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# EVERY $(\lambda^+, \varkappa^+)$-REGULAR ULTRAFILTER IS $(\lambda, \varkappa)$-REGULAR

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Abstract. We prove the following:

**Theorem A.** *If $D$ is a $(\lambda^+, \varkappa)$-regular ultrafilter, then either*
(a) *$D$ is $(\lambda, \varkappa)$-regular, or*
(b) *the cofinality of the linear order $\prod_D \langle \lambda, < \rangle$ is $\operatorname{cf} \varkappa$, and $D$ is $(\lambda, \varkappa')$-regular for all $\varkappa' < \varkappa$.*

**Corollary B.** *Suppose that $\varkappa$ is singular, $\varkappa > \lambda$ and either $\lambda$ is regular, or $\operatorname{cf} \varkappa < \operatorname{cf} \lambda$. Then every $(\lambda^{+n}, \varkappa)$-regular ultrafilter is $(\lambda, \varkappa)$-regular.*

We also discuss some consequences and variations.

The notion of a $(\lambda, \varkappa)$-regular ultrafilter has been introduced by J. Keisler in [Kei]. An ultrafilter $D$ is $(\lambda, \varkappa)$-*regular* iff there is a family of $\varkappa$ members of $D$ such that the intersection of any $\lambda$ members of the family is empty.

In [Kei] Keisler proved some cardinality results about ultraproducts taken modulo such ultrafilters. Further results were proved in the 70's: for example, the following are theorems of ZFC:

(a) Every $(\lambda^+, \lambda^+)$-regular ultrafilter is $(\lambda, \lambda)$-regular ([CC], [KP]).

(b) If $\lambda$ is singular, then every $(\lambda^+, \lambda^+)$-regular ultrafilter is $(\lambda, \lambda^+)$-regular [Ka]; moreover, it is either $(\operatorname{cf} \lambda, \operatorname{cf} \lambda)$-regular or $(\lambda', \lambda^+)$-regular for some $\lambda' < \lambda$ ([CC], [KP]).

(c) If $2^\varkappa = \varkappa^+$ and $2^{\varkappa^+} > \varkappa^{++}$, then every $(\varkappa^+, \varkappa^+)$-regular ultrafilter is $(\varkappa, \varkappa^+)$-regular ([BK], [Ket]).

It was soon realized, however, that (ir)regular ultrafilters are connected with large cardinals, inner models, and combinatorial or reflection principles; this paved the way for significant applications to set theory, but seemed to dash any hope that other results besides (a)–(c) above can be proved in ZFC alone (see e.g. [KM] or [Lp1] for further references).

However, in [Lp1] (more than twenty years later) we proved some slight improvements of (b), as well as a "down from exponents" transfer result for $(\lambda, \lambda)$-regularity; we also suggested the possibility that further results are theorems of ZFC. It is actually so; in this paper we prove the following generalization of (a):

**Theorem 1.** *If $n < \omega$, then every $(\lambda^{+n}, \varkappa^{+n})$-regular ultrafilter is $(\lambda, \varkappa)$-regular.*

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Theorem 1 solves a problem raised in [Lp1]. Actually, (the proof of) Theorem 1 has some further consequences:

**Corollary 1.** *Suppose that $\lambda \leq \varkappa$, $\lambda$ is a regular cardinal, and $D$ is a $(\lambda^+, \varkappa)$-regular ultrafilter. Then the following are equivalent*:

(i) *$D$ is $(\lambda, \varkappa)$-regular*;
(ii) *the cofinality of $\prod_D \langle \lambda, < \rangle$ is $> \varkappa$*;
(iii) *the cofinality of $\prod_D \langle \lambda, < \rangle$ is different from $\mathrm{cf}\, \varkappa$.*

**Corollary 2** ([Lp1]). *Suppose that $\mathrm{cf}\, \lambda \neq \mathrm{cf}\, \varkappa$. If $D$ is $(\lambda^+, \varkappa)$-regular and not $(\mathrm{cf}\, \lambda, \mathrm{cf}\, \lambda)$-regular, then $D$ is $(\lambda, \varkappa)$-regular.*

**Corollary 3.** *If $D$ is $(\lambda^+, \varkappa)$-regular and $(\mathrm{cf}\, \lambda, \mathrm{cf}\, \varkappa)$-regular, then $D$ is $(\lambda, \varkappa)$-regular.*

