TOWERS OF BOREL FUNCTIONS

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Abstract. We give mathematical reformulations of the cardinals \( p \) and \( t \) in terms of families of Borel functions. As an application we show that \( t \) is invariant under the addition of a single Cohen real.

1. Introduction

Definitions and notation. Let \( \mathbb{N} \) be the set of all nonnegative integers. We define a relation “almost set inclusion” on \( \mathcal{P}(\mathbb{N}) \) by

\[ A \subseteq^* B \iff |A \setminus B| < \aleph_0, \]

where \( A, B \subseteq \mathbb{N} \). And \( A \supseteq^* B \) iff \( B \subseteq^* A \). For a family \( \mathcal{F} \subseteq \mathcal{P}(\mathbb{N}), A \subseteq \mathbb{N} \) is a pseudo-intersection of \( \mathcal{F} \) iff \( A \subseteq^* F \) for every \( F \in \mathcal{F} \).

A family of (infinite) subsets of \( \mathbb{N} \) is called a filter base iff every nonempty finite subfamily has an infinite intersection. Let \( p \) be the cardinality of the smallest filter base with no infinite pseudo-intersection. Let \( \mathcal{P}_\infty(\mathbb{N}) \) denote the set of all infinite subsets of \( \mathbb{N} \). A tower is a subfamily of \( \mathcal{P}_\infty(\mathbb{N}) \) that is well-ordered by \( \supseteq^* \). Let \( t \) be the cardinality of the smallest tower with no infinite pseudo-intersection.

Now we define the corresponding notions in the realm of functions from \( \mathbb{R} \) into \( \mathcal{P}(\mathbb{N}) \). For a property of the reals \( P(x) \), we say that \( P(x) \) for almost all \( x \) in \( \mathbb{R} \) iff there is a comeager \( X \subseteq \mathbb{R} \) such that \( P(x) \) holds for all \( x \in X \). We define a relation on the set of all functions from \( \mathbb{R} \) to \( \mathcal{P}(\mathbb{N}) \) by

\[ f \subseteq^* g \iff f(x) \subseteq^* g(x) \text{ for almost all } x \in \mathbb{R}, \]

where \( f, g : \mathbb{R} \to \mathcal{P}(\mathbb{N}) \). For a family \( \mathcal{F} \) of functions from \( \mathbb{R} \) to \( \mathcal{P}(\mathbb{N}) \), \( f : \mathbb{R} \to \mathcal{P}(\mathbb{N}) \) is a pseudo-intersection of \( \mathcal{F} \) iff \( f \subseteq^* g \) for every \( g \in \mathcal{F} \).

Definition 1.1. A family \( \mathcal{F} \) of functions from \( \mathbb{R} \) to \( \mathcal{P}(\mathbb{N}) \) is filtered iff for every nonempty finite subfamily \( \mathcal{A} \subseteq \mathcal{F} \),

\[ \bigcap_{f \in \mathcal{A}} f(x) \text{ is infinite for almost all } x \in \mathbb{R}. \]

Definition 1.2. A family of functions \( \mathcal{T} \) from \( \mathbb{R} \) to \( \mathcal{P}(\mathbb{N}) \) is a tower iff

1. \( \mathcal{T} \) is well-ordered by \( \supseteq^* \), and
2. for every \( f \in \mathcal{T}, f(x) \) is infinite for almost all \( x \in \mathbb{R} \).
Denote the Cantor set, i.e. \{0,1\}^N with the product topology, by \mathcal{C}. We give \mathcal{P}(\mathbb{N}) a topology by identifying \mathcal{P}(\mathbb{N}) with \mathcal{C}.

**Definition 1.3.** Let \( p_1 \) be the cardinality of the smallest filtered family \( \mathcal{F} \) of Borel functions from \( \mathbb{R} \) to \( \mathcal{P}(\mathbb{N}) \) such that there is no Borel function \( f : \mathbb{R} \to \mathcal{P}_\infty(\mathbb{N}) \) which is a pseudo-intersection of \( \mathcal{F} \).

