rho_n |\lambda_n|^{2k}|a_n| < \infty, \quad \text{for every } k \in \mathbb{N}.
\]
The space \(\mathcal{V}\) is endowed with the topology associated to the family \(\{\beta_k\}_{k \in \mathbb{N}}\) of seminorms.

**Proposition 1.** The generalized finite Fourier transform is a homeomorphism from \(\mathcal{M}_a(\mathbb{R})\) onto the space \(\mathcal{V}\).
Proof. It can be proved making use of Theorem 3.1.1, Lemma 3.1.2 and Proposition 3.1.2 (1,2) of [T2]. □

Motivated by the works of A. H. Zemanian [Z1] and R. S. Pathak [P], in this paper we investigate the generalized finite Fourier transformation $\mathcal{F}$ of distributions. In Section 2 we define the generalized finite Fourier transformation on $\mathcal{M}_a(\mathbb{R})'$. The distributional $\#$-convolution is studied in Section 3.

Remark 1. Our study includes two important cases. On the one hand, when $A(t) = t^{2\alpha+1}$, $\gamma = 0$ and $\alpha > -1/2$, $\Delta$ reduces to the Bessel operator. On the other hand, if we choose $A(t) = 2^{2(\alpha+\beta+1)}(\sinh t)^{2\alpha+1}(\cosh t)^{2(\beta+1)}$, $\gamma = \alpha + \beta + 1$ and $\alpha, \beta > -1/2$, then $\Delta$ turns out to be the Jacobi operator.

Throughout this paper $C$ will always represent a positive constant not necessarily the same in each occurrence.

2. THE GENERALIZED FINITE FOURIER TRANSFORMATION

In this section we define the finite generalized Fourier transformation $\mathcal{F}$ on the space $\mathcal{M}_a(\mathbb{R})'$ and we establish its main properties.

The finite generalized Fourier transform $\mathcal{F}'(F)$ of $F \in \mathcal{M}_a(\mathbb{R})'$ is defined by

$$\langle \mathcal{F}'(F), (a_n)_{n \in \mathbb{N}} \rangle = \left\langle F(x), \sum_{n=0}^{\infty} \rho_n \varphi_n(x) a_n \right\rangle, \quad (a_n)_{n \in \mathbb{N}} \in \mathcal{V}. $$

Using well-known results of duality we infer that $\mathcal{F}'$ is a homeomorphism from $\mathcal{M}_a(\mathbb{R})'$ onto $\mathcal{V}'$, whenever $\mathcal{M}_a(\mathbb{R})'$ and $\mathcal{V}'$ are endowed either with the strong topology or the weak* topology.

Let $F \in \mathcal{M}_a(\mathbb{R})'$. There exist $C > 0$ and $r \in \mathbb{N}$ such that

$$|\langle F, \phi \rangle| \leq C \max_{0 \leq k \leq r} \sup_{x \in (0, a)} |\Delta^k \phi(x)|, \quad \phi \in \mathcal{M}_a(\mathbb{R}).$$

Hence, according to Theorem 1.1 in [T1], we may write

$$\left| \left\langle F(x), \sum_{n=p}^{\infty} a_n \rho_n \varphi_n(x) \right\rangle \right| \leq C \max_{0 \leq k \leq r} \sum_{n=p}^{\infty} |a_n| |\rho_n| |\lambda_n|^{2k}$$

for every $(a_n)_{n \in \mathbb{N}} \in \mathcal{V}$ and $p \in \mathbb{N}$.

Now, by invoking again Proposition 3.1.2 (2) of [T2] we obtain

$$\left| \left\langle F(x), \sum_{n=p}^{\infty} a_n \rho_n \varphi_n(x) \right\rangle \right| \leq C \sum_{n=p}^{\infty} |a_n| |\lambda_n|^{2r} (|\lambda_n|^2 + \gamma^2)^{\alpha+1/2},$$

provided that $p \geq 1$.