**Corollary 4.** *Suppose that $D$ is $(\lambda^+, \varkappa)$-regular and $(\lambda'^+, \varkappa')$-regular. If $\mathrm{cf}\, \lambda = \mathrm{cf}\, \lambda'$ and $\mathrm{cf}\, \varkappa \neq \mathrm{cf}\, \varkappa'$, then $D$ is either $(\lambda, \varkappa)$-regular or $(\lambda', \varkappa')$-regular.*

The following improves [BK, Theorem 1.3 and Corollaries 1.8 and 2.5]:

**Proposition.** *Suppose that $D$ is a $(\lambda^+, \varkappa)$-regular ultrafilter, and that the cofinality of the linear order $\prod_D \langle \lambda, < \rangle$ is different from $\mathrm{cf}\, \varkappa$. Then $D$ is $(\lambda, \varkappa)$-regular.*

Notice that, trivially, the cofinality of $\prod_D \langle \lambda, < \rangle$ equals the cofinality of $\prod_D \langle \mathrm{cf}\, \lambda, < \rangle$.

In order to discuss some further results, let us introduce some notation and definitions (see [CK] for basics about models and ultrafilters).

$S_\mu(X)$ denotes the set of all subsets of $X$ of cardinality $< \mu$. If $X$ is a well-ordered set and $\alpha$ is an ordinal, $S_\alpha(X)$ is the set of all subsets of $X$ which have order type $< \alpha$ (notice that the two definitions are consistent).

Throughout this paper, we shall use the following reformulation of $(\lambda, \varkappa)$-regularity. As Benda and Ketonen [BK] noticed, it makes sense even when $\lambda$ is allowed to be an ordinal: the next corollary deals with such a situation. See [Lp1] for further comments.

An ultrafilter $D$ is *$(\beta, \varkappa)$-regular* iff in the ultrapower $\prod_D \langle S_\beta(\varkappa), \subseteq, \{\alpha\} \rangle_{\alpha \in \varkappa}$ there is an element $x$ such that $\{\alpha\}_{\subseteq x}$, for every $\alpha \in \varkappa$ (equivalently, for $\varkappa$-many $\alpha$'s).

**Corollary 5.** *Let $D$ be an ultrafilter, and let $\varkappa$ be a cardinal. Let $\gamma$ be the least ordinal such that $D$ is $(\gamma, \varkappa)$-regular. Then either*:

(i) *$\gamma$ is a limit ordinal and $D$ is $(\mathrm{cf}\, \gamma, \mathrm{cf}\, \gamma)$-regular, or*
(ii) *$\gamma = \delta + 1$, $\delta$ is a limit ordinal and the cofinality of $\prod_D \langle \delta, < \rangle$ is $\mathrm{cf}\, \varkappa$. Hence either $D$ is $(\mathrm{cf}\, \delta, \mathrm{cf}\, \delta)$-regular and $\mathrm{cf}\, \varkappa > \mathrm{cf}\, \delta$, or $\mathrm{cf}\, \varkappa = \mathrm{cf}\, \delta$ and $D$ is not $(\mathrm{cf}\, \delta, \mathrm{cf}\, \delta)$-regular.*

**Theorem 2.** *There is a finite expansion $\mathbf{A}^+$ of the model $\langle S_{\lambda^{+n}}(\mu^{+n}), V, \subseteq, \{\alpha\} \rangle_{\alpha \in \mu^{+n}}$ (where $V(a)$ iff $a \in S_\lambda(\mu)$) such that whenever $\mathbf{B} \equiv \mathbf{A}^+$ and there is $b \in B$ such that $|\{\alpha \in \mu^{+n} | \mathbf{B} \models \{\alpha\} \subseteq b\}| = \mu^{+n}$, then there is $b' \in B$ such that $V(b')$ and $|\{\alpha \in \mu | \mathbf{B} \models \{\alpha\} \subseteq b'\}| = \mu$.*

The statement of Theorem 2 reads $\mathrm{alm}(\lambda^{+n}, \mu^{+n}) \Rightarrow \mathrm{alm}(\lambda, \mu)$, in the notation of [Lp2]. There we also discuss the interest of such results. Notice that, in view of the reformulation of $(\lambda, \mu)$-regularity we have given, Theorem 2 is stronger than Theorem 1.

Finally, let us mention that $(\lambda, \varkappa)$-regular ultrafilters have found other applications both in general topology and in the model theory for abstract logics (see [Lp1], [Lp2] for further references).

A notice to the reader wanting to consult the literature on regular ultrafilters: results like (a)–(c) above are sometimes stated in equivalent forms in terms of the closely related notions of $\lambda$-*decomposability*, $\lambda$-*descending incompleteness* and *uniformity*. We shall not need those notions in the present paper, but, for the convenience of the reader, we give below a "table of correspondence".