Our first claim is that \( p = p_1 \).

**Definition 1.4.** Let \( t_1 \) be the cardinality of the smallest tower \( \mathcal{T} \) of Borel functions from \( \mathbb{R} \) to \( \mathcal{P}(\mathbb{N}) \) such that there is no Borel function \( f : \mathbb{R} \to \mathcal{P}_\infty(\mathbb{N}) \) which is a pseudo-intersection of \( \mathcal{T} \).

The other main result of this paper is that \( t = t_1 \). No specialized knowledge of set theory is required in order to understand the proofs.

I wish to thank Stevo Todorcevic for his handwritten notes: "\( \mathcal{C}_\omega \models \hat{p} \leq \hat{\mathcal{p}} \)" from the summer of 1997.

2. The proof

To start, we need the following basic facts about \( p \) and \( t \). Let \( \mathbb{N}^\mathbb{N} \) be the set of all functions from \( \mathbb{N} \) to \( \mathbb{N} \). We define the relation \( \leq^* \) on \( \mathbb{N}^\mathbb{N} \) by

\[
 f \leq^* g \quad \text{iff} \quad \text{there is an } n \in \mathbb{N} \text{ such that } f(k) \leq g(k) \text{ for all } k \geq n,
\]

where \( f, g \in \mathbb{N}^\mathbb{N} \). Let \( b \) be the size of the smallest subfamily of \( \mathbb{N}^\mathbb{N} \) that is unbounded in \( (\mathbb{N}^\mathbb{N}, \leq^*) \).

**Theorem 2.1.** \( \omega_1 \leq p \leq t \leq b \).

**Proof.** See [2].

We also use the following trivial fact.

**Fact 2.2.** \( t \) is regular.

**Proof.** See [2].

Let \( I \) be an ideal. Define

\[
 \text{add}(I) = \min \{|A| : A \subseteq I \text{ and } \bigcup A \notin I\}.
\]

For a topological space \( X \), we let \( \mathcal{M} \) denote the \( \sigma \)-ideal of meager subsets of \( X \). We shall need the following lower bound for \text{add}(\mathcal{M}).

**Theorem 2.3** (Piotrowski–Szymański). For any Polish space, \( \text{add}(\mathcal{M}) \geq t \).

**Proof.** See [4].

**Theorem 2.4.** \( p = p_1 \).

**Theorem 2.5.** \( t = t_1 \).

We prove Theorems 2.4 and 2.5 simultaneously.

**Proofs.** \( t_1 \leq t \): Let \( \{A_\xi\}_{\xi < t} \subseteq \mathcal{P}_\infty(\mathbb{N}) \) be a tower with no infinite pseudo-intersection, such that the enumeration respects the well-ordering of the tower, i.e. \( \xi < \eta \rightarrow A_\eta \subseteq^* A_\xi \). For each \( \xi < t \), define \( f_\xi : \mathbb{R} \to \mathcal{P}_\infty(\mathbb{N}) \) by \( f_\xi(x) = A_\xi \) for all \( x \in \mathbb{R} \). Then obviously \( \{f_\xi\}_{\xi < t} \) is a tower of Borel functions. We need to show that there is no Borel function \( f : \mathbb{R} \to \mathcal{P}_\infty(\mathbb{N}) \) which is a pseudo-intersection of \( \{f_\xi\}_{\xi < t} \).
Suppose to the contrary that \( f \) is such a function. Find a comeager \( G \subseteq \mathbb{R} \) such that \( f \restriction G \) is continuous. For each \( \xi < t \) and \( n \in \mathbb{N} \), define
\[
F_{\xi n} = \{ x \in G : f(x) \cap n \subseteq A_{\xi}\}.
\]
Then each \( F_{\xi n} \) is a relatively closed subset of \( G \). Since \( \bigcup_{n=0}^{\infty} F_{\xi n} \) is a relatively comeager—and in particular, nonmeager—subset of \( G \), there is a nonempty rational interval \( I_{\xi} \) and an \( n_{\xi} \in \mathbb{N} \) such that
\[
G \cap I_{\xi} \subseteq F_{\xi n_{\xi}}.
\]
Since \( t \) is an uncountable regular cardinal (Theorem 2.1 and Fact 2.2), there are \( \pi \in \mathbb{N} \) and a rational interval \( I \) such that
\[
\Gamma = \{ \xi < t : n_{\xi} = \pi \text{ and } I_{\xi} \supseteq I \}
\]
is cofinal in \( t \). Pick \( x \in G \cap I \). Then \( f(x) \setminus \pi \subseteq \bigcap_{\xi \in \Gamma} A_{\xi} \), from which it follows that \( f(x) \) is an infinite pseudo-intersection of \( \{ A_{\xi} \}_{\xi < t} \), a contradiction.