Therefore, given an $\varepsilon > 0$, there exists a $p_0 \in \mathbb{N}$ in such a way that

$$\left| \left\langle F(x), \sum_{n=p}^{\infty} a_n \rho_n \varphi_n(x) \right\rangle \right| < \varepsilon, \quad \text{for every } p \geq p_0.$$

Next, keeping (3) in mind, we can write

$$\sum_{n=p}^{\infty} a_n \rho_n \langle F, \varphi_n \rangle \leq C \sum_{n=p}^{\infty} |a_n| (|\lambda_n|^2 + \gamma^2)^{\alpha+1/2} |\lambda_n|^{2r} < \varepsilon,$$

provided that $p$ is sufficiently large.
From (4) and (5) we conclude that
\[
\left\langle F(x), \sum_{n=0}^{\infty} a_n \rho_n \varphi_n(x) \right\rangle = \sum_{n=0}^{\infty} a_n \rho_n \left\langle F(x), \varphi_n(x) \right\rangle, \quad (a_n)_{n \in \mathbb{N}} \in \mathcal{V}.
\]
Thus we have established the following.

**Proposition 2.** Let \( F \in \mathcal{M}_a(\mathbb{R})' \). Then the sequence \( (\left\langle F(x), \varphi_n(x) \right\rangle)_{n \in \mathbb{N}} \) defines a member of \( \mathcal{V}' \) by
\[
(\left\langle F(x), \varphi_n(x) \right\rangle)_{n \in \mathbb{N}}, (a_n)_{n \in \mathbb{N}} = \sum_{n=0}^{\infty} \rho_n \left\langle F(x), \varphi_n(x) \right\rangle a_n, \quad (a_n)_{n \in \mathbb{N}} \in \mathcal{V},
\]
and \( \mathcal{F}'(F) = (\left\langle F(x), \varphi_n(x) \right\rangle)_{n \in \mathbb{N}} \) in the sense of equality in \( \mathcal{V}' \).

In the sequel we set \( \mathcal{F}'(F)(n) = (\left\langle F(x), \varphi_n(x) \right\rangle), n \in \mathbb{N} \) and \( F \in \mathcal{M}_a(\mathbb{R})' \).

**Remark 2.** If \( \phi \in \mathcal{M}_a(\mathbb{R}) \), we get \( \mathcal{F}'(\phi) = (\left\langle \phi(x), \varphi_n(x) \right\rangle)_{n \in \mathbb{N}} = (\mathcal{F}(\phi)(n))_{n \in \mathbb{N}} \).
Consequently, the classical \( \mathcal{F} \)-transformation on \( \mathcal{M}_a(\mathbb{R}) \) is a particular case of the \( \mathcal{F}' \)-transformation.

We now show a representation for the elements of \( \mathcal{M}_a(\mathbb{R})' \) that can be seen as an inversion formula for the generalized \( \mathcal{F}' \)-transformation.

**Proposition 3.** Let \( F \in \mathcal{M}_a(\mathbb{R})' \). Then
\[
F = \lim_{n \to \infty} \sum_{p=0}^{n} \rho_p \langle F, \varphi_p \rangle \varphi_p
\]
where the convergence is understood in the strong topology of \( \mathcal{M}_a(\mathbb{R})' \).

**Proof.** Let \( \in \mathcal{M}_a(\mathbb{R})' \). Note that it is sufficient to prove \( \{ \sum_{p=0}^{n} \rho_p \langle F, \varphi_p \rangle \varphi_p \}_{n \in \mathbb{N}} \) is a weak* convergent sequence to \( F \), because \( \mathcal{M}_a(\mathbb{R}) \) is a Montel space.

Let \( \phi \in \mathcal{M}_a(\mathbb{R}) \). According to [T2, Theorem 3.1.1] we get
\[
\phi(x) = \sum_{p=0}^{\infty} \rho_p \mathcal{F}(\phi)(p) \varphi_p(x), \quad x \in \mathbb{R}.
\]
Moreover the series is convergent in \( \mathcal{M}_a(\mathbb{R}) \). By taking into account that, for every \( p \in \mathbb{N} \), \( \varphi_p \) defines an element of \( \mathcal{M}_a(\mathbb{R})' \) by
\[
\langle \varphi_p, \phi \rangle = \int_{0}^{a} \varphi_p(x) \phi(x) A(x) dx, \quad \phi \in \mathcal{M}_a(\mathbb{R}),
\]
it follows that
\[
\langle F, \phi \rangle = \lim_{n \to \infty} \left\langle F, \sum_{p=0}^{n} \rho_p \mathcal{F}(\phi)(p) \varphi_p \right\rangle
\]
\[
= \lim_{n \to \infty} \sum_{p=0}^{n} \rho_p \mathcal{F}(\phi)(p) \langle F, \varphi_p \rangle
= \lim_{n \to \infty} \left\langle \sum_{p=0}^{n} \rho_p \langle F, \varphi_p \rangle \varphi_p, \phi \right\rangle.
\]