For every cardinal $\lambda$ and every ultrafilter $D$, the following hold:

(i) $D$ is $\lambda$-descendingly incomplete iff it is $(\mathrm{cf}\,\lambda, \mathrm{cf}\,\lambda)$-regular.
(ii) If $D$ is $\lambda$-decomposable, then $D$ is $(\mathrm{cf}\,\lambda, \mathrm{cf}\,\lambda)$-regular.
(iii) If $D$ is $(\mathrm{cf}\,\lambda, \mathrm{cf}\,\lambda)$-regular, then $D$ is $(\lambda, \lambda)$-regular.
(iv) $D$ is $\lambda$-decomposable iff there is a $D'$ uniform on $\lambda$ and $D' \leq D$ in the Rudin Keisler (pre)-order.

Moreover, if $\lambda$ is regular, then,

(v) if $D$ is $(\lambda, \lambda)$-regular, then $D$ is $\lambda$-decomposable.

By the above statements, for $\lambda$ a regular cardinal, the notions of $(\lambda, \lambda)$-regularity, $\lambda$-descending incompleteness and $\lambda$-decomposability are all equivalent. In particular, the notion of $(\lambda, \varkappa)$-regularity encompasses all the above notions, except for $\lambda$-decomposability when $\lambda$ is singular.

Moreover, it is trivial that if $D' \leq D$ in the Rudin Keisler order, and $D'$ is $(\lambda, \varkappa)$-regular, then $D$ is $(\lambda, \varkappa)$-regular, also.

These facts make it possible, for example, to reformulate (a) together with the second statement in (b) as: every uniform ultrafilter over $\lambda^+$ is either $\lambda$-descendingly incomplete or $(\lambda', \lambda^+)$-regular for some $\lambda' < \lambda$.

We now prove the stated results. The statement in the title of the paper is a consequence of Theorem A in the abstract (when $\varkappa$ is a successor cardinal). Theorem 1 follows now from a simple induction. Corollary 4 is immediate from the proposition.

*Proof of the Proposition.* For every $x \in S_{\lambda^+}(\varkappa)$, let $G(x, \beta)$ $(\beta < \lambda)$ be a sequence of subsets of $x$ such that:

(i) if $\beta \leq \beta' < \lambda$, then $G(x, \beta) \subseteq G(x, \beta')$;
(ii) $|G(x, \beta)| \leq |\beta|$, for every $\beta < \lambda$;
(iii) $\bigcup_{\beta<\lambda} G(x, \beta) = x$.

Consider the model $\mathbf{A} = \langle S_{\lambda^+}(\varkappa), \subseteq, U, <, G, \{\alpha\}\rangle_{\alpha\in\varkappa}$, where $\langle U, <\rangle = \langle \lambda, <\rangle$ and $G$ is the above function from $A \times U$ to $A$.

Thus, by (iii) above, for every $\alpha \in \varkappa$, $\mathbf{A}$ satisfies

$$(*) \qquad \forall x(\{\alpha\} \subseteq x \Rightarrow \exists w(U(w) \wedge \{\alpha\} \subseteq G(x, w))).$$

Let $D$ be a $(\lambda^+, \varkappa)$-regular ultrafilter; let us work in $\mathbf{B} = \prod_D \mathbf{A}$; and let $y$ witness the $(\lambda^+, \varkappa)$-regularity of $D$. By $(*)$, and since $y$ witnesses that $D$ is $(\lambda^+, \varkappa)$-regular, for every $\alpha \in \varkappa$, there is a $w_\alpha$ in $\mathbf{B}|_U = \prod_D \lambda$ such that, in $\mathbf{B}$, $\{\alpha\} \subseteq G(y, w_\alpha)$.

For every $w \in \mathbf{B}|_U$, let $X_w = \{\alpha | \, \mathbf{B} \models \{\alpha\} \subseteq G(y, w)\}$. Notice that:

(I) The union of all the $X_w$'s $(w \in \mathbf{B}|_U)$ is $\varkappa$, by the above argument.
(II) If $w \leq w' \in \mathbf{B}|_U$, then $X_w \subseteq X_{w'}$, because of clause (i) (and elementarity).

It is enough to show that there is $w \in \mathbf{B}|_U$ such that $|X_w| = \varkappa$. If this happens, then $G(y, w)$ witnesses the $(\lambda, \varkappa)$-regularity of $D$ (because of (ii), $G(y, w)$ is in $\prod_D S_\lambda(\varkappa)$).