\( p_1 \leq p \): First suppose that \( p = t \). Note that every tower of Borel functions is also filtered. Hence \( p_1 \leq t_1 \leq t = p \).

By Theorem 2.1, it remains to consider the case where \( p < t \). Let \( \{ A_{\xi} \}_{\xi < p} \subseteq \mathcal{P}(\mathbb{N}) \) be a filter base with no infinite pseudo-intersection. For each \( \xi < p \), define \( f_{\xi} : \mathbb{R} \to \mathcal{P}(\mathbb{N}) \) by \( f_{\xi}(x) = A_{\xi} \) for all \( x \in \mathbb{R} \). Obviously \( \{ f_{\xi} \}_{\xi < p} \) is a filtered family of Borel functions. Suppose towards a contradiction that \( f : \mathbb{R} \to \mathcal{P}_\infty(\mathbb{N}) \) is a pseudo-intersection of \( \{ f_{\xi} \}_{\xi < p} \). For each \( \xi < p \), choose a comeager \( G_{\xi} \subseteq \mathbb{R} \) such that \( f(x) \subseteq^* A_{\xi} \) for all \( x \in G_{\xi} \). By assumption and Theorem 2.3, \( p < \text{add}(\mathcal{M}) \), whence there is a comeager \( G \subseteq \bigcap_{\xi < p} G_{\xi} \). But if we take \( x \in G \), then \( f(x) \) is an infinite pseudo-intersection of \( \{ A_{\xi} \}_{\xi < p} \), giving a contradiction.

\( p \leq p_1 , t \leq t_1 \): We take \( \kappa < p \) (\( k < t \)), and prove that \( \kappa < p_1 \) (\( k < t_1 \)). Note that \( \mathbb{R} \) is homeomorphic to the unit interval \((0,1)\) via the standard homeomorphism. Consider the standard surjection \( \Phi : \mathcal{C} \to [0,1] \), where \( \Phi(x) \) is the base 2 expansion of \( x \) after the decimal point, for all \( x \in \mathcal{C} \). \( \Phi \) is a homeomorphism on a co-countable subset of \( \mathcal{C} \). Hence, if we replace the domains of the functions in the formulations of \( p_1 \) and \( t_1 \) with \( \mathcal{C} \), then we are proving an equivalent result.

Suppose that \( f_{\xi} : \mathcal{C} \to \mathcal{P}(\mathbb{N}) (\xi < \kappa) \) is a filtered family (tower) of Borel functions. (For \( \{ f_{\xi} \}_{\xi < \kappa} \) a tower, we assume that the enumeration respects the well-ordering of the tower.) For each \( \xi < \kappa \), we can find a dense \( G_{\delta} \)—and thus comeager—set \( G_{\xi} \subseteq \mathcal{C} \) such that \( f_{\xi} \restriction G_{\xi} \) is continuous. For each nonempty \( F \in [\kappa]^{<\aleph_0} \), write \( F = \{ \xi_1 < \xi_2 < \cdots < \xi_n \} \). Then there is a comeager \( G_{F} \subseteq \mathcal{C} \) such that \( \bigcap_{i=1}^{n} f_{\xi_i}(x) = F \) for all \( x \in G_{F} \) (and for a tower, \( f_{\xi_i}(x) \subseteq^* f_{\xi_j}(x) \) for all \( 1 \leq i < j \leq n \), for all \( x \in G_{F} \)).