Our next objective is to characterize the complex sequences that are \( \mathcal{F}' \)
transforms of an element of \( \mathcal{M}_a(\mathbb{R})' \). First, we need to characterize the elements of \( \mathcal{V}' \).
there exists a complex sequence \( H \) of Proposition 3.I.2 (2) in \([T2]\) it is not hard to see that

Even more, since \( \lambda \) is continuous when \( \ell \)

We now introduce the subset of \( V \)

Here as usual

and the mapping

Proof. First, assume that \( H \) takes the form (7) where \((b_n)_{n \in \mathbb{N}}\) satisfies (6). Because of Proposition 3.I.2 (2) in \([T2]\) it is not hard to see that \( H \in V' \).

Conversely, let \( H \in V' \). As is well-known there exists \( k \in \mathbb{N} \) such that

Even more, since \( \lambda_n \neq 0 \), \( n \in \mathbb{N} \) and \( n \geq 1 \), and 0 is not an adherent point of the set \( \{\lambda_n\}_{n \in \mathbb{N}} \) we can write

We now introduce the subset of \( V \) defined by

and the mapping

Here as usual \( \ell_1 \) stands for the space of all those complex sequences \((a_n)_{n \in \mathbb{N}}, n \geq 1\) such that \( \sum_{n=1}^{\infty} |a_n| < \infty \). It is obvious that \( J \) is one to one. Moreover, by virtue of (8), the linear mapping

is continuous when \( J(W) \) is endowed with the topology induced on it by \( \ell_1 \). Therefore, by invoking the Hahn-Banach theorem, \( L \) can be extended to \( \ell_1 \) as a member of \( \ell'_1 \). Then there exists \((\beta_n)_{n \in \mathbb{N}} \in \ell_\infty \) such that

From (9) we can conclude that

where \( e_0 = (1, 0, \ldots) \), and the proof is finished by taking \( b_0 = \frac{(H, e_0)}{\rho_0} \) and \( b_n = \frac{\beta_n \lambda_n^{2k}}{\rho_n} \), \( n \in \mathbb{N}, n \geq 1 \). \( \square \)
**Proposition 5.** For every $x \in \mathbb{R}$, $\phi \in \mathcal{M}_a(\mathbb{R})$ we define the translation operator $\tau_x$, $x \in \mathbb{R}$, on $\mathcal{M}_a(\mathbb{R})$ as

$$(\tau_x \phi)(y) = \sum_{n=0}^{\infty} \rho_n F(\phi)(n) \varphi_n(x) \varphi_n(y), \quad x, y \in \mathbb{R}.$$ 

It is clear that $F(\tau_x \phi)(n) = \varphi_n(x) F(\phi)(n), \ n \in \mathbb{N}$ and $x \in \mathbb{R}$.

**Proposition 5.** For every $x \in \mathbb{R}$, $\tau_x$ continuously maps $\mathcal{M}_a(\mathbb{R})$ into itself.

**Proof.** Let $x \in \mathbb{R}$, $\phi \in \mathcal{M}_a(\mathbb{R})$ and $k \in \mathbb{N}$. We have

$$\Delta^k(\tau_x \phi)(y) = (-1)^k \sum_{n=0}^{\infty} \rho_n F(\phi)(n) \varphi_n(x) \lambda_n^{2k} \varphi_n(y), \quad x, y \in \mathbb{R}.$$ 

Then, from [T2, Proposition 3.1.2 (2)] it is deduced that

$$\sup_{y \in (0, a)} |\Delta^k(\tau_x \phi)(y)| \leq C \sum_{n=0}^{\infty} (|\lambda_n|^2 + \gamma^2)^{n+1/2} |\lambda_n|^{2k} |F(\phi)(n)|.$$ 

Hence, since $F$ is a homeomorphism from $\mathcal{M}_a(\mathbb{R})$ onto $V$, we conclude that $\tau_x$ is a continuous mapping from $\mathcal{M}_a(\mathbb{R})$ into itself. 