Suppose by contradiction that

(III) $|X_w| < \varkappa$, for all $w \in \mathbf{B}|_U$.

Easy cofinality arguments then show that (I)–(III) imply that $\mathrm{cf}\langle \mathbf{B}|_U, < \rangle = \mathrm{cf}\, \varkappa$, contradicting the hypothesis. □

*Proof of Theorem* A. If (a) fails, then the proposition implies that $\mathrm{cf} \prod_D \langle \lambda, < \rangle$ is $\mathrm{cf}\, \varkappa$.

If $\varkappa' < \varkappa$, then $D$ is trivially $(\lambda^+, \varkappa')$-regular. If $\mathrm{cf}\, \varkappa' \neq \mathrm{cf}\, \varkappa$, then the proposition (applied to $(\lambda^+, \varkappa')$-regularity) implies that $D$ is $(\lambda, \varkappa')$-regular.

But this is enough, since for every $\varkappa'' < \varkappa$ there is $\varkappa'$ such that $\mathrm{cf}\, \varkappa' \neq \mathrm{cf}\, \varkappa$ and $\varkappa'' \leq \varkappa' < \varkappa$. □

*Proof of Corollary 1.* (i)⇒(ii) is standard (e.g. [BK, p. 233]).

(ii)⇒(iii) is trivial.

(iii)⇒(i) is immediate from the proposition. □

*Remark.* The hypothesis $\lambda$ regular is necessary in Corollary 1. If $\mu$ is strongly compact, then there exists a $\mu$-complete $(\mu, \mu^{+\omega+\omega})$-regular ultrafilter $D$. If we take $\lambda = \mu^{+\omega}$ and $\varkappa = \mu^{+\omega+\omega}$, then condition (i) holds, but (ii) and (iii) fail ($\mathrm{cf} \prod_D \langle \lambda, < \rangle = \mathrm{cf} \prod_D \langle \omega, < \rangle = \omega$, since $D$ is $\omega_1$-complete).

We wonder whether some generalization of Corollary 1 is possible when $\lambda$ is singular.

Notice that Corollary 1 and Theorem A imply that if $\lambda$ is regular, and $D$ is $(\lambda^+, \varkappa)$-regular, then $\mathrm{cf} \prod_D \langle \lambda, < \rangle \geq \varkappa$.

*Proof of Corollary* B. An induction on $n$ shows that it is enough to prove the particular case when $n = 1$. So assume that $\varkappa$ is singular, $\varkappa > \lambda$ and $D$ is $(\lambda^+, \varkappa)$-regular.

Case 1: $\lambda$ is regular. By Theorem A, $D$ is $(\lambda, \varkappa')$-regular for all $\varkappa' < \varkappa$. By, e.g., Corollary 1, the cofinality of $\prod_D \langle \lambda, < \rangle$ is $> \varkappa'$, for all $\varkappa' < \varkappa$. Since $\varkappa$ is singular, the cofinality of $\prod_D \langle \lambda, < \rangle$ must be $> \varkappa$; in particular, it is $\neq \mathrm{cf}\, \varkappa$. Hence, case (b) in Theorem A cannot occur.

Case 2: $\mathrm{cf}\, \varkappa < \mathrm{cf}\, \lambda$. Then case (b) in Theorem A cannot occur, since the cofinality of $\prod_D \langle \lambda, < \rangle$ is $\mathrm{cf}\, \lambda$ if $D$ is not $(\mathrm{cf}\, \lambda, \mathrm{cf}\, \lambda)$-regular, and is $> \mathrm{cf}\, \lambda$ otherwise. Hence, we are in case (a) and we are done. □

*Proof of Corollary 2.* $D$ is $(\mathrm{cf}\, \lambda, \mathrm{cf}\, \lambda)$-regular iff the cofinality of $\prod_D \langle \mathrm{cf}\, \lambda, < \rangle$ is different from $\mathrm{cf}\, \lambda$ (e.g., by Corollary 1). So it is enough to apply the proposition. □

*Proof of Corollary 3.* If $\varkappa \leq \lambda$, the result is trivial. The case $\mathrm{cf}\, \varkappa < \mathrm{cf}\, \lambda$ is covered by Corollary B. Otherwise, by Corollary 1, the cofinality of $\prod_D \langle \mathrm{cf}\, \lambda, < \rangle$ is $> \varkappa$. Now apply the proposition. □

*Proof of Theorem 2.* It is enough to prove the case $n = 1$.

The proofs of the proposition and of Theorem A have been devised to work as a proof for Theorem 2.