By Theorem 2.3, there is a dense \( G_{\delta} \) set \( H \subseteq \mathcal{C} \) such that
\[
H \subseteq \bigcap_{\xi < \kappa} G_{\xi} \cap \bigcap_{F \in [\kappa]^{<\aleph_0}} G_{F}.
\]
Then
\[
(1) \quad \text{for all } \xi < \kappa, f_{\xi} \restriction H \text{ is continuous, and}
\]
\[
(2) \quad \text{for all } x \in H, \{ f_{\xi}(x) \}_{\xi < \kappa} \text{ is a filter base,}
\]
\[
(2') \quad \text{for all } x \in H, \{ f_{\xi}(x) \}_{\xi < \kappa} \text{ is a tower.}
\]
Let \( \{ x_n \}_{n=0}^{\infty} \) be an enumeration of a dense subset of \( H \).
Since \( \kappa < p \) (\( \kappa < t \)), by (2) (by (2')), there is a sequence \( \{d(n)\}_{n=0}^{\infty} \subseteq \mathcal{P}(\mathbb{N}) \) such that
\[
(3) \quad d(n) \subseteq^* f_\ell(x_n) \quad \text{for all } n \in \mathbb{N}, \text{ and all } \xi < \kappa.
\]

For the remainder of the proof, we need only the fact that \( \kappa < b \). By (3), for each \( \xi < \kappa \), we can choose \( g_\xi : \mathbb{N} \to \mathbb{N} \) so that \( d(n) \setminus g_\xi(n) \subseteq f_\xi(x_n) \) for all \( n \in \mathbb{N} \).

Since \( \kappa < b \), there is a \( D : \mathbb{N} \to \mathbb{N} \) such that \( g_\xi \leq^* D \) for all \( \xi < \kappa \). For each \( \xi < \kappa \), fix \( m_\xi \in \mathbb{N} \) so that \( g_\xi(n) \leq D(n) \) for all \( n \geq m_\xi \). Then
\[
(4) \quad d(n) \setminus D(n) \subseteq f_\xi(x_n) \quad \text{for all } \xi < \kappa, \text{ for all } n \geq m_\xi.
\]

Now we measure the continuity of \( f_\xi | H \) at each \( x_n \).

**Notation.** For \( t \in 2^{<\mathbb{N}} \) and \( A \subseteq \mathcal{C}, \{t\}_A = [t] \cap A \).

**Claim 5.** There is a function \( F : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) such that for all \( \xi < \kappa \),
\[
\begin{align*}
f_\xi^\prime\prime | x_n \times F(n, \ell) | H & \subseteq [f_\xi(x_n) | \ell] \quad \text{for all but finitely many } (n, \ell) \in \mathbb{N} \times \mathbb{N}, \\
i.e. \quad [f_\xi(x_n) | \ell] = \{A \subseteq \mathbb{N} : f_\xi(x_n) \cap \ell \subseteq A\}, \text{ where } A \subseteq B \text{ means that } A \text{ is an} \\
\text{initial segment of } B \text{ for } A, B \subseteq \mathbb{N}.
\end{align*}
\]

**Proof.** Let \( \Phi : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \) be a bijection. For each \( \xi < \kappa \), since \( f_\xi | H \) is continuous, there is a function \( F_\xi : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) such that
\[
f_\xi^\prime\prime | x_n \times F_\xi(n, \ell) | H \subseteq [f_\xi(x_n) | \ell]
\]
for all \( n, \ell \in \mathbb{N} \). For each \( \xi < \kappa \), define \( g_\xi : \mathbb{N} \to \mathbb{N} \) by \( g_\xi(m) = F_\xi(\Phi(m)) \) for all \( m \in \mathbb{N} \). Since \( \kappa < b \), there is an \( h : \mathbb{N} \to \mathbb{N} \) so that \( g_\xi \leq^* h \) for all \( \xi < \kappa \). Define \( F : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) by \( F(n, \ell) = h(\Phi^{-1}(n, \ell)) \) for all \( n, \ell \in \mathbb{N} \). \( \square \)