**Corollary 1.** Let $(b_n)_{n \in \mathbb{N}}$ be a complex sequence. There exists $F \in \mathcal{M}_a(\mathbb{R})'$ such that $F'(F)(n) = b_n, \ n \in \mathbb{N}$, if, and only if, there exist $C > 0$ and $k \in \mathbb{N}$ such that

$$|b_n| \leq C|\lambda_n|^{2k}, \quad n \geq 1, \ n \in \mathbb{N}.$$ 

**Proof.** Necessity is an immediate consequence of [Z2, Theorem 1.8.1]. In order to establish the sufficiency, we consider a sequence $(b_n)_{n \in \mathbb{N}}$ satisfying (10). Then the series $\sum_{n=0}^{\infty} b_n \rho_n \varphi_n$ converges in the strong topology of $\mathcal{M}_a(\mathbb{R})'$. By denoting said limit by $F$, from Proposition 3 we can infer that $F'(F)(n) = b_n, \ n \in \mathbb{N}$. 

### 3. The Generalized Convolution

In this section we introduce and investigate a convolution operation in $\mathcal{M}_a(\mathbb{R})'$.

According to [T2] for every $\phi \in \mathcal{M}_a(\mathbb{R})$ we define the translation operator $\tau_x$, $x \in \mathbb{R}$, on $\mathcal{M}_a(\mathbb{R})$ as

$$F(\tau_x \phi)(y) = \sum_{n=0}^{\infty} \rho_n F(\phi)(n) \varphi_n(x) \varphi_n(y), \quad x, y \in \mathbb{R}.$$ 

**Proposition 5.** For every $x \in \mathbb{R}$, $\tau_x$ continuously maps $\mathcal{M}_a(\mathbb{R})$ into itself.

**Proof.** Let $x \in \mathbb{R}, \phi \in \mathcal{M}_a(\mathbb{R})$ and $k \in \mathbb{N}$. We have

$$\Delta^k(\tau_x \phi)(y) = (-1)^k \sum_{n=0}^{\infty} \rho_n F(\phi)(n) \varphi_n(x) \lambda_n^{2k} \varphi_n(y), \quad x, y \in \mathbb{R}.$$ 

Then, from [T2, Proposition 3.1.2 (2)] it is deduced that

$$\sup_{x \in (0, a)} |\Delta^k(\tau_x \phi)(y)| \leq C \sum_{n=0}^{\infty} (|\lambda_n|^2 + \gamma^2)^{n+1/2} |\lambda_n|^{2k} |F(\phi)(n)|.$$ 

Hence, since $F$ is a homeomorphism from $\mathcal{M}_a(\mathbb{R})$ onto $V$, we conclude that $\tau_x$ is a continuous mapping from $\mathcal{M}_a(\mathbb{R})$ into itself.

Proposition 5 allows us to define the $\#$-convolution $F\#\phi$ of $F \in \mathcal{M}_a(\mathbb{R})'$ and $\phi \in \mathcal{M}_a(\mathbb{R})$ as follows:

$$F(x) = (F(y), (\tau_x \phi)(y)), \quad x \in \mathbb{R}.$$ 

**Remark 3.** Note that if $f \in \mathcal{M}_a(\mathbb{R})$, then $f \in \mathcal{M}_a(\mathbb{R})'$ according to Proposition 2. Besides, $f\#\phi$ given by (11) coincides with the classical $\#$-convolution of $f$ and $\phi$.

Two interesting properties of the $\#$-convolution are shown in the following assertion.