Add to the model $\mathbf{A}$ the relations and functions necessary in order to carry over the proof of the proposition in the two cases $\varkappa = \mu^+$ and $\varkappa = \mu$. Notice that $U = \lambda$ is the same in both cases.

Let $\mathbf{B}$ be as in the statement of Theorem 2. The cofinality of $\langle \mathbf{B}|_U, < \rangle$ is different from $\mathrm{cf}\,\varkappa$ for at least one of the choices $\varkappa = \mu^+$ and $\varkappa = \mu$. Now it is enough to apply the arguments in the proof of the proposition, for the appropriate choice of $\varkappa$. ☐

Of course, a generalization of Theorem A can be given along the lines of the above proof. We can obtain the separation in cases (a) and (b); we can also obtain that, in case (b), the cofinality of $\langle U, < \rangle$ becomes $\mathrm{cf}\,\varkappa$, in $\mathbf{B}$ (but notice that, when $\varkappa$ is singular, we need $\mathrm{cf}\,\varkappa$ symbols in the expansion $\mathbf{A}^+$). Corollaries 2 and 4, too, can be generalized.

We can also generalize Corollaries 1, 3 and B in this fashion, but some technicalities are needed in the proof: we have to add Skolem functions to $\mathbf{A}^+$, and consider the substructure of $\mathbf{B}$ generated by $b$ (seemingly, $\varkappa^+$ symbols are needed in $\mathbf{A}^+$).

*Proof of Corollary 5.* (i) Let $\gamma$ be limit. For every $x \in S_\gamma(\varkappa)$ let $F(x)$ be the order type of $x$. Consider an appropriate expansion of $S_\gamma(\varkappa)$, and take its ultrapower under $D$. If $y$ witnesses $(\gamma, \varkappa)$-regularity, then $\beta < F(y)$, for every $\beta < \gamma$, since, otherwise, $D$ would be $(\beta + 1, \varkappa)$-regular. This easily implies that $D$ is $(\mathrm{cf}\,\gamma, \mathrm{cf}\,\gamma)$-regular.

Essentially, this is the argument used in [Lp2, Corollary 1.4].

(ii) If $\gamma = \delta + 1$, then it is easy to show that $\delta$ is limit.

A variation on the proof of the proposition shows that the cofinality of $\prod_D \langle \delta, < \rangle$ is $\mathrm{cf}\,\varkappa$. Just consider $U$ to be $\langle \delta, < \rangle$ and for $\beta < \delta$ let $G(x, \beta)$ be the initial subset of $x$ of order type $\beta$. Now observe that the case corresponding to (a) in Theorem A cannot occur since $\gamma$ is the least ordinal for which $D$ is $(\gamma, \varkappa)$-regular.

The last remark follows from the fact that the cofinality of $\prod_D \langle \delta, < \rangle$ is equal to the cofinality of $\prod_D \langle \mathrm{cf}\,\delta, < \rangle$. ☐

## References


[BK] M. Benda and J. Ketonen, *Regularity of ultrafilters*, Israel J. Math. **17** (1974), 231–240. MR **53:**132

[CC] G. V. Cudnovskii and D. V. Cudnovskii, *Regular and descending incomplete ultrafilters* (English Translation), Soviet Math. Dokl. **12** (1971), 901–905.

[CK] C. C. Chang and H. J. Keisler, *Model Theory*, North Holland, Amsterdam, 1977. MR **58:**27177

[Ka] A. Kanamori, *Weakly normal filters and irregular ultrafilters*, Trans Amer. Math. Soc. **220** (1976), 393–399. MR **58:**240

[KM] A. Kanamori and M. Magidor, *The evolution of large cardinal axioms in Set Theory*, in Higher Set Theory, Lecture Notes in Mathematics, 669, Springer-Verlag, Berlin, 1978. MR **80b:**03083

[Kei] J. Keisler, *On cardinalities of ultraproducts*, Bull. Amer. Math. Soc. **70** (1964), 644–647.

[Ket] J. Ketonen, *Nonregular ultrafilters and large cardinals*, Trans. Amer. Math. Soc. **224** (1976), 61–73. MR **54:**7260

[KP] K. Kunen and K. L. Prikry, *On descendingly incomplete ultrafilters*, J. Symbolic Logic **36** (1971), 650–652. MR **46:**1585

[Lp1] P. Lipparini, *More on regular ultrafilters in ZFC*, revised for J. Symbol. Logic.

[Lp2] P. Lipparini, *Ultrafilter translations*, I: $(\lambda, \lambda)$*-compactness of logics with a cardinality quantifier*, Arch. Math. Logic **35** (1996), 63–87. MR **96k:**03087



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