Let \( \{s_i\}_{i=0}^{\infty} \) enumerate \( 2^{<\mathbb{N}} \). We define \( \{n_i\}_{i=0}^{\infty}, \{k_i\}_{i=0}^{\infty} \subseteq \mathbb{N} \), and \( \{t_i\}_{i=0}^{\infty} \subseteq 2^{<\mathbb{N}} \) by recursion on \( i \) so that for all \( i \in \mathbb{N} \),
\[
\begin{align*}
(a) \quad & s_i \subseteq x_{n_i}, \\
(b) \quad & n_i \geq i, \\
(c) \quad & k_i \in d(n_i), \\
(d) \quad & k_i \geq D(n_i), \\
(e) \quad & k_{i+1} > k_i, \\
(f) \quad & t_i \subseteq x_{n_i}, \text{ and} \\
(g) \quad & |t_i| \geq \max(F(n_i, k_i + 1), |s_i|).
\end{align*}
\]

Let \( i \in \mathbb{N} \) be given. Since \( \{x_n\}_{n=0}^{\infty} \) is dense, there are infinitely many \( x_n \)'s extending \( s_i \). Hence we can find \( n_i \geq i \) such that \( s_i \subseteq x_{n_i} \). Let \( k_i \in d(n_i) \) be sufficiently large so that (d) and (e) hold. Then obviously we can find \( t_i \) as required.

Define \( G^* \subseteq \mathcal{C} \) by
\[
G^* = \bigcap_{m=0}^{\infty} \bigcup_{i=m}^{\infty} [t_i].
\]

By (a), (f), and (g), \( t_i \supseteq s_i \) for all \( i \in \mathbb{N} \). It follows that \( G^* \) is a dense \( G_\delta \) set. Hence \( G = G^* \cap H \) is also dense \( G_\delta \). Now we define \( f : G \to \mathcal{P}(\mathbb{N}) \) by
\[
f(x) = \{k_i : i \in \mathbb{N}, t_i \subseteq x\}.
\]

By (e), \( f(x) \) is infinite for all \( x \in G \), whence \( f \) is well-defined.

**Claim 6.** For every \( x \in G \), \( f(x) \subseteq^* f_\xi(x) \) for all \( \xi < \kappa \).
Suppose that \( (3.3) \) above translation yielding \( \dot{f} \) all \( \kappa < \ell \). Therefore, since \( t_i \subseteq x_n \) and \( k_i + 1 > \ell_x \),
\[
  f''_x(t_i) = f''_x(x_n, |t_i|) \subseteq [f\xi(x_n) | t_i] \subseteq [f\xi(x_n) | k_i + 1].
\]
Since \( i \geq m_\xi \), \( n_i \geq m_\xi \) by (b). Hence by (4), (c), and (d), \( k_i \in f\xi(x_n) \). Thus \( k_i \in A \) for all \( A \in [f\xi(x_n) | k_i + 1] \). Since \( t_i \subseteq x \) and \( x \in H, x \in |t_i| \). Therefore \( f\xi(x) \in [f\xi(x_n) | k_i + 1] \), whence \( k_i \in f\xi(x) \).

**Claim 7.** \( f \) is continuous.

**Proof.** Take \( x \in G \) and \( \xi < \kappa \). By Claim 5, there is an \( \ell_x \in \mathbb{N} \) such that
\[
  f''_x(x_n, |F(n, \ell)|) \subseteq [f\xi(x_n) | \ell] \quad \text{for all } n \in \mathbb{N}, \text{ and all } \ell > \ell_x.
\]
We claim that
\[
  f(x) \setminus (\ell_x \cup \{ k_i : i < m_\xi \}) \subseteq f\xi(x).
\]
Suppose that \( i \in \mathbb{N} \) is such that \( k_i \in f(x) \setminus (\ell_x \cup \{ k_i : i < m_\xi \}) \). By (g), \( |t_i| \geq F(n_i, k_i + 1) \). Therefore, since \( t_i \subseteq x_n \) and \( k_i + 1 > \ell_x \),
\[
  f''_x(t_i) = f''_x(x_n, |t_i|) \subseteq [f\xi(x_n) | k_i + 1].
\]
Since \( i \geq m_\xi \), \( n_i \geq m_\xi \) by (b). Hence by (4), (c), and (d), \( k_i \in f\xi(x_n) \). Thus \( k_i \in A \) for all \( A \in [f\xi(x_n) | k_i + 1] \). Since \( t_i \subseteq x \) and \( x \in H, x \in |t_i| \). Therefore \( f\xi(x) \in [f\xi(x_n) | k_i + 1] \), whence \( k_i \in f\xi(x) \).