**Proposition 6.** (i) For every $F \in \mathcal{M}_a(\mathbb{R})'$ and $\phi \in \mathcal{M}_a(\mathbb{R})$

$$(F\#\phi)(x) = \sum_{n=0}^{\infty} \rho_n F(\phi)(n) \varphi_n(x) (F'F)(n), \quad x \in \mathbb{R}.$$ 

(ii) For every $F \in \mathcal{M}_a(\mathbb{R})'$, the mapping $\phi \to F\#\phi$ is continuous from $\mathcal{M}_a(\mathbb{R})$ into itself.
Proof. To see (i) we have to prove that
\[
\left\langle F(y), \sum_{n=0}^{\infty} \rho_n \mathcal{F}(\phi)(n) \varphi_n(x) \varphi_n(y) \right\rangle
\]
for every \( F \in \mathcal{M}_a(\mathbb{R})' \) and \( \phi \in \mathcal{M}_a(\mathbb{R}) \).

Let \( f \in \mathcal{M}_a(\mathbb{R})' \). As is known there exists \( k \in \mathbb{N} \) such that
\[
|\langle F(y), \phi(y) \rangle| \leq C \max_{0 \leq r \leq k} |\Delta^r \phi(y)|, \quad \phi \in \mathcal{M}_a(\mathbb{R}).
\]

Hence, for every \( p \in \mathbb{N} \) and \( x \in \mathbb{R} \),
\[
\left| \left\langle F(y), \sum_{n=p}^{\infty} \rho_n \mathcal{F}(\phi)(n) \varphi_n(x) \varphi_n(y) \right\rangle \right|
\]
\[
\leq \max_{0 \leq r \leq k} \sup_{y \in (0,a)} |\Delta^r \left( \sum_{n=p}^{\infty} \rho_n \mathcal{F}(\phi)(n) \varphi_n(x) \varphi_n(y) \right)|.
\]

By invoking again [T2, Proposition 3.1.2 (2)] one has for every \( x \in \mathbb{R} \), \( y \in (0,a) \), \( k, p \in \mathbb{N} \),
\[
\sum_{n=p}^{\infty} \rho_n \mathcal{F}(\phi)(n) |\lambda_n|^{2k} |\varphi_n(x)||\varphi_n(y)| \leq C \sum_{n=p}^{\infty} |(|\lambda_n|^2 + \gamma^2)^{n+1/2} |\lambda_n|^{2k} |\mathcal{F}(\phi)(n)|.
\]

Also by proceeding in a similar way we obtain for every \( x \in \mathbb{R} \)
\[
\sum_{n=p}^{\infty} \rho_n \mathcal{F}(\phi)(n) \langle F(y), \varphi_n(y) \rangle \varphi_n(x) \leq C \sum_{n=p}^{\infty} |(|\lambda_n|^2 + \gamma^2)^{n+1/2} |\mathcal{F}(\phi)(n)| |\lambda_n|^{2r}.
\]

Since \( \mathcal{F}(\phi) \in \mathcal{V} \), by combining (13), (14) and (15) we can establish (12).

Part (ii) can be proved taking into account (i) and using [T2, Proposition 3.1.2 (2)].

We will denote by \( \mathcal{L}(\mathcal{M}_a(\mathbb{R})) \) the space of the continuous linear mappings from \( \mathcal{M}_a(\mathbb{R}) \) into itself. We now characterize the elements of \( \mathcal{L}(\mathcal{M}_a(\mathbb{R})) \) that commute with \( \tau \)-translations.

**Proposition 7.** Let \( L \in \mathcal{L}(\mathcal{M}_a(\mathbb{R})) \). The following two properties are equivalent.

(i) \( L(\tau_x \phi) = \tau_x L \phi, \phi \in \mathcal{M}_a(\mathbb{R}) \) and \( x \in \mathbb{R} \).

(ii) There exists \( F \in \mathcal{M}_a(\mathbb{R})' \) such that \( L \phi = F \phi, \phi \in \mathcal{M}_a(\mathbb{R}) \).

**Proof.** Let \( L \in \mathcal{L}(\mathcal{M}_a(\mathbb{R})) \) such that \( L(\tau_x \phi) = \tau_x L \phi \) for every \( x \in \mathbb{R} \) and \( \phi \in \mathcal{M}_a(\mathbb{R}) \). Define a functional \( F \) on \( \mathcal{M}_a(\mathbb{R}) \) by
\[
\langle F, \phi \rangle = \langle \delta, L \phi \rangle, \quad \phi \in \mathcal{M}_a(\mathbb{R}),
\]
where \( \delta \) represents the Dirac distribution.