3. **Adding a single Cohen real**

**Definitions and notation.** The poset for adding a single Cohen real is viewed as the poset of finite partial functions from \( \mathbb{N} \) into 2 which we denote by \( \mathcal{C} \). If \( f \colon \mathcal{C} \rightarrow \mathcal{P}(\mathbb{N}) \) is a Borel function, then \( \tilde{f} \) is a name for the decoding of \( f \) in the forcing extension. Fix \( \mathcal{C} \)-names \( \tilde{p} \) and \( \tilde{t} \) which are forced to be the values of \( p \) and \( t \) in the extension by one Cohen real, respectively. Let \( \dot{c} \) be the canonical name for the Cohen real.

There is a canonical correspondence between names for reals in the Cohen extension and codes for Borel functions from \( \mathcal{C} \) into \( \mathcal{P}(\mathbb{N}) \). We describe this by \( \dot{x} \mapsto f\dot{x} \), where \( \mathcal{C} \models f\dot{x} = \dot{x} \) (see [3]). And in the other direction we have: \( f \mapsto \dot{x}_f \), where \( \mathcal{C} \models \dot{x}_f = \dot{x} \subseteq \dot{x}_f \subseteq \dot{x} \). Moreover, it is an easy exercise to verify that given any two names \( \dot{x} \) and \( \dot{y} \) for reals,
\[
  (3.1) \quad \mathcal{C} \models \dot{x} \subseteq \dot{y} \quad \text{iff} \quad f\dot{x} \subseteq f\dot{y}.
\]
Also, for any finite sequence \( \dot{x}_1, \ldots, \dot{x}_n \) of names for reals,
\[
  (3.2) \quad \mathcal{C} \models \bigcap_{k=1}^n \dot{x}_k = \mathbb{N}_0 \quad \text{iff} \quad |\bigcap_{i=1}^n f\dot{x}_k(y)| = \mathbb{N}_0 \text{ for almost all } y.
\]
For example, if \( \{ f\xi \}_{\xi < \kappa} \) is a name for a tower, then by (3.1) and (3.2), \( \{ f\xi \}_{\xi < \kappa} \) is a tower of Borel functions. In this manner it is easily checked that
\[
  (3.3) \quad \mathcal{C} \models \tilde{p} = \tilde{p}_1, \quad \text{and}
\]
\[
  (3.4) \quad \mathcal{C} \models \tilde{t} = \tilde{t}_1.
\]
In effect, we are viewing \( \mathcal{C} \)-names as Borel codes (e.g. the code for \( f\dot{x} \)), with the above translation yielding \( \tilde{p} = \tilde{p}_1 \) and \( \tilde{t} = \tilde{t}_1 \). By (3.3) and (3.4), the following Corollaries are immediate from Theorems 2.4 and 2.5.
Corollary 3.5. \( C_\omega \models \check{\dot{p}} = \check{\dot{p}} \).

Corollary 3.6. \( C_\omega \models \check{\dot{i}} = \check{\dot{i}} \).

Remark 3.7. The fact that MA(\( \sigma \)-centered) is preserved under the addition of a single Cohen real is known as Roitman’s Theorem [5]. By Bell’s Theorem [1], Roitman’s Theorem states that \( C_\omega \models \check{\dot{p}} \geq \check{\dot{p}} \).

References


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