It is clear that \( F \in \mathcal{M}_a(\mathbb{R})' \). Moreover, for every \( \phi \in \mathcal{M}_a(\mathbb{R}) \)
\[
(F \phi)(x) = \langle F, \tau_x \phi \rangle = \langle \delta, L(\tau_x \phi) \rangle = \langle \delta, \tau_x L \phi \rangle = (L \phi)(x), \quad x \in \mathbb{R},
\]
because \( \langle \delta, \tau_x \phi \rangle = \phi(x) \), for every \( \phi \in \mathcal{M}_a(\mathbb{R}) \) and \( x \in \mathbb{R} \).
Conversely, if \( L \in \mathcal{L}(\mathcal{M}_a(\mathbb{R})) \) is defined by
\[
L\phi = F\#\phi, \quad \phi \in \mathcal{M}_a(\mathbb{R}),
\]
where \( F \in \mathcal{M}_a(\mathbb{R})' \), then by invoking Proposition 6
\[
\tau_x(L\phi) = \tau_x(F\#\phi) = \tau_x\left(\sum_{n=0}^{\infty} \rho_n \mathcal{F}(\phi)(n)\mathcal{F}'(F)(n)\varphi_n(y)\right)
= \sum_{n=0}^{\infty} \rho_n \mathcal{F}(\phi)(n)\mathcal{F}'(F)(n)\varphi_n(y) (x)
= \left(F(z), \sum_{n=0}^{\infty} \rho_n \mathcal{F}(\phi)(n)\varphi_n(z)\varphi_n(y)\varphi_n(x)\right)
= F\#(\tau_x \phi) = L(\tau_x \phi), \quad x \in \mathbb{R},
\]
because the series \( \sum_{n=0}^{\infty} \rho_n \mathcal{F}(\phi)(n)\varphi_n(z)\varphi_n(y)\varphi_n(x) \) is convergent in \( \mathcal{M}_a(\mathbb{R}) \) for every \( x, y \in \mathbb{R} \). \( \square \)

Let \( F \) and \( G \) be two arbitrary elements of \( \mathcal{M}_a(\mathbb{R})' \). In accordance with the preceding results, it is natural to try to define \( F\#G \) by
\[
(F\#G, \phi) = \langle F, G\#\phi \rangle, \quad \phi \in \mathcal{M}_a(\mathbb{R}).
\]
From Proposition 6 if \( F, G \in \mathcal{M}_a(\mathbb{R})' \) we have \( F\#G \in \mathcal{M}_a(\mathbb{R})' \). Also Proposition 6 allows us to infer the following.

**Proposition 8.** For every \( F, G \in \mathcal{M}_a(\mathbb{R})' \)
\[
F\#G = \sum_{n=0}^{\infty} \rho_n (\mathcal{F}'F)(n)(\mathcal{F}'G)(n)\varphi_n
\]
where the convergence of the series is understood in the strong topology in \( \mathcal{M}_a(\mathbb{R})' \).

**Proof.** By proceeding as in the proof of Proposition 6, for every \( G \in \mathcal{M}_a(\mathbb{R})' \) it can be established that
\[
(G\#\phi)(x) = \lim_{n \to \infty} \sum_{p=0}^{n} \rho_p \mathcal{F}(\phi)(p)(\mathcal{F}'G)(p)\varphi_p(x), \quad \phi \in \mathcal{M}_a(\mathbb{R}),
\]
in the sense of convergence in \( \mathcal{M}_a(\mathbb{R}) \).

Hence it follows for each \( F, G \in \mathcal{M}_a(\mathbb{R})' \) that
\[
\langle F\#G, \phi \rangle = \lim_{n \to \infty} \sum_{p=0}^{n} \rho_p (\mathcal{F}\phi)(p)(\mathcal{F}'F)(p)(\mathcal{F}'G)(p)
= \lim_{n \to \infty} \left(\sum_{p=0}^{n} (\mathcal{F}'F)(p)(\mathcal{F}'G)(p)\rho_p \varphi_p, \phi\right), \quad \phi \in \mathcal{M}_a(\mathbb{R}).
\]
Thus we have proved that \( \{\sum_{p=0}^{n} \rho_p (\mathcal{F}'F)(p)(\mathcal{F}'G)(p)\varphi_p\}_{n \in \mathbb{N}} \) is weak* convergent to \( F\#G \), as \( n \to \infty \). Since \( \mathcal{M}_a(\mathbb{R}) \) is a Montel space the proof is concluded. \( \square \)

**Remark 4.** If \( f, g \in \mathcal{M}_a(\mathbb{R}) \), then the generalized convolution \( f\#g \) defined by (16) coincides with the classical \( \# \)-convolution.
It is not hard to prove the following properties for the \(\#\)-convolution.

**Proposition 9.** Let \(F, G, H \in \mathcal{M}_a(\mathbb{R})'\). Then:

(i) \(F\#G = G\#F\).

(ii) \((F\#G)\#H = F\#(G\#H)\).

(iii) \(F\#\delta = F\), where \(\delta\) as usual denotes the Dirac distribution.

(iv) \(F'(F\#G)(n) = F'(F)(n)F'(G)(n), \ n \in \mathbb{N}\).

The \(\tau\)-translation is defined on \(\mathcal{M}_a(\mathbb{R})'\) in a usual way as the transpose of the \(\tau\)-translation on \(\mathcal{M}_a(\mathbb{R})\), i.e., for every \(F \in \mathcal{M}_a(\mathbb{R})'\) we define

\[
\langle \tau_x F, \phi \rangle = \langle F, \tau_x \phi \rangle, \quad \phi \in \mathcal{M}_a(\mathbb{R}) \text{ and } x \in \mathbb{R}.
\]

The space \(\mathcal{L}(\mathcal{M}_a(\mathbb{R})')\) denotes the set of continuous linear mappings from \(\mathcal{M}_a(\mathbb{R})'\) into itself when \(\mathcal{M}_a(\mathbb{R})'\) is endowed with the weak* topology.

We now characterize the commuting elements with \(\tau_x\) of \(\mathcal{L}(\mathcal{M}_a(\mathbb{R})')\).

**Proposition 10.** Let \(L \in \mathcal{L}(\mathcal{M}_a(\mathbb{R})')\). The two following properties are equivalent.

(i) \(L(\tau_x G) = \tau_x LG\), for every \(G \in \mathcal{M}_a(\mathbb{R})'\) and \(x \in \mathbb{R}\).

(ii) There exists \(F \in \mathcal{M}_a(\mathbb{R})'\) such that \(LG = F\#G\), for every \(G \in \mathcal{M}_a(\mathbb{R})'\).

**Proof.** To see that (i) implies (ii) we first note that the family \(\{\tau_x\}_{x \in \mathbb{R}}\) is a weakly* dense subset of \(\mathcal{M}_a(\mathbb{R})'\) ([K, Problem W(b)]). Define the mapping \(\Omega \in \mathcal{L}(\mathcal{M}_a(\mathbb{R})')\) by

\[
\Omega G = L(G) - G\#(L\delta), \quad G \in \mathcal{M}_a(\mathbb{R})'.
\]

By invoking Proposition 9 (iii), for every \(x \in \mathbb{R}\) we have

\[
\Omega(\tau_x \delta) = L(\tau_x \delta) - \tau_x \delta\#(L\delta) = \tau_x L\delta - \tau_x (\delta\#L\delta) = \tau_x L\delta - \tau_x \delta = 0.
\]

Then \(\{\tau_x \delta\}_{x \in \mathbb{R}}\) is contained in the kernel of \(\Omega\). Hence we have concluded that \(\Omega = 0\) and \(LG = L\delta\#G, G \in \mathcal{M}_a(\mathbb{R})'\).

Now let \(G \in \mathcal{M}_a(\mathbb{R})'\) and \(x \in \mathbb{R}\). If (ii) holds we can write

\[
\langle \tau_x (L G), \phi \rangle = \langle \tau_x (F \# G), \phi \rangle = \langle F \# G, \tau_x \phi \rangle = \langle F(y), \langle G(z), \tau_y (\tau_x \phi)(z) \rangle \rangle = \langle \tau_x G, \phi \rangle, \quad \phi \in \mathcal{M}_a(\mathbb{R})\).
\]

This completes the proof. \(\Box\)

**References**


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